ON A FREE BOUNDARY PROBLEM ARISING IN DETONATION THEORY: CONVERGENCE TO TRAVELLING WAVES

by

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1. INTRODUCTION.

In this note we consider the free boundary problem

$$(P_1)\begin{cases} u_t = u_{xx} + uu_x & t>0\ ,\ x\neq\zeta(t)\ ;\\ u(\zeta(t)^-,t) = u(\zeta(t)^+,t) = q & t>0\ ;\\ u_x(\zeta(t)^-,t) - u_x(\zeta(t)^+,t) = 1 & t>0\ ;\\ u(x,0) = u_0(x) & x\in\mathbf{R}\ ;\\ \zeta(0)=\zeta_0\quad , \end{cases}$$

where q is a positive constant, ζ_0 is a given real number, u_0 a given initial function satisfying $u_0(\zeta_0) = q$, and $\zeta(t)$ and $u(x,t)$ are the unknown functions to be found. This problem arises in combustion theory and was introduced by Stewart in [12] and by Ludford and Oyediran in [10]. It is a simple model of detonation waves where the reaction is supposed to occur at the detonation front $x = \zeta(t)$, and where q is the ignition temperature.

We consider three configurations for the initial function u_0.
(i) $0 \le u_o \le A$ for some constant $A > q$, $u_0(x) < q$ for $x < \zeta_0$ and $u_0(x) > q$ for $x > \zeta_0$.
(ii) $0 \le u_o \le q$ and $\zeta_0 = \sup\{x < \zeta_0 : u_0(x) < q\}$.
(iii) $0 \le u_o \le q$ and $u_0(x) < q$ for all $x \neq \zeta_0$.

In Section 2 we consider the first case and show, under some additional assumptions on the initial data, that Problem (P_1) has a unique solution (u,ζ) which is such that $0 \le u \le A$, $u(x,t) < q$ for all $x < \zeta(t)$ and $u(x,t) > q$ for all $x > \zeta(t)$.

The second case, in which Problem (P_1) reduces to the "one-phase" problem

$$(P_2)\begin{cases} u_t = u_{xx} + uu_x & t > 0 \ , \ x < \zeta(t) \ ; \\ u(\zeta(t),t) = q & t > 0 \ ; \\ u_x(\zeta(t),t) = 1 & t > 0 \ ; \\ u(x,0) = u_0(x) & x < \zeta_0 \ ; \\ \zeta(0) = \zeta_0 \quad , \end{cases}$$

is considered in Section 3. Under some mild regularity asssumptions on u_0, we show the existence and uniqueness of the solution of Problem (P_2).

In the third case, we doubt whether Problem (P_1) is well-posed: if we omit the convection term uu_x in the parabolic equation for u, and take an initial profile u_0 which is symmetric around ζ_0, then the well-posedness would imply that $\zeta(t) \equiv \zeta_0$ and that the solution remains symmetric around ζ_0. But then the jump condition in u_x across the free boundary reduces to $u_x(\zeta_0,t) = 1/2$ which makes this problem overdetermined.

One could think of solving Problem (P_1) by means of a fixed point method based on an iterative procedure of the form: given ζ solve for u, then compute ζ from u and so on. Unfortunately, ζ appears only implicitly in the free boundary conditions and it is very hard to carry out the second step. The only explicit formula one has is

$$\zeta'(t) = -\lim_{x\uparrow\zeta(t)} \left\{ \frac{u_{xx}(x,t)}{u_x(x,t)} + u(x,t) \right\} \quad ,$$

which is difficult to work with because of the second order term u_{xx}. This motivated us to look for weak formulations of the problems (P_1) and (P_2) in which the free boundary does not appear. The free boundary is characterised a posteriori as the level set of q of the weak solution u. We were able to do this for the first two cases but not for the third.

In Section 4 we study the large time behaviour of these solutions (u,ζ). It turns out that whenever the problem has a unique (up to translation) travelling wave solution, any solution converges to a travelling wave.

In [4] Brauner, Noor Ebad and Schmidt-Lainé have studied the local exponential stability of travelling wave solutions of Problem (P_2) without the convection term. For the results on the two-phase problem we refer to [1] and [2], and the one-phase problem is dealt with in [7]. Finally we mention that in [11] a related Stefan problem with supercooling is investigated. Integrating this problem with respect to the space variable one obtains our one-phase problem without the convection term.

2. WELL-POSEDNESS OF THE TWO-PHASE PROBLEM.

In this section we assume that u_0 satisfies the following hypothesis:

H1. $u_0 \in C^{0,1}(\mathbf{R})$, $0 \le u_0 \le A$, $u_0' + H(u_0 - q) \in C^{0,1}(\mathbf{R})$, where H is the Heaviside function, $u_0 - AH \in L^1(\mathbf{R})$, and, for some $\delta > 0$ and $\varepsilon > 0$, $u_0' \ge \delta$ in the set $(a,b) = \{x \ne \zeta_0, |u_0(x) - q| < \varepsilon\}$, $u_0(x) < q - \varepsilon$ for $x < a$ and $u_0(x) > q + \varepsilon$ for $x > b$;

and associate to Problem (P_1) the problem

$$(Q_1)\begin{cases} u_t = u_{xx} + uu_x + \{H(u-q)\}_x & t>0\ ,\ x\in\mathbf{R}\ ;\\ u(x,0)=u_0(x) & x\in\mathbf{R}\ ,\end{cases}$$

In [1] we prove the following results:

There exists a unique (weak) solution u of Problem (Q_1). Moreover there exists a function

$$\zeta \in C^{0,1}([0,\infty)) \cap C^1((0,\infty)) \quad ,$$

such that

$$\{(x,t)\in \mathbf{R}\times\mathbf{R}^+ : u(x,t)=q\} = \{(x,t): t>0, x=\zeta(t)\} \quad .$$

The pair (u,ζ) is a solution of Problem (P_1).

The uniqueness proof for Problem (Q_1) is rather standard and so is the idea of the existence proof: we regularize the Heaviside function in the parabolic equation. The proof of the convergence of the regularized solution to a weak solution is not standard. We really need precise information about the level curves, hence the strong assumptions on the initial data. Formally these level curves are defined by

$$u(X(c,t),t)=c \quad ,$$

for c near q where X_t turns out to satisy the equation

$$(X_t)_t = \left\{\frac{(X_t)_c}{X_c^2}\right\}_c \quad .$$

Using the assumptions we show that for the regularized solution u_x is positive near the level set of q. A maximum principle argument permits us to conclude that X_t is bounded near that same level set. One can then show that the level set of q is really a level curve $x=\zeta(t)$. Near this curve u_x can be bounded away from zero, from which we finally deduce that $\zeta(t)$ is continuously differentiable for $t>0$.

3. WELL-POSEDNESS OF THE ONE-PHASE PROBLEM.

In this section we assume that u_0 satisfies the following hypothesis:

H2. $u_0 \in C((-\infty,\zeta_0]) \cap L^1(-\infty,\zeta_0)$, $0\le u_0\le q$, $u_0(x)\to 0$ as $x\to-\infty$, $u_0(\zeta_0)=q$, $\zeta_0=\sup\{x<\zeta_0 : u_0(x)<q\}$;

and associate to Problem (P_2) the elliptic-parabolic problem

$$(Q_2)\begin{cases} c(u)_t = u_{xx} + c(u)c(u)_x & t>0\ ,\ x<R\ ;\\ u_x+\frac{1}{2}c(u)^2 = 1+\frac{1}{2}q^2 & t>0\ ,\ x=R\ ;\\ c(u(x,0))=v_0(x) & x<R\ ,\end{cases}$$

where

$$c(s)=\min(s,q) \quad ,$$

$v_0 \equiv u_0$ on $(-\infty, \zeta_0]$, and $v_0 \equiv q$ on (ζ_0, R). Note that for $u < q$ the differential equation is the same parabolic equation as before, while for $u > q$ it reduces to the elliptic equation $u_{xx} = 0$. Related problems on bounded domains have been studied by several authors, see e.g. [6,5,9,8,3]. Formally a solution of Problem (P_2) can be viewed as a (weak) solution of (Q_2) provided that $R > \zeta$ if it is extended by setting

$$u(x,t) = q + x - \zeta(t) \quad ,$$

for $x > \zeta(t)$.

Existence and uniqueness of a weak solution u of (Q_2) are established in [7] following [6,9]. However this weak solution may be strictly less than q on the whole of $(-\infty, R]$ for some $t > 0$. In order to avoid this possibility we assume that

$$R \geq \zeta_0 + q$$

Under this condition we show in [7] along the lines of [8] that there exists a continuous interface $x = \zeta(t)$ separating the regions $\{u < q\}$ and $\{u > q\}$, and that (u, ζ) is a solution of (P_2). It can then be shown that u_x is strictly positive in a neighbourhood of the interface and consequently the following transformation makes sense:

$$p = u_x \quad , \quad \xi = u \quad , \quad \tau = t \quad ,$$

For $p = p(\xi, \tau)$ we obtain

$$p_\tau = p^2(p_{\xi\xi} + 1) \quad ,$$

and the Neumann condition on the free boundary transforms into

$$p(q, \tau) = 1 \quad .$$

Standard regularity theory allows us to conclude that p is smooth up to $\xi = q$. Since

$$\frac{u_t}{u_x} = p_\xi + \xi \quad ,$$

this implies that the level curves near the free boundary and in particular the free boundary itself are smooth for positive t. Hence u is also smooth up to the free boundary.

4. CONVERGENCE TO TRAVELLING WAVES.

First we consider the two-phase problem (case (i) in Section 1). There exists a unique travelling wave velocity

$$\omega = \frac{1}{A} + \frac{A}{2} \quad ,$$

and a corresponding travelling wave solution of the form

$$u(x,t) = U(x + \omega t) \quad ,$$

unique up to translation, if and only if q and A satisfy the travelling wave condition

$$\frac{2}{A} < q < A \quad .$$

If this condition is satisfied we prove in [2] that the solution u converges to a translate of the travelling wave, and that the difference of the corresponding interfaces tends to zero as $t \to \infty$. In future work we shall come back to the case where the travelling wave condition is not satisfied.

For the one-phase problem (case (ii) in Section 1) there always exists a unique travelling wave velocity

$$\omega = \frac{1}{q} + \frac{q}{2} \quad ,$$

and a unique travelling wave profile U. The convergence of every solution to a travelling wave as $t \to \infty$, as well as the convergence of the difference of the interfaces to zero are established in [7].

In both cases the translate is determined by the condition that at $t = 0$ the integral of the difference of u_0 and the travelling wave equals zero. The proofs are based on the L^1-contractivity of the solution operators corresponding to the problems (P_1) and (P_2), and on comparison principle arguments.

Let us now come back to the the two-phase problem in the case that

$$0 < q < \frac{2}{A} \quad .$$

Although a "global" travelling wave does not exist, the asymptotic behaviour of the solution as $t \to \infty$ can be described by means of two travelling waves: the travelling wave U^- of the one-phase problem described just above with velocity

$$\omega^- = \frac{1}{q} + \frac{q}{2} \quad ,$$

and the travelling wave U^+ of Burger's equation with $U^+(-\infty) = q$ and $U^+(+\infty) = A$, which has velocity

$$\omega^+ = \frac{q + A}{2} < \omega^- \quad .$$

Then in the travelling wave coordinates corresponding to ω^-, $\min(u, q)$ converges to a translate of U^-, and in the travelling wave coordinates corresponding to ω^+, $\max(u, q)$ to a translate of U^+.

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