Similarity Solutions of the Porous Medium Equation with Sign Changes

by

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1. Introduction.

In this paper we study similarity solutions with sign changes of the porous medium equation in one space dimension:

$$u_t = (|u|^{m-1}u_x)_x, \quad x \in R, \quad t > 0, \tag{1.1}$$

with $m > 1$. Here u is a function of x and t.

For nonnegative u equation (1.1) arises in the theory of gas flow through a one-dimensional porous medium, and an extensive theory has been developed over the last decades, see e.g. [1,3,15]. More recently [4,5,12,13] applied and pure mathematicians have also become interested in solutions of (1.1) with sign changes.

The most striking property of (1.1) is that solutions may have compact support whose (free) boundaries (or *interfaces)* move outwards with a finite, possibly zero speed proportional to the one-sided x-derivative of $|u|^{m-1}$ at the free boundary, taken from the interior of the support. This is usually called the *interface condition.* The expression between brackets in (1.1), i.e. $|u|^{m-1}u_x = (\frac{1}{m}|u|^{m-1}u)_x$, which we shall think of as the *flux* because of the divergence form of (1.1), is continuous with respect to x in zeros of u. If it is nonzero we shall call this the *sign change condition.* We observe that for u bounded away from zero and infinity equation (1.1) is uniformly parabolic and local smoothness follows from standard regularity theory.

For the development of the theory of (1.1) *similarity solutions* play an important role. Essentially these are solutions whose profiles remain the same as t varies. In the nonnegative case they were studied extensively in [7,8,9]. The most familiar of them are of course the *Barenblatt-Pattle* solutions, given by

$$u(x,t) = t^{-\frac{1}{m+1}}U(\eta), \tag{1.2}$$

where

$$\eta = xt^{-\frac{1}{m+1}}, \tag{1.3}$$

and

$$U(\eta) = \pm\Big\{C - \frac{m-1}{2(m+1)}\eta^2\Big\}_+^{\frac{1}{m-1}}, \tag{1.4}$$

with $C > 0$ an arbitrary constant. Up to some scaling, every nonnegative solution of (1.1) eventually converges to a Barenblatt-Pattle solution and recently Kamin and Vasguez [13] have extended this result to solutions with positive integral by showing that they have to become nonnegative in finite time.

Another well known family of similarity solutions consists of the *"dipole"* solutions, given by

$$u(x,t) = t^{-\frac{1}{m}} U(\eta), \tag{1.5}$$

where

$$\eta = x t^{-\frac{1}{2m}}, \tag{1.6}$$

and

$$U(\eta) = \pm |\eta|^{\frac{1}{m}-1} \eta \{ C - \frac{m-1}{2(m+1)} |\eta|^{\frac{m+1}{m}} \}_{+}^{\frac{1}{m-1}}, \tag{1.7}$$

with $C > 0$ an arbitrary constant. These are the asymptotic profiles for solutions with zero integral (with a different scaling of course)[14].

Both the Barenblatt-Pattle and the dipole solutions have compact expanding support and the question as to whether there are any more is a natural one. Our result concerns similarity solutions of the form

$$u(x,t) = t^{\alpha} U(\eta), \tag{1.8}$$

where

$$\eta = x t^{-\beta}, \tag{1.9}$$

and α and β are fixed reals. It is well known that they have to be related by

$$2\beta = (m-1)\alpha + 1, \tag{1.10}$$

and that $U(\eta)$ has to satisfy the second order ordinary differential equation

$$(|U|^{m-1}U)'' + m\beta\eta U' = m\alpha U. \tag{1.11}$$

The following theorem classifies the similarity solutions of this form with compact support and includes the Barenblatt-Pattle and the dipole families.

THEOREM 1.1. *Let $m > 1$. Then there exists a strictly decreasing sequence*

$$\alpha_0 = -\frac{1}{m+1} > \alpha_1 = -\frac{1}{m} > \alpha_2 > \alpha_3 > ... \downarrow -\frac{1}{m-1}, \tag{1.12}$$

such that (1.1) *has (a one-parameter family of) compactly supported solutions of the form given by* (1.8-10) *if and only if $\alpha = \alpha_k$ for some integer $k \geq 0$. Moreover, k equals exactly the number of sign changes of $U(\eta)$, and $U(\eta)$ is symmetric (anti-symmetric) if k is even (odd).*

These solutions $u(x,t)$ all have

$$\alpha < 0 \text{ and } \beta > 0, \tag{1.13}$$

so their supports spread out and their profiles decay as t increases. Profiles with more sign changes decay faster but spread out more slowly.

In the remainder of the introduction we give an outline of the proof of Theorem 1.1 with some remarks along the way. The details of the proof will be dealt with in Sections 2 and 3. In Section 4 we discuss some corollaries of the proof, and in Section 5 finally we generalize the results on symmetric solutions to radially symmetric solutions in arbitrary space dimension.

Note that (1.11) can be studied in itself without any conditions on α and β. Every time we use (1.10) we shall state this explicitly. Just as (1.1), (1.11) is degenerate in zeros of U and the correct matching condition is the continuity of $(|U|^{m-1}U)'$.

In [5], Zhang et al. have studied the case

$$\alpha = 0 \text{ and } \beta = \frac{1}{2}, \tag{1.14}$$

by transforming (1.11) to a singular elliptic boundary value problem. For $m > 1$ they obtained that then all solutions of (1.11) are bounded, monotone and that the limits at $\pm\infty$ determine the solution uniquely.

On the other hand, when

$$\beta = 0 \text{ and } \alpha = -\frac{1}{m-1}, \tag{1.15}$$

one can set $W = |U|^{m-1}U$ and multiply (1.11) by $W'(\eta)$. Integration then yields

$$\frac{1}{2}W'(\eta)^2 + \frac{m^2}{m^2-1}|W(\eta)|^{\frac{m+1}{m}} = C, \tag{1.16}$$

with C an arbitrary positive constant, so that all solutions of (1.11) are periodic in $x = \eta$.

Our method relies on a number of transformations of (1.11). The first one goes back to [11]. Set

$$\tau = \log|\eta|, \quad x(\tau) = \frac{\eta U'(\eta)}{U(\eta)}, \quad y(\tau) = \eta^2 |U(\eta)|^{1-m}. \tag{1.17}$$

For U and η nonzero (1.17) transforms (1.11) into a quadratic autonomous system in the open upper half (x,y)-plane H^+:

$$(Q)\begin{cases} \dot{x} = x - mx^2 + \alpha y - \beta xy; \\ \dot{y} = y(2 + (1-m)x), \end{cases}$$

where the dot denotes derivative with respect to τ. Note that H^+ is invariant and that (Q) also acts on the open lower half plane H^-. This latter action however is irrelevant for our purposes here. In fact the orbits in H^- contain similarity profiles of (1.1) with t replaced by $-t$.

Because of the symmetry with respect to U and η as well as the τ-invariance of (Q), every orbit of (Q) contains four one-parameter families of either positive or negative solutions $U(\eta)$ defined on maximal open subintervals of either $\mathbf{R}^+$ or $\mathbf{R}^-$.

Orbits coming out of critical points of (Q) contain solutions which exist but do not oscillate as $\eta \to 0$. There are three of those critical points, namely the origin $O = (0,0)$, the point $(\frac{1}{m}, 0)$ and the point $(\frac{2}{m-1}, -\frac{2}{m-1}\frac{m+1}{2\beta-(m-1)\alpha})$. For $m > 1$ and α and β satisfying (1.10) or (1.13) this last one lies in H^- and is not of interest here. Therefore we shall not discuss it.

The critical point $(x, y) = (0, \frac{1}{m})$. For positive m this one is always a saddle with one orbit γ_a coming out into H^+, one coming out into H^-, and two orbits going in contained in the positive x-axis. The latter three do not contain solutions of (1.11), but the orbit γ_a is important. It contains all the solutions $U(\eta)$ for which

$$\lim_{\eta\to 0} U(\eta) = 0, \tag{1.18}$$

while

$$\lim_{\eta\to 0} (|U|^{m-1}U)'(\eta) \neq 0. \tag{1.19}$$

Thus we have found the solutions which satisfy the *sign change condition* in $\eta = 0$. Loosely speaking γ_a contains the *anti-symmetric solutions*. We have to be cautious here because we shall see that for α and β both negative there are still other anti-symmetric solutions which have a zero in $\eta = 0$ but do not satisfy (1.19) and are not contained in γ_a.

The critical point $(x, y) = (0, 0)$. The linearization of (Q) in O has eigenvalues 1 and 2, no matter what the parameters are. Thus the origin is a repeller. The orbits coming out of it into H^+ contain exactly all solutions for which

$$\lim_{\eta\to 0} U(\eta) \neq 0. \tag{1.20}$$

One orbit is of special interest, namely the unique orbit γ_s coming out of O into H^+ along the eigenvector corresponding to 2. It contains all the solutions which in addition to (1.20) satisfy

$$\lim_{\eta\to 0} U'(\eta) = 0. \tag{1.21}$$

Loosely speaking again we say that γ_s contains the *symmetric solutions* but just as above the situation is a little more complicated again for α and β both negative. This will become clear when we discuss the critical points at infinity.

All the other orbits coming out of O are tangent to the x-axis and in pairs they contain all solutions satisfying (1.21) with

$$\lim_{\eta\to 0} U'(\eta) \neq 0. \tag{1.22}$$

Our next observation is that for m positive (Q) does not have periodic orbits. More precisely, every orbit in H^+ comes out of one of the two critical points above or out of infinity and, since there are no orbits in H^+ that go into either of these two, every orbit eventually escapes to infinity.

Thus we have to analyze the behaviour of (Q) near infinity. Following Poincaré we set

$$x = \frac{\rho}{1-\rho}\cos\phi, \quad y = \frac{\rho}{1-\rho}\sin\phi, \tag{1.23}$$

which transforms (Q) into

$$(\tilde{Q}) \begin{cases} (1-\rho)\dot{\rho} = \rho(1-\rho)^2\{2 - \cos\phi(\cos\phi - \alpha\sin\phi)\} \\ \qquad\qquad -\rho^2(1-\rho)\cos\phi\{m - 1 + \cos\phi(\cos\phi + \beta\sin\phi)\}; \\ (1-\rho)\dot{\phi} = \rho\sin\phi\cos\phi(\cos\phi + \beta\sin\phi) + (1-\rho)\sin\phi(\cos\phi - \alpha\sin\phi), \end{cases}$$

where $(\rho,\phi) \in R = [0,1]\times[0,\pi]$. Note that for the orbits the factor $(1-\rho)$ on the left hand sides of $(\tilde{Q})$ is irrelevant, and will therefore be ignored.

The interior of R, being a diffeomorphic image of H^+, contains no critical points. At $\rho = 1$ we find four critical points, namely at angles $\phi = \frac{\pi}{2}$, $\phi =$arccotanβ, $\phi = 0$ and $\phi = \pi$. These are the so-called *critical points at infinity.*

The critical point $(\rho,\phi) = (1, \frac{\pi}{2})$. This one is a local attractor for R when $m > 1$ and β is positive. The orbits going into it from the interior of R contain precisely the solutions for which

$$x = \frac{\eta U'(\eta)}{U(\eta)} \to \frac{\alpha}{\beta}, \tag{1.24}$$

as $\tau \to \infty$, i.e. as $\eta \to \pm\infty$, depending on the sign of η. All these solutions have *noncompact support.* If α and β satisfy (1.13) it follows from (1.24) that $U(\eta) \to 0$. On the other hand, if both α and β are negative we have a local repeller for R which means that the solutions contained in the orbits coming out of this critical point satisfy (1.24) as $\eta \to 0$. This implies that both U and the flux vanish as $\eta \to 0$. We call this *the infinite waiting time condition* since we can extend any of these solutions to the other side of $\eta = 0$ by setting it equal to zero there, and thus obtain a solution with a frozen free boundary. Note that any two of these solutions defined for η positive and η negative respectively, can be glued to one another to give one single solution. Thus we can create a multitude of symmetric and anti-symmetric solutions, which are not contained in γ_s or γ_a, as well as asymmetric solutions with a zero in $\eta = 0$ but not contained in the orbits coming out of O. In Section 4 we shall make this picture complete.

The critical point $(\rho,\phi) = (1,$arccotan$(-\beta))$. For $m > 1$ and $\beta \neq 0$ this one is a saddle. If β is positive the two orbits coming out are contained in $\rho = 1$ and therefore of no further interest to us here, while there is one unique orbit γ_i going into this critical point from the interior of R. This happens in a finite τ-time and γ_i contains the solutions for which

$$|(|U|^{m-1})'(\eta)| \to |(m-1)\beta\eta^*|, \tag{1.25}$$

as $\eta \uparrow \eta^*$ when η^* is positive, and $\eta \downarrow \eta^*$ when η^* is negative. This is the *interface condition* with the *interface moving away from* $x = 0$.

On the other hand, if β is negative, everything turns around. In stead of going in, γ_i comes out of the critical point and the corresponding solutions satisfy (1.25) as $\eta \downarrow \eta^*$ when η^* is positive, and $\eta \uparrow \eta^*$ when η^* is negative. This is the *interface condition for interfaces moving towards* $x = 0$. Note that these similarity solutions vanish pointwise as $t \downarrow 0$.

Before we discuss the other two critical points at infinity we observe that for $\alpha = \alpha_0$ in Theorem 1 the orbit γ_i coincides with γ_s and contains the *Barenblatt-Pattle* family, while for $\alpha = \alpha_1$ it coincides with γ_a and contains the *dipole* family. In both cases γ_i is a straight line in the (x, y)-coordinats, and the solutions can be computed explicitly.

The critical points $(\rho, \phi) = (1, 0)$ *and* $(\rho, \phi) = (1, \pi)$. For $m > 1$ the second one is always an attractor and the first one a repeller. Orbits going into the attractor do this in finite τ-time and the solutions $U(\eta)$ they contain go to zero as η approaches some finite *positive* value *from below*, or some finite *negative* value *from above*, but their fluxes remain nonzero. Thus they satisfy the *sign change condition.* Similarly, the orbits coming out of the repeller do this starting at some finite τ-value. The only difference is that the solutions they contain go to zero with nonzero flux as η approaches some finite *positive* value *from above*, or some finite *negative* value *from below.* In fact there is a *pairing* between the orbits going into the *attractor* and orbits coming out of the *repeller* such that *every solution with a sign change is contained in one orbit pair.* What we need therefore is a transformation which *blows up* these critical points and *identifies* them in such away that each of these orbit pairs can be considered as one single orbit varying continuously with the parameters. We have found two of these transformations, one that is more suited for the study of symmetric profiles, the other for anti-symmetric profiles. Below we shall describe the first one.

We set

$$a = (x + \beta y) y^{\frac{m}{1-m}}, \quad b = y^{\frac{1}{1-m}}, \tag{1.26}$$

which is a diffeomorphism of H^+ onto itself transforming (Q) into

$$(Q^*) \begin{cases} b^{m-1}\dot{a} = b^{m-1}(\frac{1+m}{1-m} a + (\alpha + \beta) b); \\ b^{m-1}\dot{b} = a - \beta b - \frac{2}{m-1} b^m, \end{cases}$$

where we ignore the factor b^{m-1} on the left hand sides again. We shall denote the open upper half (a, b)-plane by H^*. It is easy to see from (1.10) or (1.13) that apart from the origin there are no finite critical points of (Q^*) in H^*. It turns out that the orbits in H^+ going to infinity along $\phi = \pi$ are mapped onto the orbits in H^* which leave H^* through the negative a-axis while the orbits in H^+ coming out of infinity along $\phi = 0$ are mapped onto the orbits entering H^* through the positive a-axis. In other words, these singularities have been blown up and removed! This is the first part of what we set out to do. The other part, namely their identification, is now simple. We just identify the positive and negative a-axis. The result is that every orbit going through it contains solutions with a sign change. Although there is some lack of smoothness at each crossing, the well-posedness at the positive and negative a-axis is not difficult to prove.

In order to trace orbits both forwards and backwards we have to investigate the behaviour of (Q^*) near the origin and near infinity. We set

$$a = \Big(\frac{r}{1-r}\Big)^{\frac{1}{m-1}} \cos\theta, \quad b = \Big(\frac{r}{1-r}\Big)^{\frac{1}{m-1}} \sin\theta, \tag{1.27}$$

which is is a diffeomorphism of the open rectangle $R^* = \{(r, \theta) : r \in (0, 1), \theta \in (0, \pi)\}$ onto H^* transforming (Q^*) into

$$(\tilde{Q}^*) \begin{cases} r \sin^{m-1}\theta \quad \dot{r} = (m-1) r (1-r)^2 \sin\theta (\cos\theta - \beta \sin\theta) - \\ \qquad\qquad r^2 (1-r) \sin^{m-1}\theta (2 + (m-1)\cos\theta(\cos\theta - (\alpha+\beta)\sin\theta)); \\ r \sin^{m-1}\theta \quad \dot{\theta} = r \sin^m\theta (\cos\theta - (\alpha+\beta)\sin\theta) + \\ \qquad\qquad (1-r)\cos\theta(\cos\theta - \beta\sin\theta), \end{cases}$$

where the factor $r\sin^{m-1}\theta$ on the left hand sides of $(\tilde{Q}^*)$ is ignored again.

Now let C be the cylinder obtained from R^* by indentifying $\theta = 0$ and $\theta = \pi$. We find that at the bottom of C, i.e. at $r = 0$, there is a critical point P_i at $\theta = \frac{\pi}{2}$ which is a saddle if $m > 1$ and $\beta \neq 0$. If $\beta > 0$ one single orbit, the image of γ_i, goes into this saddle from the interior of C. On top of C, i.e. at $r = 1$, there is a critical point P_s at θ =arccotan$(\alpha + \beta)$ which is always a saddle with one orbit, the image of γ_s, coming out into C. We shall show that, for certain bifurcation values of the parameters, there exist *connections* between these two saddles which go around C any prescribed number of times. If α and β are related by (1.10) these connections are unique and they contain the compactly supported *symmetric* similarity profiles in Theorem 1.

In order to find the *anti-symmetric* profiles we modify (1.27) and set

$$a = (x + \beta y - \frac{1}{m})y^{\frac{m}{1-m}}, \quad b = y^{\frac{1}{1-m}}, \tag{1.28}$$

and essentially the only thing that changes is that there is a different saddle on top of C with one orbit, the image of γ_a, coming out. The compactly supported *anti-symmetric* similarity profiles in Theorem 1.1 are again contained in saddle-saddle connections.

2. The autonomous systems.

In this section we analyze the critical points of the autonomous systems (Q), $(\tilde{Q})$, (Q^*) and $(\tilde{Q}^*)$ we encountered in the introduction in more detail. It is convenient to introduce some notation. If γ is an orbit of (Q) in H^+, we denote the corresponding orbits of $(\tilde{Q})$, (Q^*) and $(\tilde{Q}^*)$ by $\tilde{\gamma}$, γ^* and $\tilde{\gamma}^*$, and vice versa. Each of these orbits contains four one-parameter families of solutions U.

Figure 2.1. The phase plane.

Throughout this section we shall only consider $\eta > 0$ and $U > 0$ which leaves us with just one solution family in each orbit to consider. The remaining three families are obtained by reflections in the η- and U-axis.

We begin with (Q) and its polar coordinates version $(\tilde{Q})$. Unless otherwise stated, $m > 1$, $\alpha < 0$ and $\beta > 0$. For reasons explained in the introduction we restrict ourselves to the closure of the upper half plane H^+. In Figure 2.1 we sketch the regions where $\dot{x}$ and $\dot{y}$ are positive and negative.

PROPOSITION 2.1. *There are no periodic orbits of* (Q) *in* H^+.

PROOF. This is immediate from Figure 2.1.

Next we analyze the two critical points of (Q) on the x-axis.

PROPOSITION 2.2. *The critical point* $(0,0)$ *is a repeller. The linearization of* (Q) *around* $(0,0)$ *has matrix*

$$A(0,0) = \begin{pmatrix} 1 & \alpha \\ 0 & 2 \end{pmatrix},$$

which has eigenvalues $\lambda_1 = 1$ *and* $\lambda_2 = 2$ *with corresponding eigenvectors*

$$\bar{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and } \bar{v}_2 = \begin{pmatrix} \alpha \\ 1 \end{pmatrix}.$$

(i) *One orbit,* γ_s*, comes out of* $(0,0)$ *along* $\bar{v}_2$. *It contains the solutions* $U(\eta)$ *of* (1.11) *with* $U(0) > 0$ *and* $U'(0) = 0$.

(ii) *For each* $p > 0$ *there is one orbit coming out of* $(0,0)$ *along* $\bar{v}_1$ *, and one along* $-\bar{v}_1$*, such that*

$$p = \lim_{\tau \downarrow -\infty} \frac{y(\tau)}{x(\tau)^2}. \tag{2.1}$$

Each of these orbit pairs contains the solutions $U(\eta)$ *of* (1.11) *with* $U(0) > 0$ *and*

$$U(0)^{-2}|U'(0)^{3-m}| = p. \tag{2.2}$$

There are no other orbits coming out of $(0,0)$ *into* H^+.

PROOF. The computation of $A(0,0)$, λ_1, λ_2, $\bar{v}_1$ and $\bar{v}_2$ is straightforward. Since $2 > 1$ only one orbit (γ_s) comes out of $(0,0)$ along $\bar{v}_2$, and one along $-\bar{v}_2$. All other orbits come out of $(0,0)$ along the positive or negative x-axis, and we need their precise asymptotics near the origin. Since zero is not in the complex hull of the spectrum of $A(0,0)$ and since (Q) does not contain the resonant term x^2 in the differential equation for y, it follows from the Poincaré-Dulac Theorem ([2]), that there is a biholomorphic equivalence near the origin between (Q) and its linear part

$$(Q_l) \begin{cases} \dot{x} = x + \alpha y; \\ \dot{y} = \quad 2y. \end{cases}$$

With the exception of the two orbits coming out along $\pm\bar{v}_2$, the integral curves of this system are given by a one-parameter family of parabolas

$$y = \tilde{p}(x - \alpha y)^2,$$

parametrized by $\tilde{p}$. It is not difficult to check that the biholomorphic equivalence maps this family surjectively to a one-parameter family of curves locally given by

$$y = px^2 + F_p(x), \tag{2.3}$$

with

$$F_p(x) = o(x^2) \text{ as } x \to 0,$$

and $p - \tilde{p} =$ constant. For fixed p the graph of $y = px^2 + F_p(x)$ contains an orbit pair, with one orbit going to the left, the other going to the right. Obviously $p = 0$ corresponds to the positive and negative x-axis whereas negative values of p correspond to orbits in the lower half plane. Relevant to us are of course only the orbit pairs with positive p.

Next we show that the solutions of (1.11) with $U(0) > 0$ are contained in these orbit pairs. By the standard theory for ordinary differential equations there exists for every $u_0 > 0$ and $u_0' \neq 0$ a unique solution U of (1.11) with

$$U(0) = u_0 \text{ and } U'(0) = u_0'.$$

For the solution with $u_0' = 0$ we obviously have

$$\lim_{\tau \downarrow -\infty} x(\tau) = \lim_{\tau \downarrow -\infty} y(\tau) = 0,$$

but also

$$\lim_{\tau \downarrow -\infty} \frac{x(\tau)}{y(\tau)} = \lim_{\eta \downarrow 0} \frac{(U^m)'(\eta)}{\eta} \frac{1}{mU(\eta)} = \frac{(U^m)''(0)}{mU(0)} = \alpha.$$

This implies that U is contained in γ_s. Moreover,

$$u_0^{1-m} = \lim_{\eta \downarrow 0} \frac{\eta^2 U(\eta)^{1-m}}{\eta^2} = \lim_{\tau \downarrow -\infty} y(\tau)e^{-2\tau}, \tag{2.4}$$

so that a scaling of u_0 amounts to a shift in τ.

Now suppose that U is a solution with $u_0 > 0$ and $u_0' \neq 0$. Again it is obvious that U is contained in an orbit coming out of the origin. Computing

$$p = \lim_{\tau \downarrow -\infty} \frac{y(\tau)}{x(\tau)^2} = \lim_{\eta \downarrow 0} U(\eta)^{3-m} U'(\eta)^{-2} = U(0)^{3-m} U'(0)^{-2},$$

we see that all solutions U with $u_0 > 0$ and $u_0' \neq 0$ are contained in the orbit pairs found above. The sign of u_0' determines whether U is contained in the left or the right orbit. By the same argument as above, varying u_0 and u_0' with p being fixed amounts to a shift in τ. This completes the proof.

By Proposition 2.2(i), the symmetric solutions with $U(0) \neq 0$ are contained in γ_s. Proposition 2.2(ii) tells us where to find solutions of (1.11) with $U(0) \neq 0 \neq U'(0)$. If U is such a solution, then, modulo symmetry, the parts of U defined on $\mathbf{R}^+$ and $\mathbf{R}^-$ are contained in the two orbits of one orbit pair of the form (2.3), with p given by (2.2).

PROPOSITION 2.3. *The critical point $(\frac{1}{m}, 0)$ is a saddle. The linearization of (Q) around $(\frac{1}{m}, 0)$ has matrix*

$$A(\frac{1}{m}, 0) = \begin{pmatrix} -1 & \alpha - \frac{\beta}{m} \\ 0 & \frac{m+1}{m} \end{pmatrix},$$

which has eigenvalues $\mu_1 = -1$ and $\mu_2 = \frac{m+1}{m}$ with corresponding eigenvectors

$$\bar{w}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \bar{w}_2 = \begin{pmatrix} m\alpha - \beta \\ 2m+1 \end{pmatrix}.$$

The orbits going into this saddle lie on the x-axis. Only one orbit, γ_a, comes out of $(1/m, 0)$ into H^+. It contains the solutions $U(\eta)$ of (1.11) with $U(0) = 0$ and $(U^m)'(0^+) > 0$ arbitrary, and this derivative parametrizes these solutions.

PROOF. The computation of $A(\frac{1}{m}, 0)$, μ_1, μ_2, $\bar{w}_1$ and $\bar{w}_2$ is again straightforward. So only one orbit (γ_a) comes out of $(\frac{1}{m}, 0)$ into the open upper half plane. Any solution U contained in γ_a has $U'(\eta) > 0$ as $\eta \downarrow 0$ so that $\lim_{\eta \downarrow 0} U(\eta)$ exists. We claim that it is zero, for suppose it is positive, then $\lim_{\tau \downarrow -\infty} y(\tau) e^{-2\tau} = \lim_{\eta \downarrow 0} U(\eta)^{1-m}$ is also positive. Since

$$\lim_{\tau \downarrow -\infty} \frac{\dot{y}(\tau)}{y(\tau)} = 2 + (1-m)\frac{1}{m} = \frac{m+1}{m},$$

this would imply that $\frac{m+1}{m} = 2$, which is absurd. So

$$\lim_{\eta \downarrow 0} U(\eta) = 0.$$

Furthermore we have

$$\frac{(U^m)''(\eta)}{U(\eta)} = m(\alpha - \beta x(\tau)) \to m\alpha - \beta \text{ as } \eta \downarrow 0,$$

forcing

$$(U(\eta)^m)'' \to 0 \text{ as } \eta \downarrow 0.$$

Hence $\lim_{\eta \downarrow 0} (U(\eta)^m)'$ exists and again we claim it is positive. Suppose not then by l'Hôspital's rule,

$$1 = m \lim_{\eta \downarrow 0} \frac{\eta U'(\eta)}{U(\eta)} = \lim_{\eta \downarrow 0} \frac{(U(\eta)^m)'}{\frac{U(\eta)^m}{\eta}} = \lim_{\eta \downarrow 0} \frac{(U(\eta)^m)''}{(\frac{U(\eta)^m}{\eta})'} = \lim_{\tau \downarrow -\infty} y(\tau) \frac{\alpha - \beta x(\tau)}{x(\tau) - \frac{1}{m}} = \frac{2m+1}{m},$$

which is again absurd. Thus

$$0 < \lim_{\eta\downarrow 0} (U(\eta)^m)' = m \lim_{\tau\downarrow-\infty} x(\tau)y(\tau)^{\frac{m}{1-m}} e^{\frac{m+1}{m-1}\tau}, \tag{2.5}$$

and a scaling of $(U^m)'(0)$ amounts to a shift in τ. This completes the proof.

From Proposition 2.3 we see that the anti-symmetric solutions with $(|U|^{m-1}U)'(0) \neq 0$ are contained in γ_a.

Next we look at orbits coming out of or going into infinity ($\rho = 1$). Observe that orbits of $(\tilde{Q})$ contained in $\rho = 1$ do not contain solutions of (1.11).

We discuss the four critical points of $(\tilde{Q})$ at $\rho = 1$. We call these points the critical points of (Q) at infinity. For the linearization of $(\tilde{Q})$ we introduce the notation

$$B(\rho,\phi) = \begin{pmatrix} (\partial/\partial\rho)(1-\rho)\dot{\rho} & (\partial/\partial\phi)(1-\rho)\dot{\rho} \\ (\partial/\partial\rho)(1-\rho)\dot{\phi} & (\partial/\partial\phi)(1-\rho)\dot{\phi} \end{pmatrix}.$$

PROPOSITION 2.4. *The critical point* $(\rho,\phi) = (1, \frac{\pi}{2})$ *is an attractor for the rectangle* $R = [0,1] \times [0,\pi]$ *and has*

$$B(1, \frac{\pi}{2}) = \begin{pmatrix} 0 & 0 \\ \alpha & -\beta \end{pmatrix}.$$

The orbits going in to this attractor from the interior of R *correspond to the orbits in* H^+ *that enter the positively invariant set*

$$S(\alpha,\beta) = \{(x,y)|x,y \in \mathbf{R}, x < \frac{\alpha}{\beta}, y > \frac{x(1-mx)}{\beta x - \alpha}\}.$$

All these orbits can be continued as $\tau \to \infty$ *and have* $x(\tau) \uparrow \frac{\alpha}{\beta}$ *and* $y(\tau) \uparrow +\infty$. *The solutions* $U(\eta)$ *contained in these orbits satisfy*

$$\lim_{\eta\uparrow+\infty} \frac{\eta U'(\eta)}{U(\eta)} = \frac{\alpha}{\beta} < 0, \tag{2.6}$$

and decay to zero as $\eta \to \infty$.

PROOF. The computation of $B(1, \frac{\pi}{2})$ is trivial. The invariance of $S(\alpha,\beta)$ is obvious from Figure 2.1 and hence x is bounded for every orbit that enters $S(\alpha,\beta)$. This neutralizes the quadratic terms in (Q) and establishes the global existence as $\tau \to \infty$, the limit behaviour being straightforward. Furthermore it is easy to see that no other orbits γ come out of or go into infinity with

$$\frac{x(\tau)}{y(\tau)} \to 0,$$

so we have simultaneously identified all orbits $\tilde{\gamma}$ in the interior of R coming out of or going into $(\rho,\phi)=(1,\frac{\pi}{2})$. The two orbits contained in $\rho=1$ also go into $(1,\frac{\pi}{2})$ and thus $(1,\frac{\pi}{2})$ is an attractor for R. This completes the proof.

PROPOSITION 2.5. *The critical point $(\rho,\phi)=(1,\operatorname{arccot}(-\beta))$ is a saddle and has*

$$B(1,\operatorname{arccot}(-\beta))=\begin{pmatrix}\frac{(1-m)\beta}{(1+\beta^2)^{\frac{1}{2}}} & 0\\ \frac{\alpha+\beta}{1+\beta^2} & \frac{\beta}{(1+\beta^2)^{\frac{1}{2}}}\end{pmatrix}.$$

The two orbits coming out are contained in $\rho=1$. Only one orbit, $\tilde{\gamma}_i$, goes into this saddle from the interior of R. It corresponds to a unique orbit γ_i of (Q) in H^+ escaping to infinity in finite τ-time with

$$\frac{x(\tau)}{y(\tau)}\to-\beta,$$

and contains the solutions $U(\eta)$ of (1.11) for which the interface condition (1.25) holds as $\eta\uparrow\eta^*$, where $\eta^*>0$ is arbitrary and determines the solution.

PROOF. The local analysis is left to the reader. Observe that γ_i escapes to infinity in finite time because of the quadratic terms in (Q). We can thus take $\eta^*=e^{\tau^*}$ where τ^* is the moment at which γ_i reaches infinity, as parameter. Since $y=\eta^2U^{1-m}\to\infty$ as $\tau\uparrow\tau^*$, it follows that $U(\eta)\to 0$ as $\eta\uparrow\eta^*$. Moreover

$$(U(\eta)^{m-1})'=(m-1)\frac{x(\tau)}{y(\tau)}e^{\tau}\to(1-m)\beta\eta^* \text{ as } \eta\uparrow\eta*, \tag{2.7}$$

which completes the proof.

PROPOSITION 2.6. *The critical point $(\rho,\phi)=(1,0)$ is a repeller and has*

$$B(1,0)=\begin{pmatrix}m & 0\\ 0 & 1\end{pmatrix}.$$

For every $a^+>0$ there is a unique orbit γ coming out of infinity in finite τ-time such that $\tilde{\gamma}$ comes out of $(\rho,\phi)=(0,0)$, and γ^* enters H^* through the point $(a^+,0)$. Except for the two orbits contained in $\rho=1$ and $\phi=0$, there are no other orbits coming out of $(\rho,\phi)=(0,0)$. Varying a^+ over $\mathbf{R}^+$ we obtain all orbits of (Q) in H^+ which do not come out of $(0,0)$ or $(1/m,0)$. The solutions $U(\eta)$ of (1.11) contained in these orbits satisfy

$$U(\eta)\to 0 \text{ and } (U(\eta)^m)'\to ma^+(\eta^+)^{\frac{m+1}{m-1}}, \tag{2.8}$$

as $\eta\downarrow\eta^+>0$, where the free parameters $\eta^+>0$ and $a^+>0$ determine the solution.

PROOF. The computation of $B(1,0)$ is trivial and clearly $(1,0)$ is a repeller. All orbits $\tilde{\gamma}$ coming out of it into the open rectangle are tangent to $\rho=1$, but this is not enough to classify the solutions U contained in these orbits. We therefore turn to (Q^*) and sketch its phase plane in Figure 2.2.

Figure 2.2. The (a, b)-phase plane.

Any orbit γ^* of (Q^*) starting on the positive a-axis, say in $(a^+, 0)$, can be represented near $(a^+, 0)$ by

$$a = A(b),$$

where A is a solution of

$$\begin{cases} A'(b) = \dfrac{b^{m-1}(\frac{1+m}{1-m}A+(\alpha+\beta)b)}{A-\beta b-\frac{2}{m-1}b^m}; \\ A(0) = a^+, \end{cases}$$

which has a unique local solution provided a^+ is nonzero. We denote this orbit by $\gamma^*(a^+)$, and its corresponding orbits of (Q) and $(\tilde{Q})$ by $\gamma(a^+)$ and $\tilde{\gamma}(a^+)$. Now observe that

$$(\frac{x}{y} + \beta)y^{\frac{1}{1-m}} = (x + \beta y)y^{\frac{m}{1-m}} = a \to a^+ \text{ and } b = y^{\frac{1}{1-m}} \to 0$$

imply

$$y \to \infty \text{ and } \frac{x}{y} \to \infty.$$

Consequently, $\tilde{\gamma}(a^+)$ comes out of $(\rho, \phi) = (1, 0)$ into the open rectangle. Because of the quadratic terms in (Q), $\gamma(a^+)$ comes out of infinity in a finite time. As parameter we choose $\eta^+ = e^{\tau^+}$ where τ^+ the moment at which $\gamma(a^+)$ leaves infinity. We then have

$$U(\eta) \to 0 \text{ as } \eta \downarrow \eta^+,$$

and

$$a^+ = \lim_{\tau\downarrow\tau^+}(x + \beta y)y^{\frac{m}{1-m}} = \lim_{\tau\downarrow\tau^+} xy^{\frac{m}{1-m}} = \frac{1}{m}(\eta^+)^{\frac{1+m}{1-m}} \lim_{\eta\downarrow\eta^+}(U(\eta)^m)'.$$

It remains to show that every orbit γ not coming out of $(0, 0)$ or $(\frac{1}{m}, 0)$ is of the form $\gamma(a^+)$ with $a^+ > 0$. Clearly, from Figure 2.1, such an orbit γ must come out of infinity into the first quadrant, so that in view of the lack of other critical points at $\rho = 1$, $\tilde{\gamma}$ must come out of $(1, 0)$. Thus we have

$$x \to \infty \text{ and } \frac{x}{y} \to \infty,$$

but also, because of the linearization,

$$y(\tau) = \frac{\phi(\tau)}{1 - \rho(\tau)} \frac{\rho(\tau) \sin \phi(\tau)}{\phi(\tau)} \to \infty,$$

as τ decreases to the infimum $\bar{\tau}$ of the interval on which $(x(\tau), y(\tau))$ is defined. Consequently, the quadratic terms in (Q) force $\bar{\tau}$ to be finite, and because

$$\dot{a} = \frac{m+1}{1-m} a + (\alpha + \beta) y^{\frac{1}{1-m}},$$

it follows that $\lim_{\tau \downarrow \bar{\tau}} a(\tau)$ exist as a nonnegative, possibly zero number. The proof will be completed by eliminating the latter possibility. Since

$$\frac{a}{b} = \frac{x}{y} + \beta \to \infty \text{ and } b = y^{\frac{1}{1-m}} \to 0$$

as $\tau \downarrow \bar{\tau}$, we have

$$\frac{\dot{a}}{\dot{b}} = b^{m-1} \frac{\frac{1+m}{1-m} \frac{a}{b} - (\alpha + \beta)}{\frac{a}{b} - \beta - \frac{2}{m-1} b^{m-1}} \to 0.$$

But then it is impossible for a to converge to zero because of l'Hôspital's rule. This concludes the proof.

PROPOSITION 2.7. *The critical point $(\rho, \phi) = (1, \pi)$ is an attractor and has*

$$B(1, \pi) = \begin{pmatrix} -m & 0 \\ 0 & -1 \end{pmatrix}.$$

For every $a^- < 0$ there is a unique orbit γ escaping to infinity in finite τ-time such that $\tilde{\gamma}$ goes into $(\rho, \phi) = (0, \pi)$, and γ^* leaves H^* through the point $(a^-, 0)$. Except for the two orbits contained in $\rho = 1$ and $\phi = \pi$, there are no other orbits going into $(\rho, \phi) = (0, \pi)$. Varying a^- over $\mathbf{R}^-$ we obtain all orbits of (Q) in H^+ which do not go into $(1, \frac{\pi}{2})$ or $(1, \operatorname{arccot}(-\beta))$. The solutions $U(\eta)$ of (1.11) contained in these orbits satisfy

$$U(\eta) \to 0 \text{ and } (U(\eta)^m)' \to m a^- (\eta^-)^{\frac{m+1}{m-1}}, \tag{2.9}$$

as $\eta \uparrow \eta^- > 0$, where the free parameters $\eta^- > 0$ and $a^- < 0$ determine the solution.

PROOF. Similar to that of Proposition 2.7 and therefore omitted.

So now we know how to represent solutions U of (1.11) with a sign change in $\eta = \bar{\eta} > 0$. The modulus of the part of the solution before the sign change is contained in an orbit going into $(\rho, \phi) = (1, \pi)$, while the modulus of the part after the sign change is contained in an orbit coming out of $(\rho, \phi) = (1, 0)$. With the notation of the last two propositions it is clear that

$$\eta^- = \eta^+ = \bar{\eta} \text{ and } a^+ + a^- = 0,$$

and this is precisely why the positive and negative a-axis have to be indentified.

In Figure 2.3 we sketch the local behaviour of (Q) near the critical points in the compactified upper half plane. Observe that every orbit is heteroclinic, i.e. connects two different critical points.

Figure 2.3. The compactified upper half (x, y)-phase plane.

Next we turn to (Q^*) and $(\tilde{Q}^*)$. Note that the positive a-axis corresponds with $\{(r, 0) : 0 < r < 1\}$ and the negative a-axis with $\{(r, \pi) : 0 < r < 1\}$. Obviously (Q^*) does not have critical points, except for the origin. We continue with the critical points of $(\tilde{Q}^*)$ in $R^* = [0, 1] \times [0, \pi]$. They are

$$(r, \theta) = (0, \text{arccotan}\beta), \quad (0, \frac{\pi}{2}), \quad (1, 0), \quad (1, \text{arccotan}(\alpha + \beta)) \text{ and } (1, \pi).$$

For the linear analysis of these critical points we use the notation

$$D(r, \theta) = \begin{pmatrix} (\partial/\partial r)(r \sin^{m-1}\theta \quad \dot{r}) & (\partial/\partial \theta)(r \sin^{m-1}\theta \quad \dot{r}) \\ (\partial/\partial r)(r \sin^{m-1}\theta \quad \dot{\theta}) & (\partial/\partial \theta)(r \sin^{m-1}\theta \quad \dot{\theta}) \end{pmatrix}.$$

PROPOSITION 2.8. *The critical point $(r, \theta) = (0, \frac{\pi}{2})$ has*

$$D(0, \frac{\pi}{2}) = \begin{pmatrix} (1-m)\beta & 0 \\ -(\alpha + \beta) & \beta \end{pmatrix},$$

and is a saddle. The two orbits coming out are contained in $r = 0$. Only one orbit, $\tilde{\gamma}_i^*$, which corresponds to γ_i and γ_i^*, goes into this saddle from the interior of R^*. Asymptoticly it is tangent to the eigenvector of the negative eigenvalue $(1-m)\beta$, which is

$$\begin{pmatrix} m\beta \\ \alpha + \beta \end{pmatrix}.$$

PROOF. The local analysis is straightforward. We only show that, with the notation introduced at the beginning of this section, the orbit going into this saddle is $\tilde{\gamma}_i^*$. Namely, if $(x(\tau), y(\tau))$ is a solution with trajectory γ_i, then

$$x(\tau) \to -\infty, \quad y(\tau) \to \infty \text{ and } \frac{x(\tau)}{y(\tau)} \to \beta \text{ as } \tau \uparrow \tau^*,$$

implying

$$b \to 0, \quad \frac{a}{b} = \frac{x}{y} + \beta \to 0 \text{ and thus } a \to 0.$$

This proves that $\tilde{\gamma}_i^*$ is the orbit going into $(0, \frac{\pi}{2})$.

PROPOSITION 2.9. *The critical point* $(r, \theta) = (1, \operatorname{arccotan}(\alpha + \beta))$ *has*

$$D(1, \operatorname{arccotan}(\alpha+\beta)) = \begin{pmatrix} 2(1+(\alpha+\beta)^2)^{\frac{1-m}{2}} & 0 \\ -\beta(\alpha+\beta)(1+(\alpha+\beta)^2)^{-1} & -(1+(\alpha+\beta)^2)^{\frac{1-m}{2}} \end{pmatrix},$$

and is a saddle. The two orbits going in are contained in $r = 1$. *Only one orbit,* $\tilde{\gamma}_s^*$, *which corresponds to* γ_s *and* γ_s^*, *comes out of this saddle going into the interior of* R^*.

PROOF. Again the local analysis is straightforward, so we only show that, with the notation introduced at the beginning of this section, the orbit coming out of this saddle is $\tilde{\gamma}_s^*$. Namely, if $(x(\tau), y(\tau))$ is a solution with trajectory γ_s, then

$$x(\tau) \to 0, \quad y(\tau) \to 0 \text{ and } \frac{x(\tau)}{y(\tau)} \to \alpha \text{ as } \tau \downarrow -\infty,$$

implying

$$b \to \infty \text{ and } \frac{a}{b} = \frac{x}{y} + \beta \to \alpha + \beta.$$

This proves that $\tilde{\gamma}_s^*$ is the orbit coming out of $(1, \operatorname{arccotan}(\alpha + \beta))$.

PROPOSITION 2.10. *The critical point* $(r, \theta) = (0, \operatorname{arccotan}\beta)$ *has*

$$D(0, \operatorname{arccotan}\beta) = \begin{pmatrix} 0 & 0 \\ -\alpha(1+\beta^2)^{-\frac{m+1}{2}} & -\beta \end{pmatrix}.$$

No orbits come out of this critical point but there is a one-parameter family of orbits going into this critical point coming from the interior of R^*. *These orbits correspond to the orbits in Proposition 2.4.*

PROOF. We only show that every orbit $\tilde{\gamma}$ going into or coming out of $(1, \frac{\pi}{2})$ corresponds to an orbit $\tilde{\gamma}^*$ going into or coming out of $(0, \operatorname{arccotan}\beta)$. But this follows from the equivalence of

$$\frac{x}{y} \to 0 \text{ and } y \to \infty,$$

and

$$\frac{a}{b} \to \beta \text{ and } b \to 0,$$

which completes the proof.

We have now recovered all orbits of (Q) and $(\tilde{Q})$ as orbits of $(\tilde{Q}^*)$ except for the ones in Proposition 2.2(ii) and γ_a in Proposition 2.3. Clearly they have to correspond to orbits of $(\tilde{Q}^*)$ coming out of $(1,0)$ and $(1,\pi)$. In general the right hand side of $(\tilde{Q}^*)$ is not smooth in these points and a formal linearization only yields

$$D(0,1) = D(0,\pi) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

PROPOSITION 2.11. *There are no orbits of* $(\tilde{Q}^*)$ *in* $(0,1) \times (0,\pi)$ *going into* $(1,0)$ *or* $(1,\pi)$. *The orbits in* Proposition 2(ii) *correspond in pairs to orbits of* $(\tilde{Q}^*)$ *coming out of* $(0,1)$ *and* $(0,\pi)$, *while* γ_a *corresponds to an orbit coming out of* $(1,0)$. *There are no other orbits coming out of these critical points except for orbits contained in* $\{r = 1\}$.

PROOF. Similar to the previous two proofs and left to the reader.

This completes our analysis of the local behaviour. In the next section we shall carry out the global analysis.

3. The cylinders.

In this section we give the bifurcation analysis of the autonomous system $(\tilde{Q}^*)$ acting on the cylinder

$$C = \{(r,\theta) | r \in (0,1), \theta \in \mathbf{R} \bmod \pi\}.$$

Unless otherwise stated we assume that $m > 1$, $\alpha < 0$ and $\beta > 0$.

Clearly the vectorfield defined by the right hand side of $(\tilde{Q}^*)$ is smooth on the complement of the segment

$$l = \{(r,\theta) \in C | \theta = 0 \bmod \pi\},$$

and from Propositions 2.6 and 2.7 it follows that through every point $(r,\theta) \in l$ there goes precisely on "generalized" orbit Γ. This orbit intersects l perpundicularly. Before the intersection it corresponds to the orbit in Proposition 2.7 with

$$a^- = -(\frac{r}{1-r})^{\frac{1}{1-m}}$$

and after the intersection to the orbit in Proposition 2.6 with $a^+ = -a^-$. Each Γ is a union of orbits of $(\tilde{Q}^*)$. The number of intersections of Γ and l is called the *rotation number* of Γ, denoted by $n(\Gamma)$.

The boundary of C consists of two topological circles

$$S_1 = \{(1,\theta)|\theta \in \mathbf{R} \bmod \pi\} \text{ and } S_0 = \{(0,\theta)|\theta \in \mathbf{R} \bmod \pi\},$$

As we have seen in Section 2 they both contain one saddle, namely

$$P_s = (1, \operatorname{arccotan}(\alpha+\beta)) \in S_1 \text{ and } P_i = (0, \frac{\pi}{2}) \in S_0.$$

The unique orbit coming out of P_s containing the symmetric solutions of (1.11) is denoted by Γ_s and the unique orbit going into P_i containing the solutions of (1.11) with an interface by Γ_i.

The only other critical point on S_0 we found in Section 2 is the attractor

$$P_0 = (0, \operatorname{arccotan}\beta).$$

It contains all solutions $U(\eta)$ which decay to zero as $\eta \to \infty$.

Figure 3.1. The cylinder C.

In order to find compactly supported symmetric similarity profiles $U(\eta)$ we must look for saddle-saddle connections between P_s and P_i. In other words, when do Γ_s and Γ_i coincide? Note that so far we have not used that α and β are related by (1.10). It turns out that (1.10) implies an important monotonicity property which will help us to carry out the bifurcation analysis as α runs from $-\frac{1}{m+1}$ to $-\frac{1}{m-1}$, so from now on we shall always assume that (1.10) holds, i.e.

$$2\beta = (m-1)\alpha + 1.$$

This leaves us with one free parameter α, since $m > 1$ is fixed.

The results in Theorems 1.1 concerning the symmetric profiles are a direct consequence of the following theorem.

THEOREM 3.1. *There exists a sequence*

$$\alpha_0 = -\frac{1}{m+1} > \alpha_2 > \alpha_4 > \alpha_6 > \ldots \downarrow -\frac{1}{m-1},$$

such that Γ_s and Γ_i coincide if and only if $\alpha = \alpha_{2n}$ for some $n \in \{0, 1, 2, \ldots\}$. If this is so then $n = n(\Gamma_s) = n(\Gamma_i)$.

The proof of Theorem 3.1 is based on the analysis of the action of $(\tilde{Q}^*)$ on C when $\alpha = -\frac{1}{m+1}$ and when $\alpha = -\frac{1}{m-1}$ combined with the monotonicity result announced above.

LEMMA 3.2. *Let $\alpha = -\frac{1}{m+1}$, then $\Gamma_s = \Gamma_i = \{(r, \frac{\pi}{2}) : r \in (0,1)\}$ and $n(\Gamma_s) = n(\Gamma_i) = 0$. This connection corresponds to the straight line*

$$(m+1)x + y = 0 \tag{3.1}$$

in the (x, y)-upper half plane H^+.

PROOF. By verification.

LEMMA 3.3. *Let $\alpha = -\frac{1}{m-1}$, then $\beta = 0$ and the critical points P_0 and P_s coincide. The orbit Γ_i no longer exists, and with the exception of orbits contained in the circles S_0 and S_1 every orbit Γ goes around the cylinder forever having S_0 as ω-limit set. In particular, $n(\Gamma_s) = +\infty$.*

PROOF. This follows directly from the fact that all solutions U of (1.11) are periodic when $\alpha < 0$ and $\beta = 0$.

In order to formulate our monotonicity result we rewrite $(\tilde{Q}^*)$ as

$$(\tilde{Q}^*)\begin{cases} r\sin^{m-1}\theta & \dot{r} = (m-1)r(1-r)F(r,\theta,\alpha); \\ r\sin^{m-1}\theta & \dot{\theta} = G(r,\theta,\alpha). \end{cases}$$

This defines F and G.

LEMMA 3.4. *Let F and G be as above. Then*

$$F_\alpha G - FG_\alpha = -r\sin^{m+1}\theta(\frac{1-r}{2} + \frac{m+1}{m-1}r\sin^{m-1}\theta),$$

which is negative if $r \neq 0$ *and* $\theta \neq 0 \bmod \pi$.

PROOF. By verification.

LEMMA 3.5. *Let*
$$-\frac{1}{m-1} < \alpha < -\frac{1}{m+1}.$$
Denote the r*-value of the last intersection point of* Γ_i *and* l *by* $r^-(\alpha)$. *Then*

$$r^- : (-\frac{1}{m-1}, -\frac{1}{m+1}) \to (0,1]$$

is a well defined nondecreasing continuous function, strictly increasing on $(-\frac{1}{m-1}, -\frac{1}{m}]$ *and identically equal to one on* $(-\frac{1}{m}, -\frac{1}{m+1})$.

PROOF. First we check that r^- is well defined. Note that $\alpha + \beta < 0$ so that $\tilde{\gamma}_i^*$ cannot have entered the rectangle $\{(r,\theta) : 0 < r < 1, 0 < \theta < \frac{\pi}{2}\}$ through the segment $\{(r, \frac{\pi}{2}) : 0 < r < 1\}$. Since P_0 is an attractor, it either comes out the critical point $(1,0)$, or enters through $\{(r,0) : 0 < r < 1\}$. We claim that it can be written as the graph of a function

$$R_\alpha^- : [0, \frac{\pi}{2}] \to (0,1],$$

where the subscript α indicates the dependence on α. This will be clear if $\dot{\theta} > 0$ along $\tilde{\gamma}_i^*$. Let us examine where $\dot{\theta}$ is positive and negative. We introduce the sets

$$D_1^- = \{(r,\theta) : \text{arccotan}\beta < \theta < \frac{\pi}{2}, 0 < \frac{r}{1-r} < -\frac{\cos\theta(\cos\theta - \beta\sin\theta)}{\sin^m\theta(\cos\theta - (\alpha+\beta)\sin\theta)}\},$$

$$D_2^- = \{(r,\theta) : \text{arccotan}(\alpha+\beta) < \theta < \pi, -\frac{\cos\theta(\cos\theta - \beta\sin\theta)}{\sin^m\theta(\cos\theta - (\alpha+\beta)\sin\theta)} < \frac{r}{1-r} < \infty\},$$

and
$$D^+ = R^* - (\bar{D}_1^- \cup \bar{D}_2^-).$$
Then $\dot{\theta} > 0$ in D^+ and $\dot{\theta} < 0$ in $D_1^- \cup D_2^-$. Furthermore,

$$\dot{r} = r(1-r)\big\{(1-r)(\text{cotan}\theta - \beta) - 2r\sin^{m-1}\theta\big\} < 0 \text{ on } \delta D^+ \cup R^*.$$

Thus D_1^- is positively invariant and any orbit in it has to go to P_0. Hence $\tilde{\gamma}_i^*$ must lie in D^+ and be of the form $r = R_\alpha^-(\theta)$ with $r^-(\alpha) = R_\alpha^-(0)$.

Next we show that $R_\alpha^-(\theta)$ is strictly increasing in α for $0 < \theta < \frac{\pi}{2}$. From Proposition 2.8 we conclude that
$$(R_\alpha^-)'(\frac{\pi}{2}) = \frac{m\beta}{\alpha+\beta} = m\frac{(m-1)\alpha+1}{(m+1)\alpha+1},$$
which is strictly decreasing from zero to $-\infty$ as α runs from $-\frac{1}{m-1}$ to $-\frac{1}{m-1}$.

This implies that if $\tilde{\gamma}_i^*(\alpha^-)$ and $\tilde{\gamma}_i^*(\alpha^+)$ are the interface orbits corresponding to $-\frac{1}{m-1} < \alpha^- < \alpha^+ < -\frac{1}{m-1}$, then $\tilde{\gamma}_i^*(\alpha^-)$ is below $\tilde{\gamma}_i^*(\alpha^+)$ near $\theta = \frac{\pi}{2}$. Now suppose they intersect somewhere between $\theta = 0$ and $\theta = \frac{\pi}{2}$. Then there has to be a last intersection, say in (r^*, θ^*), before they both go into P_s. We claim that this is impossible in view of Lemma 3.4. To be precise, in (r^*, θ^*) we must have $\dot{\theta}$ positive for both α^- and α^+, and one easily checks that D^+ varies monotonically with α. So $\dot{\theta}(r^*, \theta^*) > 0$ for all $\alpha \in (\alpha^-, \alpha^*)$. By Lemma 3.4,

$$\frac{\partial}{\partial \alpha} \frac{\dot{r}}{\dot{\theta}} < 0 \text{ in } (r^*, \theta^*),$$

implying

$$(R^-_{\alpha^+})'(\theta^*) < (R^-_{\alpha^-})'(\theta^*).$$

But then $\tilde{\gamma}_i^*(\alpha^-)$ would be above $\tilde{\gamma}_i^*(\alpha^+)$ near $\theta = \frac{\pi}{2}$, contradiction.

We conclude that $R^-_\alpha(\theta)$ is strictly increasing in α for $0 < \theta < \frac{\pi}{2}$, so that consequently $r^- : (-\frac{1}{m-1}, -\frac{1}{m+1}) \to (0, 1]$ is nondecreasing. Now suppose there exist $-\frac{1}{m-1} < \alpha^- < \alpha^+ < -\frac{1}{m+1}$ with $r^-(\alpha^-) = r^-(\alpha^+) = \hat{r} \in (0, 1)$. Fix α between α^- and α^+ and take any point between $\tilde{\gamma}_i^*(\alpha^-)$ and $\tilde{\gamma}_i^*(\alpha^+)$. Tracing its orbit backward in time we see by the same argument as before that it cannot have intersected any of these two, so it must have come from $(\hat{r}, 0)$. But this contradicts the fact that there is only orbit starting at this point. Thus r^- is strictly increasing in α-values where it is less then one.

For which α-values can we have that $r^-(\alpha) = 1$? For $\alpha = -\frac{1}{m}$ we know that γ_i coincides with γ_a, so that $r^-(-\frac{1}{m}) = 1$, and hence $r^-(\alpha) = 1$ for all α between $-\frac{1}{m}$ and $-\frac{1}{m+1}$. For α between $-\frac{1}{m-1}$ and $-\frac{1}{m}$ however it is not difficult to show that $\tilde{\gamma}_a$ goes to $(\rho, \phi) = (1, \pi)$, which makes it impossible for γ_i to have come out of $(x, y) = (0, 0)$. But then it can only have come out of infinity in the first quadrant, so that $r^-(\alpha) < 1$.

Leaving the continuity of r^- as an exercise to the reader this completes the proof.

LEMMA 3.6. *Let*

$$-\frac{1}{m-1} \leq \alpha < -\frac{1}{m+1}.$$

Denote the r-value of the first intersection point of Γ_s and l by $r^+(\alpha)$. Then

$$r^+ : [-\frac{1}{m-1}, -\frac{1}{m+1}) \to (0, 1)$$

is a well defined strictly increasing continuous function.

PROOF. As in the previous proof it is easy to check that the orbit $\tilde{\gamma}_s^*$ can be represented as the graph of a function

$$R^+_\alpha : [\operatorname{arccotan}(\alpha + \beta), \pi] \to (0, 1),$$

with $r^+(\alpha) = R^+_\alpha(\pi) \in (0, 1)$. Obviously, if $-\frac{1}{m-1} \leq \alpha^- < \alpha^+ < -\frac{1}{m-1}$, then $\tilde{\gamma}_i^*(\alpha^-)$ starts above $\tilde{\gamma}_i^*(\alpha^+)$, and as in the previous proof one shows that they cannot intersect. The rest of the proof is then similar to that of Lemma 3.5.

LEMMA 3.7.

$$\lim_{\alpha \uparrow -\frac{1}{m+1}} r^+(\alpha) = 0.$$

PROOF. Suppose this limit equals $\epsilon > 0$. Then, since $P_s \to (1, \frac{\pi}{2})$ as $\alpha \uparrow -\frac{1}{m+1}$, using the strict monotonicity of the family $\tilde{\gamma}_s^*(\alpha)$ combined with standard continuity arguments, we would have for $\alpha = -\frac{1}{m+1}$ the existence of an orbit $\tilde{\gamma}^*$ coming from $(1, \frac{\pi}{2})$ going to (ϵ, π). But only one orbit comes out of P_s, namely $\tilde{\gamma}_s^*$ itself, and it coincides with $\tilde{\gamma}_i^*$ when $\alpha = -\frac{1}{m+1}$. Contradiction. Thus the limit is zero.

LEMMA 3.8.

$$\lim_{\alpha \downarrow -\frac{1}{m-1}} r^-(\alpha) = 0.$$

PROOF. Suppose not, then, by similar arguments as in the previous proof, we obtain, for $\alpha = -\frac{1}{m+1}$, the existence of an orbit $\tilde{\gamma}^*$ going into $(0, \frac{\pi}{2})$ coming from $(\epsilon, 0)$ with $\epsilon > 0$. This again contradicts Lemma 3.3.

LEMMA 3.9. *There exists a unique number α_2 such that if $\alpha = \alpha_2$ then $\Gamma_i = \Gamma_s$ and $n(\Gamma_i) = n(\Gamma_s) = 1$. For $\alpha \in (\alpha_2, -\frac{1}{m+1})$ the orbit Γ_s goes into P_0 with $n(\Gamma_s) = 1$, while $n(\Gamma_s) \geq 2$ for $\alpha \in (-\frac{1}{m-1}, \alpha_2)$.*

PROOF. This is a direct consequence of the previous four lemmas. Note that α_2 is the unique α-value for which

$$r^-(\alpha) = r^+(\alpha).$$

We shall now finish the proof of Theorem 3.1 by induction with respect to the rotation number. Let us formulate the induction hyphothesis.

(H_n). There exists $\alpha_{2n} \in (-\frac{1}{m-1}, -\frac{1}{m+1})$ such that the following two conditions hold:

(i) if $\alpha = \alpha_{2n}$ then $\Gamma_i = \Gamma_s$ and $n(\Gamma_i) = n(\Gamma_s) = n$;
(ii) if $\alpha \in (-\frac{1}{m-1}, \alpha_{2n})$ then $n(\Gamma_s) \geq n + 1$.

LEMMA 3.10. *Suppose (H_n) holds. Then there exists a unique number $\alpha_{2n+2} \in (-\frac{1}{m-1}, \alpha_{2n})$ such that*

(I) *if $\alpha = \alpha_{2n+2}$ then $\Gamma_i = \Gamma_s$ and $n(\Gamma_i) = n(\Gamma_s) = n + 1$;*
(II) *if $\alpha \in (\alpha_{2n+2}, \alpha_{2n})$ then $n(\Gamma_s) = n + 1$ and Γ_s goes into P_0;*
(III) *if $\alpha \in (-\frac{1}{m-1}, \alpha_{2n+2})$ then $n(\Gamma_s) \geq n + 2$.*

PROOF. Suppose that (H_n) holds and let $\alpha \in (-\frac{1}{m-1}, \alpha_{2n})$. Then $n(\Gamma_s) \geq n$. For $1 \leq k \leq n+1$ let $r_k(\alpha)$ denote the r-value of the k^{th} intersection of Γ_s and l. Then

$$r_k : (-\frac{1}{m-1}, \alpha_{2n}) \to (0, 1)$$

is a well defined function. By exactly the same arguments as before one shows with induction that these functions are strictly increasing and continuous. Let α_{2n+2} be the unique α-value for which

$$r^-(\alpha) = r_{n+1}(\alpha).$$

Then it is easy to see that (I), (II) and (III) hold.

LEMMA 3.11.

$$\lim_{n\uparrow\infty} \alpha_{2n} = -\frac{1}{m-1}.$$

PROOF. Suppose not, then

$$\alpha_{2n} \downarrow \tilde{\alpha} > -\frac{1}{m-1},$$

as $n \uparrow \infty$ and $n(\Gamma_s)(\tilde{\alpha})$ has to be infinite. We claim that the sequence

$$r_1(\tilde{\alpha}), r_2(\tilde{\alpha}), r_3(\tilde{\alpha}), ...$$

is strictly decreasing. Otherwise, using the notation of Propositions 2.6 and 2.7, there would exist an orbit γ^* starting on the positive a-axis and ending on the negative a-axis with $a^+ \leq -a^-$. But such an orbit contains a solution $U(\eta)$ of (1.11) defined and positive on an interval $[\eta^+, \eta^-] \subset (0, \infty)$, and zero on the boundary. Multiplying by $(U^m)'$ and integrating over this interval it is immediate that $(U^m)'(\eta^+) > -(U^m)'(\eta^-)$ for negative α and positive β, implying $a^+ > -a^-$, contradiction. So indeed the sequence is decreasing.

Thus its limit r^* exists, and it cannot be smaller than $r^-(\tilde{\alpha})$, for then we would have for n large that $r_n(\tilde{\alpha}) < r^-(\tilde{\alpha})$, forcing $\Gamma_s(\tilde{\alpha})$ to go into P_0 and $n(\Gamma_s(\tilde{\alpha}))$ to be finite.

So r^* is positive. But then, by continuity arguments again, there has to be an orbit $\tilde{\gamma}^*$ connecting $(r^*, 0)$ and (r^*, π), which would come down to the existence of on orbit γ^* going from the positive to the negative a-axis with $\alpha^- = \alpha^+$. As we just saw, this is impossible. This completes the proof of Lemma 3.11, as well as that of Theorem 3.1, and thereby of Theorems 1.1 as far as the symmetric profiles are concerned.

Let us indicate how the analagous statements for the anti-symmetric profiles are proved.

We replace (1.26) by

$$a = (x + \beta y - \frac{1}{m})y^{\frac{m}{1-m}}, \quad b = y^{\frac{1}{1-m}},$$

which yields

$$(Q^*) \begin{cases} b^{m-1}\dot{a} = b^{m-1}(\frac{2m}{1-m}a + (\alpha + 2\beta)b); \\ b^{m-1}\dot{b} = a - \beta b - \frac{m+1}{m(m-1)}b^m. \end{cases}$$

Substituting (1.27) we obtain

$$(\tilde{Q}^*) \begin{cases} r\sin^{m-1}\theta \quad \dot{r} = (m-1)r(1-r)^2\sin\theta(\cos\theta - \beta\sin\theta) - \\ \qquad\qquad r^2(1-r)\sin^{m-1}\theta(\frac{m+1}{m} + (m-1)\cos\theta(\frac{2m+1}{m}\cos\theta - (\alpha+2\beta)\sin\theta)); \\ r\sin^{m-1}\theta \quad \dot{\theta} = r\sin^m\theta(\frac{2m+1}{m}\cos\theta - (\alpha+2\beta)\sin\theta) + \\ \qquad\qquad (1-r)\cos\theta(\cos\theta - \beta\sin\theta). \end{cases}$$

The critical points of $(\tilde{Q}^*)$ are now

$$(r,\theta) = (0, \text{arccotan}\beta), \quad (0, \frac{\pi}{2}), \quad (1,0), \quad (1, \text{arccotan}\frac{m(\alpha+2\beta)}{2m+1}) \text{ and } (1,\pi),$$

the linear analysis of which is similar as before. The only essential change is in Proposition 2.9. The critical point $(r,\theta) = (1, \text{arccotan}\frac{m(\alpha+2\beta)}{2m+1}) = P_a$ has

$$D(P_a) = \begin{pmatrix} \frac{m+1}{m}(1+(\frac{m(\alpha+2\beta)}{2m+1})^2)^{\frac{1-m}{2}} & 0 \\ \frac{(\frac{\beta}{m}-\alpha)(\alpha+2\beta)}{(\frac{2m+1}{m})^2+(\alpha+2\beta)^2} & -(\frac{2m+1}{m})^{\frac{m+1}{2}}(1+(\frac{m(\alpha+2\beta)}{2m+1})^2)^{\frac{1-m}{2}} \end{pmatrix},$$

and is a saddle. The two orbits going in are again contained in $r=1$. Only one orbit, $\tilde{\gamma}_a^*$, which corresponds to γ_a and γ_a^*, comes out of this saddle going into the interior of $[0,1]\times[0,\pi]$.

Again all orbits of (Q) and $(\tilde{Q})$ are recovered as orbits of $(\tilde{Q}^*)$ except for the ones in Proposition 2.2(ii) and γ_s in Proposition 2.3. Clearly they have to correspond to orbits of $(\tilde{Q}^*)$ coming out of $(1,0)$ or $(1,\pi)$. As it turns out, they now all come out of $(1,\pi)$.

As before, $(\tilde{Q}^*)$ acts on C and the generalized orbits are denoted by Γ. On S_1 we still find one saddle. The orbit Γ_a coming out of it now contains the anti-symmetric profiles. On S_0 the situation is unchanged. Defining F and G as before Lemma 3.4 remains unaltered, so again we have monotonicity. Now we start with $\alpha = -\frac{1}{m}$ and find that, as in Lemma 3.2, $\Gamma_a = \Gamma_i = \{(r,\frac{\pi}{2}) : r \in (0,1)\}$ and $n(\Gamma_a) = n(\Gamma_i) = 0$. This connection corresponds to the straight line

$$x + \frac{y}{2m} = \frac{1}{m} \tag{3.2}$$

in the (x,y)-upper half plane H^+, and contains the "dipole" profiles. For $\alpha = -\frac{1}{m-1}$ Lemma 3.3 holds again. Carrying out exactly the same bifurcation analysis we obtain the existence of a sequence

$$\alpha_1 = -\frac{1}{m} > \alpha_3 > \alpha_5 > \ldots \downarrow -\frac{1}{m-1},$$

such that Γ_a and Γ_i coincide if and only if $\alpha = \alpha_{2n+1}$ for some $n \in \{0,1,2,\ldots\}$. If this is so then $n = n(\Gamma_a) = n(\Gamma_i)$. Moreover, if $\alpha \in (\alpha_{2n+1}, \alpha_{2n+3})$, then $n(\Gamma_s) = n+1$ and Γ_a goes into P_0.

Finally, it is left to the reader to show that

$$\alpha_0 > \alpha_1 > \alpha_2 > \alpha_3 > \alpha_4 > \alpha_5 > \ldots,$$

and this completes the proof of Theorem 1.1.

4. Corollaries.

The machinery developed in the proof of Theorem 1.1 allows us to give a complete classification of similarity solutions of the form (1.8-9). Below we list all possibilities (modulo symmetry). Because the proofs are relatively simple we omit them.

Let us introduce some notation. If U is a solution of (1.11), then z, z^- and z^+ denote the numbers of sign changes of U on $\mathbf{R}$, $\mathbf{R}^-$ and $\mathbf{R}^+$ respectively.

(I) $-\frac{1}{m-1} < \alpha < 0$.

As we have seen in Section 2 all solutions U of (1.11) decay to zero as $\eta \to \pm\infty$ with

$$\frac{\eta U'(\eta)}{U(\eta)} \to \frac{\alpha}{\beta} < 0,$$

unless U is identically equal to zero near infinity.

The global behaviour of the orbits Γ_s, Γ_a and Γ_i, and therefore of all orbits, as well as the solutions they contain, follows from the results in the previous section. We consider the following four different cases.

(i) $\alpha_{k+1} < \alpha < \alpha_k$ with $k \geq -1$ odd ($\alpha_{-1} = 0$).

Both the (symmetric) solutions of (1.11) with $U(0) \neq 0$ and $U'(0) = 0$ as well as the (anti-symmetric) solutions of (1.11) with $U(0) = 0$ and $(U^m)'(0) \neq 0$ have support $\mathbf{R}$ with $z = k+1$ and $z = k+2$ respectively.

There exists precisely one number $p^* = p^*_k(\alpha) > 0$ such that $U(0)U'(0) > 0$ and $U(0)^{-2}|U'(0)|^{3-m} = p^*$ imply that the support of U is bounded from below with $z = k+1$ and $z^+ = \frac{k+1}{2}$. For $U(0)^{-2}|U'(0)|^{3-m} < p^*$ and $U(0)U'(0) > 0$, the support of U is $\mathbf{R}$, with $z = k+2$ and $z^+ = \frac{k+1}{2}$. For $U(0)^{-2}|U'(0)|^{3-m} > p^*$ and $U(0)U'(0) \neq 0$, the support of U is $\mathbf{R}$, with $z = k+1$ and $z^+ = \frac{k+1}{2}$.

(ii) $\alpha = \alpha_k$ with $k \geq 0$ even.

The (symmetric) solutions of (1.11) with $U(0) \neq 0$ and $U'(0) = 0$ are compactly supported and have $z = k$. The (anti-symmetric) solutions of (1.11) with $U(0) = 0$ and $(U^m)'(0) \neq 0$ have support $\mathbf{R}$ and $z = k+1$. The same holds for the solutions of (1.11) with $U(0) \neq 0$ and $U'(0) \neq 0$, with $z^+ = \frac{k}{2}$ if $U(0)U'(0) > 0$.

(iii) $\alpha_{k+1} < \alpha < \alpha_k$ with $k \geq 0$ even.

Both the (symmetric) solutions of (1.11) with $U(0) \neq 0$ and $U'(0) = 0$ as well as the (anti-symmetric) solutions of (1.11) with $U(0) = 0$ and $(U^m)'(0) \neq 0$ have support $\mathbf{R}$ with $z = k+2$ and $z = k+1$ respectively.

There exists precisely one number $p^* = p^*_k(\alpha) > 0$ such that $U(0)U'(0) > 0$ and $U(0)^{-2}|U'(0)|^{3-m} = p^*$ imply that the support of U is bounded from above with $z = k+1$ and $z^+ = \frac{k}{2}$. For $U(0)^{-2}|U'(0)|^{3-m} < p^*$ and $U(0)U'(0) > 0$, the support of U is $\mathbf{R}$, with $z = k+1$ and $z^+ = \frac{k}{2}$. For $U(0)^{-2}|U'(0)|^{3-m} > p^*$ and $U(0)U'(0) \neq 0$, the support of U is $\mathbf{R}$, with $z = k+2$ and $z^+ = \frac{k}{2} + 1$.

(iv) $\alpha = \alpha_k$ with $k > 0$ odd.

The (anti-symmetric) solutions of (1.11) with $U(0) = 0$ and $(U^m)'(0) \neq 0$ are compactly supported and have $z = k$. The (symmetric) solutions of (1.11) with $U(0) \neq 0$ and $U'(0) = 0$ have support $\mathbf{R}$ and $z = k+1$. The same holds for the solutions of (1.11) with $U(0) \neq 0$ and $U'(0) \neq 0$, with $z^+ = \frac{k+1}{2}$.

Some of the results below were discovered independently by Vazquez [V].

(II) $\alpha > 0$.

The nature of the critical points is the same. The main difference is that solutions contained in orbits going into P_0, i.e. orbits with $x \to \frac{\alpha}{\beta}$ now contain solutions that blow up as $\eta \to \infty$, whereas before they decayed to zero. It is not difficult to see that both $\tilde{\gamma}_s$ and $\tilde{\gamma}_a$ go into the critical point $(\rho, \phi) = (1, \frac{\pi}{2})$. Consequently all the symmetric and anti-symmetric solutions have support $\mathbf{R}$ and $z = 0$ and $z = 1$ respectively, and $U(\eta) \to \pm\infty$ as $\eta \to \pm\infty$.

There exists precisely one number $\bar{p} = \bar{p}_n(\alpha) > 0$ such that $U(0)U'(0) > 0$ and $U(0)^{-2}|U'(0)|^{3-m} = \bar{p}$ imply that the support of U is bounded from below with $z = 0$ and $U(\eta) \to +\infty$ as $\eta \to +\infty$. For $U(0)^{-2}|U'(0)|^{3-m} < \bar{p}$ and $U(0)U'(0) > 0$, the support of U is $\mathbf{R}$, with $z = z^- = 1$ and $U(\eta) \to \pm\infty$ as $\eta \to \pm\infty$. For $U(0)^{-2}|U'(0)|^{3-m} > \bar{p}$ and $U(0)U'(0) > 0$, the support of U is $\mathbf{R}$, with $z = 0$ and $U(\eta) \to \pm\infty$ as $\eta \to \pm\infty$.

Whenever $U(\eta) \to \pm\infty$ as $\eta \to \pm\infty$, one has

$$\frac{\eta U'(\eta)}{U(\eta)} \to \frac{\alpha}{\beta}.$$

Note that $\frac{\alpha}{\beta}$ runs from 0 to $\frac{2}{m-1}$ as α runs from 0 to $+\infty$. If $\alpha = \frac{1}{m+1}$ and $\beta = \frac{m}{m+1}$, then γ_a is the straight line $x = \frac{1}{m}$ containing the solutions $U(\eta) = C\text{sign}\eta|\eta|^{\frac{1}{m}}$, yielding the stationary solutions $u(x,t) = C\text{sign}x|x|^{\frac{1}{m}}$.

Whenever an orbit γ of (Q) is a straight line it contains a family of explicit solutions. There is however at least one well known family of solutions for which the converse of this statement is false. Namely, let

$$\alpha = \frac{1}{m-1} \text{ and } \beta = 1.$$

Then

$$U(\eta) = \pm\{\frac{C}{m-1} - C^2\eta\}_+^{\frac{1}{m-1}},$$

with C arbitrary solves (1.11), but it is contained in the orbit pair given by

$$x^2 + xy = \frac{1}{m-1}y,$$

which is not a straight line.

REMARK. As a “limit case” we mention similarity solutions of the form

$$u(x,t) = e^{\alpha t}U(\eta) \text{ with } \eta = xe^{-\beta t}. \tag{4.1}$$

One finds that α and β have to satisfy $2\beta = (m-1)\alpha$, but apart from that the equation for U remains the same. Thus for α and β both positive we obtain the same picture as just before, with $\frac{\alpha}{\beta} = \frac{2}{m-1}$. For α and β both negative we find the same behaviour as in the case below.

(III) $\alpha < -\frac{1}{m-1}$.

Here the local behaviour near the critical points P_i and P_0 is different. First of all, P_0 is now a repeller. Orbits coming out of it contain solutions with

$$U(\eta) \to 0 \text{ from one side and } \frac{\eta U'(\eta)}{U(\eta)} \to \frac{\alpha}{\beta} > \frac{2}{m-1} \text{ as } \eta \to 0 \text{ from one side.}$$

The corresponding solutions are defined on a half space and have infinite waiting time. Another change is that the orbit γ_i now comes out of infinity in finite time. It contains solutions with support in a half space and bounded away from zero. Note that this could be expected because $\beta < 0$, and that the corresponding similarity solutions converge to zero pointwise as $t \downarrow 0$. Any two of these new solutions defined on a left and a right half space together form a solution on the whole space.

Again there are the symmetric and anti-symmetric profiles contained in Γ_s and Γ_a respectively as well as the profiles with $U(0)U'(0) \neq 0$ contained in the other orbits coming out of $(x, y) = (0, 0)$.

What happens to all these solutions as $\eta \to \pm\infty$? There are no orbits Γ on the cylinder C that go into a critical point on the boundary. It is not difficult to show that in forward τ-time every orbit has to go around the cylinder forever and converge to a periodic orbit on C. Consequently all the solutions above remain oscillating as $\eta \to \infty$, with $U(\eta)^{m-1}\eta^{-2}$ approximately periodic in $\log \eta$. Roughly speaking they oscillate between $+\eta^{\frac{2}{m-1}}$ and $-\eta^{\frac{2}{m-1}}$.

The periodic orbit above contains solutions defined on a half space with $U(\eta)^{m-1}\eta^{-2}$ periodic in $\log \eta$. In particular these solutions have infinitely many sign changes both near $\eta = 0$ and $\eta = \infty$ and show that the lapnumber of a solution of the porous medium equation does not necessarily have to be locally finite. The corresponding similarity solutions are of the form

$$u(x,t) = x^{\frac{2}{m-1}} t^{-\frac{1}{m-1}} g(\log x - \beta \log t), \tag{4.2}$$

where $g = b$ is a periodic function. For solutions of the form (4.1) a periodic orbit on C must exist provided α and β are both negative. It contains solutions of the form

$$u(x,t) = x^{\frac{2}{m-1}} g(\log x - \beta t), \tag{4.3}$$

with g periodic. Thus these latter solutions are periodic in t!

We end this section with a conjecture.

CONJECTURE. For fixed $\alpha < -\frac{1}{m-1}$ there is only one periodic orbit on C, and it is the ω-limit set of every other orbit on C.

Note that if the conjecture is false there must be a whole family of orbits going around C forever. Every solution contained in such an orbit oscillates near zero and infinity.

5. Radially symmetric similarity solutions.

In arbitrary space dimension the porous medium equation reads

$$u_t = \frac{1}{m}\Delta(|u|^{m-1}u), \quad x \in R^n, \quad t > 0. \tag{5.1}$$

For $m > 1$ one still has the Barenblatt-Pattle solutions, given by

$$u(x,t) = t^{-\frac{n}{n(m-1)+2}}U(\eta) \text{ with } \eta = |x|t^{-\frac{1}{n(m-1)+2}}, \tag{5.2}$$

and

$$U(\eta) = \pm\{C - \frac{m-1}{2}\frac{\eta^2}{n(m-1)+2}\}_+^{\frac{1}{m-1}}. \tag{5.3}$$

Again we want to classify all compactly supported similarity solutions of the form

$$u(x,t) = t^{\alpha}U(\eta) \text{ with } \eta = |x|t^{-\beta}, \tag{5.4}$$

with α negative and β positive. Substitution yields (1.10) and

$$(|U|^{m-1}U)'' + \frac{n-1}{\eta}(|U|^{m-1}U)' + m\beta\eta U' = m\alpha U.$$

In order for u to be a solution one needs $(U^m)'(0) = 0$.

THEOREM 5.1. *Let $m > 1$. Then there exists a strictly decreasing sequence*

$$\alpha_0^n = -\frac{n}{n(m-1)+2} > \alpha_1^n > \alpha_2^n > \alpha_3^n > \ldots \downarrow -\frac{1}{m-1},$$

such that there is a one-parameter family of radially symmetric compactly supported solutions of the form above if and only if $\alpha = \alpha_k^n$ for some integer $k \geq 0$. Moreover, k equals exactly the number of sign changes of $U(\eta)$ on $\mathbf{R}^+$.

As before the proof is based on the substitution of (1.17). The quadratic autonomous system in the open upper half (x,y)-plane H^+ is now

$$(Q_n)\begin{cases} \dot{x} = (2-n)x - mx^2 + \alpha y - \beta xy; \\ \dot{y} = y(2 + (1-m)x). \end{cases}$$

The origin is still a critical point with one unique orbit γ_r coming out into the open upper half plane along the eigenvector

$$\begin{pmatrix} \alpha \\ n \end{pmatrix}$$

of the linearization

$$A(0,0) = \begin{pmatrix} 2-n & \alpha \\ 0 & 2 \end{pmatrix}$$

of (Q_n) around the origin. The (positive) solutions U contained in γ_r satisfy

$$\frac{\alpha}{n} = \lim_{\tau\downarrow-\infty} \frac{x(\tau)}{y(\tau)} = \frac{1}{m-1} \lim_{\eta\downarrow 0} \frac{(U^{m-1})'(\eta)}{\eta},$$

so that

$$\lim_{\eta\downarrow 0} (U^{m-1})'(\eta) = 0.$$

Since obviously $U'(\eta)$ is negative for η near zero, it follows that

$$\lim_{\eta\downarrow 0} U(\eta) \text{ exists as a positive number,}$$

and that

$$\lim_{\eta\downarrow 0} U'(\eta) = 0.$$

Scaling of $U(0)$ corresponds again to a shift in τ. Thus γ_r contains the profiles that give rise to radially symmetric similarity solutions of the above form with $u(0,t) \neq 0$.

As before (1.23) transforms (Q_n) into

$$(\tilde{Q}_n)\begin{cases} (1-\rho)\dot{\rho} = \rho(1-\rho)^2\{2-\cos\phi(n\cos\phi-\alpha\sin\phi)\} \\ \qquad\qquad -\rho^2(1-\rho)\cos\phi\{m-1+\cos\phi(\cos\phi+\beta\sin\phi)\}; \\ (1-\rho)\dot{\phi} = \rho\sin\phi\cos\phi(\cos\phi+\beta\sin\phi)+(1-\rho)\sin\phi(n\cos\phi-\alpha\sin\phi). \end{cases}$$

The image of the orbit γ_r is the orbit $\tilde{\gamma}_r$ coming out of the saddle $(\rho,\phi) = (0, \operatorname{arccotan}\frac{\alpha}{n})$. At infinity we find exactly the same picture as before, with one orbit $\tilde{\gamma}_i$ going into the saddle $(\rho,\phi) = (1, \operatorname{arccotan}(-\beta))$ containing the profiles with an interface. Again the singularities at $(\rho,\phi) = (1,0)$ and $(\rho,\phi) = (1,\pi)$ are blown up by (1.26), which yields

$$(Q_n^*)\begin{cases} b^{m-1}\dot{a} = b^{m-1}((\frac{2}{1-m}-n)a+(\alpha+n\beta)b); \\ b^{m-1}\dot{b} = a-\beta b-\frac{2}{m-1}b^m, \end{cases}$$

with exactly the same interpretations concerning solutions U with sign changes. Substituting (1.27) we obtain

$$(\tilde{Q}_n^*)\begin{cases} r\sin^{m-1}\theta \quad \dot{r} = (m-1)r(1-r)^2\sin\theta(\cos\theta-\beta\sin\theta)- \\ \qquad r^2(1-r)\sin^{m-1}\theta(2+(m-1)\cos\theta(n\cos\theta-(\alpha+n\beta)\sin\theta)); \\ r\sin^{m-1}\theta \quad \dot{\theta} = r\sin^m\theta(n\cos\theta-(\alpha+n\beta)\sin\theta)+ \\ \qquad (1-r)\cos\theta(\cos\theta-\beta\sin\theta), \end{cases}$$

which acts again on the cylinder C. With the same notation as in Lemma 3.4 one finds

$$F_\alpha G - FG_\alpha = -r\sin^{m+1}\theta(\frac{1-r}{2}+(\frac{2}{m-1}+n)r\sin^{m-1}\theta).$$

which is negative for $\theta \neq 0,\pi$ and $r \neq 0$. It is not difficult to show that for $\alpha = -\frac{1}{m-1}$ every orbit goes around C forever having S_0 as ω-limit set. Letting α run from $-\frac{n}{n(m-1)+2}$

to $-\frac{1}{m-1}$ the bifurcation analysis in Section 3 carries over immediately, and the theorem follows.

We conclude this section with some remarks on the other radially symmetric solutions. Everything we said in Section 4 about symmetric profiles generalizes to the radially symmetric case if α_{2k} is replaced by α_k^n.

Moreover, for $\alpha < -\frac{1}{m-1}$, every orbit on C converges again to a periodic orbit, which we conjecture to be unique. Hence every solution oscillates and blows up as $\eta \to \infty$, with $U(\eta)^{m-1}\eta^{-2}$ approximately periodic in $\log\eta$, and the periodic orbit itself contains solutions for which $U(\eta)^{m-1}\eta^{-2}$ is periodic in $\log\eta$. Finally, the orbits coming out of P_0 contain solutions with

$$U(\eta) \to 0 \text{ and } \frac{\eta U'(\eta)}{U(\eta)} \to \frac{\alpha}{\beta} > \frac{2}{m-1} \text{ as } \eta \downarrow 0,$$

and the interface orbit Γ_i contains profiles with support bounded away from zero corresponding to similarity solutions with pointwise limit zero as $t \downarrow 0$.

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Addendum 1. Shortly after the manuscript was completed, the author discovered that the time periodic solution of the form (4.3) is unique modulo the obvious scalings and symmetries. This can be made precise as follows. Substitution of (4.3) into (1.1) yields

$$\left(\frac{m+1}{m-1} + \frac{d}{d\eta}\right)\left(\frac{2m}{m-1} + \frac{d}{d\eta}\right)(|g(\eta)|^{m-1}g(\eta)) = \lambda m \frac{dg(\eta)}{d\eta}, \tag{A.1}$$

where $\lambda = -\beta > 0$, and

$$\eta = \log x + \lambda t. \tag{A.2}$$

Now observe that λ can be scaled out of $(A.1)$ by setting

$$g_0(\eta) = \lambda^{\frac{1}{m-1}} g(\eta). \tag{A.3}$$

Obviously the function g_0 is a solution of $(A.1)$ with $\lambda = 1$, while (4.3) transforms in

$$u(x,t) = x^{\frac{2}{m-1}} \lambda^{\frac{1}{1-m}} g_0(\log x + \lambda t). \tag{A.4}$$

We have seen in Section 4 that there is at least one nontrivial periodic solution of $(A.1)$. Below we indicate why it is uniquely determined upto a translation.

Suppose there are two periodic solutions of (A.1) with $\lambda = 1$ such that one is not a translate of the other. Then their maxima have to be different, so scaling one of them by $(A.3)$ we obtain two periodic solutions of $(A.1)$ for two different values of λ but with the same maxima. Since $g = b$, where b is given by (1.26), and with the positive and negative a-axis identified, we thus obtain two periodic orbits of (Q^*) corresponding to different values of the parameters α and β, but with the same maximal values for b. Note that α and β are now related by $2\beta = (m-1)\alpha$. Obviously these two orbits have to intersect at least twice, and the same holds for the corresponding orbits on the cylinder C. On the other hand we have that Lemma 3.4 simplifies to

$$F_\alpha G - F G_\alpha = -\frac{m+1}{m-1} r^2 \sin^{2m}\theta,$$

which is still negative for $r \neq 0$ and $\theta \neq 0 \bmod \pi$. Consequently there can be at most one intersection. This contradicts the conclusion above and completes the proof.

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