A VERY SINGULAR SOLUTION FOR THE DUAL POROUS MEDIUM EQUATION AND THE ASYMPTOTIC BEHAVIOUR OF GENERAL SOLUTIONS

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1. Introduction

This paper is devoted to study the nonlinear parabolic equation

$$z_t = |z_{xx}|^{m-1} z_{xx} \tag{E}$$

with $m > 0$. Our contribution consists in establishing the existence and properties of certain solutions of selfsimilar type which are then used to obtain information on the behaviour of general classes of solutions. Actually, for $m = 1$, i.e. for the *classical heat equation* the solutions to look for are the so-called fundamental solutions and a complete theory can be based on representation of general solutions in terms of them.

In this paper we show that for $m > 1$ equation (E) admits nonnegative selfsimilar solutions $\tilde{z}(x,t)$ of the form

$$\tilde{z} = t^{-\gamma} F(xt^{-\delta}) \quad \text{with} \quad \gamma, \delta > 0\,, \tag{1.1}$$

with the property that $\tilde{z}(x,t)$ is compactly supported as a function of x for every $t > 0$. The exponents depend on m. Moreover, such a solution is uniquely determined if we impose the normalization condition $\int \tilde{z}(x,1)\,dx = c$ for some fixed $c > 0$. In this way the concept of fundamental solution of the heat equation $u_t = u_{xx}$ is generalized to equation (E) with $m > 1$. Our solutions share some of the properties of fundamental solutions of the heat equation ; thus, the asymptotic behaviour of any solution with initial data having compact support is given in first approximation by the family $\tilde{z}_{m,c}$. On the other hand, the functions $\tilde{z}_{m,c}$ have a strikingly different behaviour as $t \to 0$: while the fundamental solutions of the heat equation tend to a point mass as time goes to 0 (*source-type* solutions), our solutions are of *a very singular type.* In fact, the exponents in (1.1) are such that γ is always larger than δ. This implies not only that the *total mass* $\int \tilde{z}(x,t)\,dx = Ct^{\delta-\gamma}$ decreases in time (against the conservation law in the heat equation case) but also that

$$\int \tilde{z}(x,t)\,dx \to \infty \quad \text{as } t \to 0\,. \tag{1.2}$$

This is a quite unusual phenomenon for a diffusion equation without lower-order terms. We only know of a similar abnormal behaviour for the Barenblatt equation of elasto-plastic filtration, $u_t = u_{xx} + a|u_{xx}|$, which was proved by Kamin [K3]. The fact that these equations are not of divergence form is crucial.

For every fixed $t > 0$ the function $\tilde{z}_{m,c}(x,t)$ is compactly supported as a function of $x \in \mathbf{R}$. This property is also in contrast with the fundamental solutions of the heat equation, but agrees with the other two well-known nonlinear versions of the heat equation, namely the porous medium equation (PME) $u_t = (u^m)_{xx}$, $m > 1$, and the so-called p-Laplacian equation (PLE) $u_t = (|u_x|^{p-2}u_x)_x$ for $p > 2$. But it must be observed that in both equations the total mass is conserved so that their nonnegative solutions of the form (1.1) satisfy $\gamma = \delta$ and their initial trace is a point source. Anyway, the three equations are intimately related since formal differentiation of (E) gives (PLE) with $p = m+1$ and a second differentiation step gives the PME. This is why we call equation (E) the *dual porous medium equation* . Actually, we shall use this relationship to formulate and prove our technical results in terms of equation PME.

The interest of the particular selfsimilar solution $\tilde{z}$ lies in its application to describe the large-time behaviour of *all* nonnegative solutions of (E) defined in $Q = \{(x,t) : x \in \mathbf{R},\, t > 0\}$, whose initial data are continuous and compactly supported. This is a property that fundamental solutions of diffusion equations typically have.

A similar study can be performed under the assumption $0 < m < 1$. In this case a family of positive selfsimilar solutions $\tilde{z}_{m,c}(x,t)$ with optimal decay as $|x| \to \infty$ exists. Though the situation is similar in many respects, two differences are worth noting. Firstly, we now find that $\gamma < \delta$ so that the solutions have zero trace in the L^1-norm (in $L^1_{\rm loc}$ if $m \leq 1/3$) as $t \to 0$. However, the singularity is apparent in some L^p-norm, $p > 1$, it is a "small singularity". The total mass goes to infinity as $t \to \infty$. Secondly, the support of the solutions extends to the whole $Q = \mathbf{R} \times \mathbf{R}_+$ in this case.

Let us finally mention that, apart from its mathematical interest as a simple example of a fully nonlinear and degenerate parabolic equation, equations like (E) appear in some problems in elasticity with damping [DL] as well as in problems of Bellman-Dirichlet type [L]. The existence and uniqueness of solutions of equations of the general type

$$\partial u/\partial t = \beta(\Delta u), \qquad x \in \mathbf{R}^N, t > 0\,,$$

for some continuous and increasing real-valued function β has been studied by Kamin [K1], Strauss [S], Bénilan and Ha [BH], [Ha], and Konishi [Ko]. The last three papers are written in the context of semigroup theory. On the other hand, the paper [KPV2] establishes the asymptotic convergence of a wide class of nonnegative solutions of the Barenblatt equation of elasto-plastic filtration towards a selfsimilar solution whose singularity at $(0,0)$ is not of source type, a situation exactly as the one described here.

2. Preliminaries and statement of main results for $m > 1$

The study of the large-time behaviour of solutions to the Cauchy Problem

$$u_t = (|u|^{m-1}u)_{xx} \quad \text{in } Q = \mathbf{R} \times (0,\infty) \tag{2.1}$$

$$u(x,0) = u_0(x) \quad \text{for } x \in \mathbf{R} \tag{2.2}$$

can be reduced to the description of the selfsimilar solutions of this equation, i.e. solutions of the form

$$u(x,t) = t^{-\alpha}U(\beta^{1/2}xt^{-\beta}) \tag{2.3}$$

with $U \not\equiv 0$. Necessarily, the parameters α and β have to satisfy

$$\alpha(m-1) + 2\beta = 1\,. \tag{2.4}$$

It is convenient to introduce the ratio $k = \alpha/\beta$ so that (2.4) can be written as

$$\alpha = \frac{k}{k(m-1)+2} \qquad \beta = \frac{1}{k(m-1)+2} \tag{2.5}$$

and the profile $U = U(y)$ is a solution of the ODE

$$(|U|^{m-1}U)'' + yU' + kU = 0\,. \tag{2.6}$$

In the case of the classical heat equation, which corresponds to setting $m = 1$ in (2.1), it is well-known that exactly for the sequence $k = 1, 2, \cdots, n, \cdots$ we obtain particular solutions U_n of (2.6) which decay exponentially as $|y| \to \infty$ while for other values of k such a decrease rate is only power-like. Moreover, the corresponding solution $u_n(x,t) = t^{-n/2}U_n(x(2t)^{-1/2})$ takes on the initial data

$$u_n(x,t) \to C_n \frac{d^{n-1}}{dx^{n-1}}\delta(x) \quad \text{as} \quad t \to 0\,, \tag{2.7}$$

where δ is the Dirac mass and C_n is an arbitrary positive constant which uniquely determines U_n and u_n, and the limit is taken in the distributional sense. Actually, since (2.1) is a linear equation with constant coefficients all these solutions are obtained (up to a constant factor) by differentiation of the fundamental solution.

For $m > 1$ the situation is not essentially different. In fact, it has been proved by Hulshof [Hu] that there exists an infinite increasing sequence $\{k_n\}$, $n \geq 1$ such that $k_n \to \infty$ and for every n there exists a one-parameter family of nontrivial solutions of (2.6) with $k = k_n$ which are *compactly supported* , while for other values of k the solutions of (2.6) have power-like decay. We shall call the k_n *eigenvalues* of (2.1) while the corresponding solutions are called *eigenfunctions.* The n-th eigenfunctions are characterized by having $n-1$ zeroes in their support. It is maybe worth mentioning that all eigenfunctions corresponding to the same eigenvalue can be obtained from a single one by performing a change of sign and/or the scaling transformation

$$\hat{U}(y) = \lambda^{-2}U(\lambda^{m-1}y)\,, \tag{2.8}$$

which produces out of every $\lambda > 0$ and every solution U a new solution $\hat{U} = U_\lambda$. We may thus without loss of generality normalize for instance U_n to take the value $U_n(0) = 1$ if n is odd (then U_n is even) or $U_n'(0) = 0$ if n is even (then U_n is odd).

The eigenfunctions play an important role in the theory of the Cauchy Problem (2.1)-(2.2). In particular, the first eigenvalue is $k_1 = 1$ (hence $\alpha_1 = \beta_1 = 1/(m+1)$), and the corresponding eigenfunction has the explicit form

$$U_1(y) = \left(c - \tfrac{m-1}{2m}y^2\right)_+^{\frac{1}{m-1}} \tag{2.9}$$

with $c > 0$ arbitrary. Its initial data is a Dirac mass, $u(x,0) = M\delta(x)$ where M (the mass) is a function of c. It is usually called the *source-type* solution or *Barenblatt* solution, [Ba],

[ZK]. Any solution of (2.1)-(2.2) with $u_0 \in L^1(\mathbf{R}), u_0 \geq 0$ behaves for large times much as the function u_1 having the same mass as u_0: $\int u_0(x)\,dx = \int u_1(x,t)\,dx$. This determines the c in (2.9), see [K1]. The result has been recently extended to all $u_0 \in L^1(\mathbf{R})$ without sign restriction if the mass is not 0, [KV1].

The second eigenfunction corresponds to $k_2 = 2$, $\alpha_2 = 1/m$, $\beta_2 = 1/(2m)$. It takes initial data of *dipole* form

$$u_2(x,t) \to C\delta'(x) \quad \text{as } t \to 0 \tag{2.10}$$

with $C \neq 0$. This means that formula (2.7) holds for $m > 1$ if $n = 1, 2$. The profile U_2, antisymmetric and compactly supported, is also given in explicit form by

$$U_2(y) = y^{1/m}\left(c - \tfrac{m-1}{m+1} y^{\frac{m+1}{m}}\right)_+^{\frac{1}{m-1}} \tag{2.11}$$

for $y > 0$. c is an arbitrary constant related to the initial constant C. The functions u_2 describe the asymptotic behaviour of solutions of (2.1)-(2.2) such that

$$\int u_0(x)\,dx = 0 \quad \text{and} \quad \int x u_0(x)\,dx \neq 0\,, \tag{2.12}$$

see [KV1]. An equivalent and interesting form of interpreting this result consists in integrating the problem in x which gives for the new variable

$$w(x,t) = \int_{-\infty}^{x} u(s,t)\,ds \tag{2.13}$$

the problem

$$w_t = (|w_x|^{m-1} w_x)_x \quad \text{in } Q \tag{2.14}$$

$$w(x,0) = w_0(x) \equiv \int_{-\infty}^{x} u_0(s)\,ds \qquad \text{for } x \in \mathbf{R}\,. \tag{2.15}$$

Then w_2, defined through (2.13) from u_2, is a source-type solution of (2.14) and gives the asymptotic behaviour for solutions of (2.14)-(2.15) if $w_0 \in L^1(\mathbf{R})$ and the mass $\int w_0(x)\,dx$ does not vanish, [KV2].

The purpose of this paper is to investigate the properties of the third eigenvalue k_3 and its eigenfunctions U_3 and to relate them to the study of the behaviour of the solutions of the problem

$$z_t = |z_{xx}|^{m-1} z_{xx} \tag{2.16}$$

$$z(x,0) = z_0(x)\,, \tag{2.17}$$

which can be obtained from (2.14)-(2.15) by still another integration in space

$$z(x,t)=\int_{-\infty}^{x} w(s,t)\,ds\,. \tag{2.18}$$

Contrary to equations (2.1) and (2.14) which have been intensely studied in recent years, cf. [A], [KV1], their 'cousin' equation (2.16) has been somewhat disregarded in the literature. It is not an equation in divergence form for $m\neq 1$. It is known that (2.16) generates a contraction semigroup in a subspace of $L^\infty(\mathbf{R})$, cf. [BH], [Ha], [Ko]. However, no L^p-norm, $1\le p\le\infty$, is conserved by the semigroup as we shall show below. The property of finite speed of propagation holds: initial data with compact support give rise to solutions which for every fixed $t>0$ enjoy the same property.

We began our investigation by considering the problem of determining the value of k_3. The first conjecture that came to our minds was that perhaps $k_3=3$, in which case formula (2.7) would also hold for $n=3$ like in the linear case. This turned out to be false.

Theorem 2.1. *For $m>1$ we have $k_3>3$.*

In fact, we also prove that $k_3\to 4$ as $m\to\infty$ (Section 4) and k_3 is also a continuous function of $m\in[1,\infty)$ (Section 5). We have numerical evidence that this dependence is monotone increasing so that

$$3\le k_3<4 \quad \text{for} \quad 1\le m<\infty\,. \tag{2.19}$$

In order to see the importance of Theorem 2.1 for equation (E) we recall that an eigenfunction $U_3(y)$ gives rise to a solution $u_3(x,t)$ given by (2.3)-(2.4), and twice integrating in x this solution gives a solution of (E) in self-similar form

$$z_3(x,t)=t^{-\gamma}F(xt^{-\delta}) \quad \text{with} \quad \gamma=(k_3-2)\beta_3\,,\ \delta=\beta_3\,, \tag{2.20}$$

so that $k_3>3$ implies that $\gamma>\delta$. We shall also show that z_3 is compactly supported in $\mathbf{R}$ for every $t>0$: its positivity set takes the form

$$\{(x,t)\in Q: u(x,t)>0\}=\{(x,t)\in Q: |x|<Ct^{\beta_3}\}\,, \tag{2.21}$$

where $C>0$ can be chosen arbitrarily (by rescaling). Besides, after a possible change of sign z_3 can be taken to be nonnegative. Finally, the L^p-norm, $1\le p\le\infty$, decays in time like

$$\|z_3(\cdot,t)\|_p=C_p t^{-\mu} \quad \text{with} \quad \mu=(k_3-2-\tfrac{1}{p})\beta_3>0\,. \tag{2.22}$$

Therefore, z_3 does not behave as a source-type solution as $t\to 0$, but exhibits a very singular behaviour in the terminology now usual in parabolic equations of diffusion-absorption type where this kind of singularity also occurs, see for instance [BPT], [KPV1].

Our selfsimilar solutions z_3 can be used to describe the asymptotic behaviour of general solutions of (2.16)-(2.17), much as the fundamental solutions of the heat equation are used. The following result is proved in Section 6.

Theorem 2.2. *Every solution of* (2.16)-(2.17) *with* $z_0 \in C(\mathbf{R})$, $z_0 \geq 0$ *and compactly supported, decays in time like* $t^{-\gamma}$ *with* $\gamma = (k_3 - 2)\beta_3 > 0$, *namely there exist constants* C_1 *and* C_2 *such that for every* $t >> 0$

$$C_1 t^{\gamma} \leq \sup z(x,t) \leq C_2 t^{\gamma}\,. \tag{2.23}$$

The support grows like t^{β_3}: *there exist* C_3, C_4 *such that for* $t >> 0$ *the support of* $z(\cdot, t)$ *contains the ball of center 0 and radius* $C_3 t^{\beta_3}$ *and is contained in the ball with radius* $C_4 t^{\beta_3}$. *The upper bounds are still true without the assumption* $z_0 \geq 0$.

The above asymptotic information can be stated in terms of classes of solutions of (PLE) or (PME), by differentiating once and twice respectively. Thus, in the first case we obtain the asymptotic behaviour of solutions w of equation (2.14)-(2.15) such that $w_0 \in L^1(\mathbf{R})$ has compact support and

$$\int_{\mathbf{R}} w_0 = 0\,, \tag{2.24}$$

$$\int_{-\infty}^{x} w_0 \geq 0 \quad \text{for every } x \in \mathbf{R}\,, \tag{2.25}$$

while $u_3(x,t)$ describes the large-time behaviour of solutions of (2.1)-(2.2) with $u_0 \in L^1(\mathbf{R})$, u_0 with compact support and

$$\int_{\mathbf{R}} u_0(r)\,dr = 0\,, \quad \int_{\mathbf{R}} \int_{-\infty}^{s} u_0(r)\,dr\,ds = 0 \quad \text{and} \tag{2.26}$$

$$\int_{-\infty}^{x} \int_{-\infty}^{s} u_0(r)\,dr\,ds \geq 0\,, \quad \text{for every } x \in \mathbf{R}\,. \tag{2.27}$$

We point out that (2.26) can be reformulated as the vanishing of the first two moments

$$\int_{\mathbf{R}} u_0(r)\,dr = \int_{\mathbf{R}} r\,u_0(r)\,dr = 0\,. \tag{2.28}$$

In this way we see how this result complements those obtained in [KV1] for the case when these moments do not vanish simultaneously. Again, the upper bounds hold without the sign assumptions (2.25), (2.27).

3. Proof of Theorem 2.1

In this section we assume that $m > 1$ and consider the eigenvalue k_3 and the eigenfunction U_3, which is an even function in $\mathbf{R}$, [Hu].We normalize U_3 so that $U_3(0) = 1$. Thus, $U_3 = U_3(x)$ is the solution to the following initial-value problem

$$-\left(|U|^{m-1}U\right)'' = xU' + kU \tag{3.1}$$

$$U(0) = 1, \quad U'(0) = 0\,, \tag{3.2}$$

with $k = k_3$.We recall that although U is not C^1 and $|U|^{m-1}U$ is not C^2, equation (3.1) makes sense by requiring that U and $(|U|^{m-1}U)'$ be absolutely continuous. Given any $k > 0$ and $m > 1$ it i s known that (3.1)-(3.2) has a unique solution defined on the whole line.

We summarize in the following Lemma some properties of U_3 proven in [Hu].

Lemma 3.1. (i) $k_3 > 2$.

(ii) *The support of* U_3 *is a bounded interval of the form* $[-A, A]$ *with* $A > 0$.

(iii) U_3 *has a unique positive zero,* $x = a$, *inside its support* $(0 < a < A)$.

(iv) $(|U|^{m-1}U)' < 0$ *at* $x = a$.

We introduce the functions

$$V_3 = |U_3|^{m-1}U_3 \tag{3.3}$$

$$Z_3(x) = \int_x^A (t-x)U_3(t)\,dt \tag{3.4}$$

$$W_3(x) = Z_3'(x) = -\int_x^A U_3(t)\,dt \tag{3.5}$$

Thus, $Z_3'' = U_3$. Integrating (3.1) twice from x to A we obtain

$$-V_3' = xU_3 + (k_3-1)W_3 = xZ_3'' + (k_3-1){Z'}_3 \quad \text{and} \tag{3.6}$$

$$-V_3 = x{Z'}_3 + (k_3-2)Z_3\,. \tag{3.7}$$

Lemma 3.2. (i) $W_3(0) = 0$. (ii) $W_3 > 0$ *on* $(0, A)$.

(iii) $Z_3 < 0$ *on* $[0, A)$. (iv) $W_3(x) = Z_3(x) = 0$ *for all* $x \geq A$.

Proof. (i) is a consequence of (3.2) and (3.6), (ii) is implied by (i) and Lemma 3.1, (iii) follows from (ii) since ${Z'}_3 = W_3$ and finally (iv) is clear from (3.4), (3.5) and Lemma 3.1. #

We remark that Z_3 can be also defined by the conditions $Z_3'' = U_3$, ${Z'}_3(0) = 0$ and $Z_3(0) = -1/(k_3-2)$. It is easily seen that both definitions are equivalent.

Lemma 3.3. *We have*

$$-\int_0^A V_3(x)\,dx = (k_3-3)\int_0^A Z_3(x)\,dx\,. \tag{3.8}$$

The proof consists in integrating (3.7). #

The next result is the crucial step in the proof of Theorem 2.1.

Lemma 3.4. *If* $m > 1$ *then*

$$\int_0^A V_3(x)\,dx > 0\,. \tag{3.9}$$

Proof. For simplicity we drop the subindex 3. Observe that by (3.5) W is increasing in $(0, a)$ and decreasing in (a, A). Taking this into account we can express V as a function of W, $V = V_1(W)$ in the interval $(0, a)$, and as another function of W, $V = V_2(W)$

in the interval (a, A). Both functions are defined for $0 < W < W(a) = \sigma$. Moreover, $V_1(\sigma) = V_2(\sigma) = V_2(0) = 0$ while $V_1(0) = 1$. Since

$$\frac{dW}{dx} = U = |V|^{\frac{1}{m}} \operatorname{sign}(V)\,, \tag{3.10}$$

we obtain

$$\begin{aligned} &\int_0^A V(x)\,dx = \int_0^a V(x)\,dx + \int_a^A V(x)\,dx = \\ &\int_0^\sigma |V_1(W)|^{1-\frac{1}{m}}\,dW - \int_0^\sigma |V_2(W)|^{1-\frac{1}{m}}\,dW\,. \end{aligned} \tag{3.11}$$

Notice that $V_1 > 0$ and $V_2 < 0$ in $(0, \sigma)$.

We **claim** that

$$V_1(W) > |V_2(W)| = -V_2(W) \qquad \text{for all} \quad W \in (0, \sigma)\,. \tag{3.12}$$

From this claim and formula (3.11) our lemma follows since $m > 1$.

Let us proof the claim and thus establish Lemma 3.4. Suppose that (3.12) is false. Then $V_1 + V_2$ has a negative or zero minimum at a point W_* inside $(0, \sigma)$. At this point $V'_1(W_*) + V'_2(W_*) = 0$. On the other hand, it follows from (3.6) and (3.10) that at W_*

$$\begin{aligned} &-dV_1/dW = x_1 + (k-1)W_* V_1^{-\frac{1}{m}} \\ &-dV_2/dW = x_2 + (k-1)W_* |V_2|^{-\frac{1}{m}} \operatorname{sign}(V_2) = x_2 - (k-1)W_* |V_2|^{-\frac{1}{m}}\,, \end{aligned}$$

where x_1 and x_2 are determined as the two points where the function $W(x)$ takes on the value W_*, $0 < x_1 < x_2 < A$. From the above we deduce that at $W = W_*$

$$-d(V_1 + V_2)/dW = x_1 + x_2 + (k-1)W_*(V_1^{-\frac{1}{m}} - |V_2|^{-\frac{1}{m}})\,.$$

Now, this quantity is positive since x_1, x_2 and W_* are so, and $V_1 \le |V_2|$ at the minimum. This contradiction proves our claim (3.12). #

The preceding Lemmas imply Theorem 2.1. In fact, we know from Lemma 3.2 that $Z_3 < 0$ on $[0, A)$. Use now formula (3.8) and Lemma 3.4 to conclude that $k_3 - 3 > 0$. #

We end the section with another property of the eigenfunction U_3 which will be used later on.

Lemma 3.5. *V'_3 has a unique zero, b, in the interval $(0, A)$ and $b > a$.*

Proof. Recalling the notation (3.3) and Lemma 3.1, it is clear that V'_3 has at least a zero in $(0, A)$. From equation (3.1) it follows that $V_3'' > 0$ at the zeros of V'_3 in (a, A). Thus, V_3 has no local maximum in (a, A) so that V'_3 has only one zero in (a, A). A similar argument proves that V'_3 has no zeros in $(0, a)$. #

4. The limit $m \to \infty$

In this section we establish the asymptotic behaviour of k_3 as $m \to \infty$.

Theorem 4.1. *Let $k_3 = k_3(m)$ be the third eigenvalue of* (2.6). *Then $k_3(m) \to 4$ as $m \to \infty$.*

Before we concern ourselves with the limit behaviour of k_3 it is convenient to pass to the limit in the problem. Once we have identified the correct limit problem we shall show that $k = 4$ is the only acceptable solution for it. Then we will be able to prove that k_3 converges to this limit value as $m \to \infty$.

4.1 The limit problem. We recall that integrating twice equation (2.6) we obtain for $k \neq 2$

$$-V = xZ' + (k-2)Z\,, \qquad V = |U|^{m-1}U\,, \tag{4.1}$$

where Z is related to U by

$$Z'' = U\,, \qquad Z'(0) = 0\,, \qquad Z(0) = -\frac{1}{k-2}\,. \tag{4.2}$$

Problem (4.1)-(4.2) can be viewed as an ODE for the variable Z with initial data at $x = 0$. Since the problem is even we may consider only the positive half-line $x \geq 0$.

Next we take a formal limit $m \to \infty$ in the above problem and arrive at the following problem $(\mathbf{P}_\infty)$: To find a function $Z \in C^1(\mathbf{R}_+)$ such that its second derivative is a function of bounded variation, $Z'' \in BV(\mathbf{R}_+)$, and a continuous function V in $\mathbf{R}_+$ such that

$$-\,V = xZ' + (k-2)Z \tag{4.3}$$

$$Z''(x) \in \operatorname{sign}(V(x)) \quad \text{for a.e. } x > 0 \tag{4.4}$$

$$Z(0) = -\frac{1}{k-2}\,, \quad Z'(0) = 0 \tag{4.5}$$

$$\text{the quantity } k|V| + \tfrac{1}{2}V'^2 \text{ is nonincreasing a.e. in } \mathbf{R}_+ \tag{4.6}$$

The space BV and condition (4.6) are introduced because they play a role in ensuring the uniqueness of solutions to this problem. Following previous notations we put $U = Z''$. Derivatives are understood in the weak sense when necessary. Note that, as in Section 3, we still have for the solutions of (P_∞):

$$-\,V' = xU + (k-1)Z' \tag{4.7}$$

$$-\,V'' = xU' + kU \tag{4.8}$$

$$V(0) = 1\,, \quad V'(0) = 0\,. \tag{4.9}$$

4.2. Solutions of the limit problem. Our first result is

Theorem 4.2. *Given any $k > 2$ there exists a unique solution to problem* (P_∞).

Observe that in an interval where $V \neq 0$ we have $U = \operatorname{sign}(V) \in \{-1, 1\}$, hence $U' = 0$ a.e. and from (4.8) it follows that $V'' = -k \operatorname{sign}(V)$. Therefore, both Z and V are given by second-degree polynomials, i.e. they are parabolic arcs. Existence and uniqueness questions are thus reduced to the points where $V = 0$. Theorem 4.2 is then a consequence of the next result.

Lemma 4.3. *Let Z be a solution of Problem (P_∞) defined in an interval $[0, c)$ with $c > 0$. Assume that $V(c) = 0$ and $V > 0$ in a left neighbourhood of c, $(c - \delta, c)$. Then V and Z exist and are uniquely determined in a right neighbourhood of c. Furthermore,*

(i) $V'(c-) < 0$ *and* $V'(c+) \leq 0$

(ii) If $|V'(c-)| \leq 2c$ *then* $V'(c+) = 0$ *and* $V(x) = 0$ *for all* $x \geq c$

(iii) If $|V'(c-)| > 2c$ *then*

(4.10)
$$V'(c+) = 2c + V'(c-) < 0$$
and $V < 0$ *in a right neighbourhood of* c.

Proof. (i) It is clear that on $(c-\delta, c)$ V is given by a parabolic arch and $V'(c-) < 0$. From (4.7) it follows that $V' \in BV_{loc}(\mathbf{R}_+)$ and hence the one-sided limit $V'(c+)$ exists. Finally, if $V'(c+) > 0$ then V would be positive to the right of c, hence $U = 1$, and (4.7) would then imply that $V'(c+) = V'(c-)$, a contradiction.

(ii) If $V'(c+) < 0$ then $V < 0$ in a right neighbourhood of c and from (4.7) $V'(c+) = 2c + V'(c-) > 0$. This contradiction proves that $V'(c+) = 0$. Then condition (4.6) implies that $V(x) = 0$ for all $x \geq c$ and Z has a unique C^1 extension determined by $xZ' + (k-2)Z = 0$ for all $x \geq c$.

(iii) If $V'(c+) = 0$ we deduce from (4.7) and $U(c-) = 1$ that

$$U(c+) = -(k-1)\frac{Z'(c)}{c} = \frac{c + V'(c-)}{c} < \frac{-c}{c} = -1\,,$$

which contradicts $|U| \leq 1$ (recall (4.4)). Thus, $V'(c+) < 0$. In this case Z and V are parabolic arcs to the right of c determined by the equations $Z'' = U = -1$ and $V'' = k$. #

A similar result is obtained when the assumption $V > 0$ on $(c - \delta, c)$ is replaced by $V < 0$ by simply applying the Lemma to the function $-V$. Since the occurrence of (iii) in the above situations implies a net decrease of $2c$ for $|V'|$ accross a zero c, it is clear that after a finite number of isolated zeroes the function V settles down to the situation (ii), i.e. V has compact support. We will say that k is an eigenfunction of (P_∞) when Z is also a function with compact support. Since near infinity Z is given by the equation

$$xZ' + (k-2)Z = 0 \tag{4.11}$$

this is equivalent to asking that $U = Z''$ or $W = Z'$ have compact support.

The result we need about problem (P_∞) is the following

Theorem 4.4. *Let $k > 2$. Problem (P_∞) admits a nonpositive solution Z with compact support if and only if $k = 4$. The solution is an even function supported in $[-\sqrt{2}, \sqrt{2}]$.*

Proof. Step 1. Clearly the solutions of (P_∞) are even functions, hence we restrict our calculations to $x \ge 0$. It is easy to see that near 0 and as long as $V > 0$ we have $U = 1$, hence V is given by the arc $V(x) = 1 - (k/2)x^2$, so that V has a first zero at

$$a = \sqrt{2/k}\,.$$

Moreover, since $V'(a-) = -ka$ and $-ka + 2a < 0$, from Lemma 4.3-(iii) it follows that $V'(a+) = -(k-2)a$ and V has at least a second parabolic arch, stretching from $x = a$ to, say, $x = A$. Then $V(A) = 0$ and $V < 0$ on (a, A). Using the symmetry of this arch we conclude that

$$V'(A-) = -V'(a+) = (k-2)a\,. \tag{4.12}$$

In fact, since by (4.8) $V'' = k$ in (a, A) we have in this interval

$$V'(x) = kx - 2(k-1)a\,. \tag{4.13}$$

(4.12) and (4.13) imply that $V'(A-) = (k-2)a = kA - 2(k-1)a$, hence

$$A = \frac{3k-4}{k}a\,. \tag{4.14}$$

Step 2. We turn now to the determination of Z. Since $Z'' = 1$ for $0 < x < a$ and $Z'' = -1$ for $a < x < A$ and $Z(0) = -1/(k-2)$ we arrive at the values

$$Z'(A) = -a\frac{k-4}{k} \qquad \text{and} \qquad Z(A) = aA\frac{k-4}{k(k-2)}\,. \tag{4.15}$$

Thus, for $k > 4$ we arrive at a contradiction since $Z(A)$ becomes positive. Let us also exclude the case $2 < k < 4$: this follows from the observation that

$$V'(A-) - 2A = a\frac{k^2 - 8k + 8}{k} \tag{4.16}$$

which is negative for those values of k, hence by Lemma 4.3-(ii) we have $V(x) \equiv 0$ on (A, ∞). Now, since by (4.15) the value of $Z(A)$ is not zero (more precisely negative) and Z satisfies equation (4.11) whenever V vanishes, we then would have $Z < 0$ for all $x \ge A$.

Step 3. Finally, if $k = 4$ it is easy to see that $a = 1/\sqrt{2}$, $A = 2a$, $Z(A) = 0$ and both V and Z are prolonged to the right of A as the null function. It is clearly seen that $Z < 0$ on $(0, A)$ and V has exactly one zero in this interval. #

4.3. Convergence as $m \to \infty$. We may now proceed with the proof of Theorem 4.1. We begin with some estimates for the solutions of problem (4.1)-(4.2).

Lemma 4.5. *Let Z be the solution of Problem* (4.1)-(4.2). *Then the following functions are nonincreasing in* $\mathbf{R}+$*:*

$$\begin{aligned}
\Phi_1 &\equiv \frac{km}{m+1}|V|^{\frac{m+1}{m}} + \frac{1}{2}V'^2 \\
\Phi_2 &\equiv \frac{m}{m+1}|V|^{\frac{m+1}{m}} + \frac{k-1}{2}Z'^2 \\
\Phi_3 &\equiv \frac{1}{2}Z'^2 + (k-2)^{\frac{1}{m}}\frac{m}{m+1}|Z|^{\frac{m+1}{m}} .
\end{aligned}$$

In the last case $k > 2$ is assumed.

Proof. These estimates can be checked by computing the derivatives and using equations (4.8), (4.7) and (4.1) respectively. We give a detailed derivation of Φ_3. Set $g(s) = |s|^{1/m}\operatorname{sign} s$. From (4.1) it follows that $-Z'' = g(xZ' + (k-2)Z)$. Then

$$\begin{aligned}
&\Phi_3' = Z'(Z'' + (k-2)^{1/m}|Z|^{1/m}\operatorname{sign} Z) = \\
&- Z'\left[g(xZ' + (k-2)Z) - g((k-2)Z)\right] \le 0 ,
\end{aligned}$$

because g is increasing and x is nonnegative. #

Lemma 4.6. *Let $k > 2$ and let Z be the solution of Problem* (4.1)-(4.2). *Then the following inequalities hold in* $\mathbf{R}_+$*:*

$$\begin{aligned}
&\text{(a) } |V| \le 1 , \qquad \text{(b) } |Z''| \equiv |U| \le 1 . \qquad \text{(c) } |Z| \le \frac{1}{k-2} \\
&\text{(d) } V'^2 \le \frac{2km}{m+1} , \qquad \text{(e) } Z'^2 \le \frac{2m}{(m+1)(k-1)}
\end{aligned}$$

Proof. Since $|V| = |U|^m$, (a) and (b) are equivalent. (a) for instance follows from $V(0) = 1$, $V'(0) = 0$ and the monotonicity of Φ_1, from which (d) also follows. A similar argument with Φ_3 proves (c). For (e) we use the monotonicity of Φ_2. #

Lemma 4.7. *$k_3(m)$ remains bounded as $m \to \infty$.*

Proof. Assume for contradiction that this is not the case and let m_j be a sequence, $m_j \to \infty$, such that $k_j = k_3(m_j)$ goes to infinity. Let Z_j be the corresponding solution of Problem (4.1)-(4.2). We introduce a convenient rescaling

$$Z_j(x) = \frac{1}{k_j}\hat{Z}_j(\sqrt{k_j}x) , \qquad V_j(x) = \hat{V}_j(\sqrt{k_j}x) \; , , \tag{4.17}$$

which satisfies the ODE

$$-\hat{V}_j(t) = \frac{t}{k_j}\hat{Z}'{}_j(t) + \left(1 - \frac{2}{k_j}\right)\hat{Z}_j(t) \tag{4.18}$$

and the estimates

$$|\hat{Z}_j| \leq \frac{k_j}{k_j - 2}, \quad \hat{Z'}_j^2 \leq \frac{2m_j k_j}{(m_j+1)(k_j-1)}, \quad |\hat{Z'}_j'| \leq 1\,.$$

From the above relations it follows that (a subsequence of) $\hat{Z}_j$ tends in $C^1_{loc}(\mathbf{R}_+)$ to a certain function Z and $Z(0) = 1$, $Z'(0) = 0$ and $Z \leq 0$ since $Z_j \leq 0$ for all j (see Lemma 3.2). Passing to the limit in equations (4.18) and in $\hat{Z}_j'' = \hat{U}_j$ we deduce that Z must satisfy *a.e.* in $\mathbf{R}_+$ the relation $-Z''(x) \in \mathrm{sign}\,(Z(x))$. But this implies that Z has infinitely many sign changes, a contradiction. #

Therefore, for every sequence $m_j \to \infty$ there must be a subsequence m_{j_n} such that along it k_j converges to a finite limit $\bar{k}$. By Lemma 4.6 there exists a further subsequence along which we have with the preceding notations the following convergences

$$Z_j \to \bar{Z}, \quad Z'_j \to \bar{Z}', \quad V_j \to \bar{V} \quad \text{uniformly on bounded sets}\,. \tag{4.19}$$

We have

Lemma 4.8. *Under the above assumptions $\bar{Z}$ is a nonpositive solution of Problem* (P_∞)
.

Proof. It follows from (4.19), $U_j = |V_j|^{1/m_j}\,\mathrm{sign}\,V_j$ and the estimate $|U_j| \leq 1$ that $\bar{Z}$ and $\bar{V}$ satisfy (4.3)-(4.5).

Since U_j has two nontrivial extrema (by Lemma 3.5), the uniform bound $|U_j| \leq 1$ implies that

$$U_j' \text{ is bounded in } L^1(\mathbf{R}_+)\,.$$

Thus, $\bar{U}'$ is a measure and $\bar{U} \in BV(\mathbf{R}_+)$. By compact embedding (and extraction of a further subsequence that we will still denote by U_j) we obtain that

$$U_j \to \bar{U} \text{ strongly in } L^p_{loc}(\mathbf{R}_+),\ 1 \leq p < \infty \ \text{ and}$$
$$U_j \to \bar{U} \quad a.e. \text{ in } \mathbf{R}_+\,.$$

This,(4.7) and (4.19) imply that $\bar{V}$ also satisfies condition (4.6), which completes the proof of the Lemma. #

Lemma 4.9. *Both $\bar{V}$ and $\bar{Z}$ have compact support.*

Proof. Using the estimates of Lemma 4.6 on equation (4.7) we see that $|xU_j(x)|$ is uniformly bounded independently of j and x. This means that there exists a constant C such that

$$|V_j(x)| \leq \left(\frac{C}{x}\right)^{m_j}\,. \tag{4.20}$$

Letting $m_j \to \infty$ we obtain $\bar{V}(x) = 0$ for all $x > C$.

To show that $\bar{Z}$ is compactly supported we write equation (4.1) in the form

$$-x^{k-3}V = (x^{k-2}Z)'\,.$$

Since Z_j has bounded support we have

$$x^{k_j-2}Z_j(x) = \int_x^\infty t^{k_j-3}V_j(t)\,dt \quad \text{for all } x \geq 0\,.$$

By (4.19), (4.20) and $|V_j| \leq 1$ we may pass to the limit in this integral to obtain

$$x^{\bar{k}-2}\bar{Z}(x) = \int_x^\infty t^{\bar{k}-3}\bar{V}(t)\,dt$$

for all $x \geq 0$. Since $\bar{V}$ has compact support the conclusion follows. #

The *conclusion of Theorem 4.1* is now obvious. By Lemmas 4.8 and 4.9 $\bar{Z}$ must be a nonpositive solution of (P_∞) with compact support. By Theorem 4.4 this implies that $\bar{k} = 4$. Since the limit is unique the whole sequence m_j must converge.

Remark. Similar arguments allow to prove that

$$\lim_{m\to\infty} k_4(m) = 4 + 2\sqrt{2} \approx 6.8284$$
$$\lim_{m\to\infty} k_5(m) = 6 + 2\sqrt{5} \approx 10.4721\,.$$

Recall that k_4 corresponds to an odd eigenfunction while k_5 corresponds to an even one.

5. Continuity

In the last section we have proved that $k_3(m)$ is continuous at $m = \infty$ if we put $k_3(\infty) = 4$. Here we prove the continuity for finite values of m.

Theorem 5.1. *The function $k_3(m)$ is continuous for $1 \leq m < \infty$.*

The proof will follow the line of argument used for the case $m \to \infty$ in Section 4. The only essential novelty is the obtention of suitable estimates which allow to prove that the limit function has a good decay rate as $x \to \infty$, so it can be identified with the corresponding eigenfunction. Therefore, we begin this section with the derivation of some powerful uniform bounds, which will be also useful in the study of the case $0 < m < 1$ in Section 7.

Let us remark that a quite different proof can be done in terms of the phase-plane analysis as performed in [Hu], by studying the continuity of certain connecting orbits with respect to the parameters m and k. This phase-plane analysis will be used below in Section 7 to establish the properties of the eigenvalues and eigenfunctions for $0 < m < 1$.

To prepare for the proof, let us introduce a parameter λ and the conditions

$$m > 0\,, \quad k > 0\,, \quad 2 + k(m-1) > 0 \tag{5.1}$$

$$\lambda \geq k\,, \qquad 2 + \lambda(m-1) \geq 0\,. \tag{5.2}$$

Notice that (5.1) and (5.2) together imply

$$2 + \lambda m - k > 0\,. \tag{5.3}$$

Lemma 5.2. *Assume that* (5.1) *and* (5.2) *hold. Let* $0 \leq \bar{a} < a < \infty$ *and let* U *be a solution of* (3.1) *such that in* $(\bar{a}, a)$ U *is bounded and* $U \neq 0$ *and*

$$\bar{a}^k U(\bar{a}) = U(a) = 0\,.$$

Then

$$M \equiv \sup_{\bar{a}<x<a} x^\lambda |U(x)| \leq F(m,k,\lambda) |U|_\infty^{\frac{2+\lambda(m-1)}{2}}\,, \tag{5.4}$$

where $|U|_\infty$ *is the supremum of* $|U|$ *in* $(\bar{a}, a)$ *and* F *is a continuous function defined in the region of the* (m,k,λ)*-space determined by conditions* (5.1)-(5.2)*. Furthermore,* F *is independent of* U*,* $\bar{a}$ *and* a*.*

We remark that when $m \geq 1$ then λ and k can be chosen arbitrarily large. On the contrary, if $0 < m < 1$, then (5.1) and (5.2) imply that $k < 2/(1-m)$ and $\lambda \leq 2/(1-m)$. On the other hand, observe that for $\lambda = k$ we have

Corollary 5.3. *Assume that* (5.1) *holds and let* $\bar{a}$*,* a *and* U *be as in Lemma 5.2. Then*

$$\sup_{\bar{a}<x<a} x^k |U(x)| \leq G_1(m,k) |U|_\infty^{\frac{2+k(m-1)}{2}}\,, \tag{5.5}$$

where G_1 *is continuous in the region determined by* (5.1) *and is independent of* U*,* $\bar{a}$ *and* a*.*

Proof of Lemma 5.2. We may assume that $U > 0$ in $(\bar{a}, a)$, the case $U < 0$ being analogous. There exists $\rho \in (\bar{a}, a)$ such that $M = \rho^\lambda U(\rho)$ and

$$(x^\lambda U(x))'_{|x=\rho} = \rho^{\lambda-1}(\rho U'(\rho) + \lambda U(\rho)) = 0$$

Recalling that $V = U^m$ we get

$$\rho |V'(\rho)| = \lambda m V(\rho) \tag{5.6}$$

$$V(x) \leq M^m / x^{\lambda m} \quad \text{for all} \quad x \in (\bar{a}, a)\,. \tag{5.7}$$

Writing equation (3.1) in the form $-x^{k-1}V'' = (x^k U)'$, integrating by parts and taking into account that $V'(a) \leq 0$ we derive that

$$\rho^k U(\rho) = \int_\rho^a x^{k-1} V''(x)\, dx \leq$$
$$\leq - \rho^{k-1}V'(\rho) + (k-1)\rho^{k-2}V(\rho) + (k-1)(k-2)\int_\rho^a x^{k-3}V(x)\, dx\,.$$

From (5.6), (5.7) and (5.3) it follows that

$$\rho^k U(\rho) \leq F_1 M^m \rho^{-(2+\lambda m-k)}\,,$$

with

$$F_1 = \lambda m + (k-1)_+ + \frac{((k-1)(k-2))_+}{2+\lambda m - k}\,.$$

Hence

$$M = \rho^\lambda U(\rho) \leq F_1 M^m \rho^{-(2+\lambda(m-1))}\,.$$

Observing that $M \leq \rho^\lambda |U|_\infty$ and consequently $\rho^{-\lambda} \leq |U|_\infty / M$, from $2+\lambda(m-1) \geq 0$ it follows that the conclusion of the lemma holds with $F = F_1^{\lambda/2}$. #

Remark. Lemma 5.2 and Corollary 5.3 hold also when $a = \infty$ if we assume in addition that, as $x \to \infty$ we have $x^\lambda U(x) \to 0$ and $x^k U(x) \to 0$, respectively.

We may now proceed with the

Proof of Theorem 5.1. Step 1. As in Section 4 we let m_j be a sequence converging to a value $\bar{m} \in [1, \infty)$ and let $k_j = k_3(m_j)$ and U_j be the corresponding solution of (3.1)-(3.2). Let $k_j \to \bar{k} \in [3, \infty]$ (possibly after passing to a subsequence). Recalling Section 4 from Lemma 4.5 onwards, by similar - in fact simpler arguments - we deduce that for some subsequence, which we still call $\{j\}$, we have

$$U_j \to \bar{U}\,, \quad V_j \to \bar{V}\,, \quad \text{and} \quad {V'}_j \to \bar{V}'$$

uniformly on bounded subsets of $\mathbf{R}$ and $\bar{k} < \infty$. Hence, $\bar{V} = |\bar{U}|^{\bar{m}-1}\bar{U}$ and $\bar{U}$ is the solution of (3.1)-(3.2) with $k = \bar{k}$ and $\bar{W} \geq 0$ on $(0, \infty)$. Now $\bar{W}$ cannot have a local minimum at a point where $\bar{W} > 0$ (see equation (4.7)). Besides, $\bar{W}(x_0) = \bar{W}'(x_0) = 0$ imply $W(x) = 0$ for all $x \geq x_0$, because of the monotonicity of the function Φ_2 in Lemma 4.5. From these two facts it follows that $\bar{U} = \bar{W}'$ has a unique isolated zero in $(0, \infty)$.

Step 2. $\bar{U}$ is absolutely continuous because $\bar{U} = |\bar{V}|^{\frac{1}{m}-1}\bar{V}$ and $\bar{V}$ is C^1, the function $s \mapsto |s|^{1/\bar{m}} \operatorname{sign} s$ is absolutely continuous and $\bar{V}$ has a finite number of monotone arches (the last assertion is proved as in Lemma 3.5). By (3.1) $\bar{V}'$ is also absolutely continuous.

Step 3. It remains to prove that $\bar{U}$ has bounded support for $\bar{m} > 1$ and exponential decay for $\bar{m} = 1$. Since $m_j > 1$ we may Lemma apply 5.2 to U_j with λ arbitrary large. Hence

$$|U_j(x)| \leq C(\lambda)x^{-\lambda} \qquad \text{for all } x > 0\,,$$

where $C(\lambda)$ is independent of j. Recall also that $|U_j| \leq 1$ on $\mathbf{R}$. We conclude that $\bar{U}$ decays at infinity faster than any power of x^{-1}. Now, if $\bar{m} > 1$ it has been proved in [Hu] that solutions decaying faster than $|x|^{-k}$ as $|x| \to \infty$ must be compactly supported. Hence, $\bar{U}$ has compact support, and $\bar{k} = k_3(\bar{m})$ because of the uniqueness of the eigenvalues proved in [Hu].

In the case $\bar{m} = 1$ it follows from standard linear methods that the solutions of $-U'' = xU' + kU$ decaying faster than $|x|^{-k}$ as $|x| \to \infty$ actually have exponential decay. Thus, $\bar{k} = k_3(1) = 3$. #

6. Asymptotic behaviour for equation (E). Proof of Theorem 2.2

Let us begin by reviewing the existence and uniqueness theory for the Cauchy Problem for equation (E), i.e. (2.16)-(2.17). Since (E) can be obtained by twice integrating the PME (2.1), we may use the well-known theory for this latter equation to obtain existence results for our equation.

Thus, we recall [A] that for every $u_0 \in L^1(\mathbf{R})$ there exists a unique function $u = u(x,t) \in C(Q) \cap C([0,\infty) : L^1(\mathbf{R})) \cap L^\infty(\mathbf{R} \times (\tau,\infty))$, where $Q = \mathbf{R} \times \mathbf{R}_+$ and any $\tau > 0$, which satisfies (2.1) in the sense of distributions in Q and takes on the initial data $u(\cdot,0) = u_0$. Moreover, if u_0 has compact support, then for every $t > 0$ the function $u(\cdot,t)$ has compact support.

Integration of this result allows to construct a solution $z = z(x,t) \in C^{2,1}_{x,t}(Q)$ of equation (E) for every initial data

$$z_0 \in E = \{v \in C^1(\mathbf{R}) : v' \text{ and } v'' \in L^1(\mathbf{R})\}\,. \tag{6.1}$$

The uniqueness of such *classical* solution is an easy consequence of the Maximum Principle. Moreover, the same argument allows to show the following Contraction Principle.

Lemma 6.1. *Let z_i, $i = 1,2$ be classical solutions of (E) with initial data $z_{0,i}$ satisfying the above hypothesis. Then the quantity*

$$\|(z_1(\cdot,t) - z_2(\cdot,t))_+\|_\infty \tag{6.2}$$

is nonincreasing in time.

In fact, this kind of property allows for the construction of *generalized* solutions of (2.16)-(2.17) as uniform limits of classical ones for all continuous initial data tending to 0 at infinity, $z_0 \in C_0(\mathbf{R})$, which is a subset of the closure of $E \cap L^\infty(\mathbf{R})$ with respect to the uniform norm. In this way we obtain for every $z_0 \in C_0(\mathbf{R})$ a function $z(x,t) \in C(Q)$ with $z(x,0) = z_0(x)$ and the Contraction Principle is still true. In particular, this Principle implies the following Comparison result.

Lemma 6.2. *Let z_i, $i = 1, 2$ be two generalized solutions of* (2.16)-(2.17) *with respective initial data $z_{0,i}$ and assume that $z_{0,1} \geq z_{0,2}$ in* **R**. *Then $z_1 \geq z_2$ in Q.*

Taking $z_2 \equiv 0$ we conclude that if z is a solution such that $z(x, 0) \geq 0$ in **R** then $z(x, t) \geq 0$ in Q. On the other hand, taking as z_2 an arbitrary solution z and as z_1 the selfsimilar solution z_3 constructed in Section 2, cf. (2.20), rescaled if necessary by means of the formula

$$\hat{Z}(y) = \lambda^{-2m} Z(\lambda^{m-1} y),$$

cf. (2.8), so that $-z_3(x, 1) \leq z(x, 0) \leq z_3(x, 1)$, we obtain

$$-z_3(x, t+1) \leq z(x, t) \leq z_3(x, t+1) \quad \text{for every } x \in \mathbf{R} \text{ and every } t > 0\,. \tag{6.3}$$

In this way we conclude

Lemma 6.3. *For every solution $z(x, t)$ of* (2.16)-(2.17), *where $z_0 \in C(\mathbf{R})$ has compact support, and for every $t > 0$ we have:*

(i) The function $z(\cdot, t)$ has compact support and there exists $C > 0$ such that the support of $z(\cdot, t)$ is contained in the ball of radius Ct^{β_3} centered at 0.

(ii) There exists a constant $C_2 > 0$ such that

$$|z(x, t)| \leq C_2 t^{(k_3-2)\beta_3}\,. \tag{6.4}$$

C and C_2 depend only on z_0.

To complete the *proof of Theorem 2.2* we still have to derive lower estimates for nonnegative solutions. This is easily done by the same comparison technique. #

The precise statement and derivation of the corresponding estimates for equations (PLE) and (PME) under conditions (2.24)-(2.25) and (2.26)-(2.27) respectively should offer no difficulty and is left to the reader. #

Comment on Existence Theory. The preceding results can be viewed in the framework of a more abstract theory, though we have preferred the above presentation for its simplicity. Thus, Bénilan and Ha [BH] and Konishi [Ko] have shown that equation (E) generates a semigroup of contractions in $L^\infty(\mathbf{R})$. This result is based on the fact that for every bounded interval $I \subset \mathbf{R}$ the nonlinear operator $\mathcal{A}$ defined in the domain

$$D(\mathcal{A}) = \{v \in L^\infty(I) \cap H_0^1(I) \cap H^2(I) : v'' \in L^\infty(I)\} \tag{6.5}$$

by the formula

$$\mathcal{A}(v) = |v''|^{m-1} v'' \tag{6.6}$$

is m-dissipative in $L^\infty(I)$. Using this property the Crandall-Liggett Semigroup Generation allows to obtain for every $z_0 \in \overline{D(\mathcal{A})}$ a function $z \in C([0, \infty) : L^\infty(I))$ which solves equation (E) in a *mild* sense and the map

$$S_t : z_0 \mapsto z(t) = z(\cdot, t)$$

is a contraction in the L^∞-norm, see [C] or [E]. Mild solutions do not necessarily solve the equation in any usual (distribution) way. Nevertheless, classical solutions and uniform limits of classical solutions are mild solutions and in that case the solution is unique, hence our generalized solutions are included in this concept.

7. The case $0 < m < 1$. Eigenvalues

In the sequel we will concern ourselves with similar problems for equation (E) under the assumption that $0 < m < 1$. In this section we study the existence of eigenfunctions and eigenvalues to equation (2.6). For reasons of convenience in the phase-plane analysis to be performed below we will slightly change the equation by using a new space variable

$$y = \beta^{\frac{1}{2}}\eta\,, \tag{7.1}$$

and replace k by the exponents α and β given by (2.5), assuming always that $k(1-m) \le 2$. Then $k = \alpha/\beta$. By means of this we transform equation (2.6) into the form

$$V''(\eta) + \beta\eta U'(\eta) + \alpha U(\eta) = 0\,. \tag{7.2}$$

where $\alpha > 0$ is a free parameter and $2\beta = \alpha(1-m)+1$ (cf. (2.4)). This corresponds to writing the solutions u of (E) in the form $u(x,t) = t^{-\alpha}U(\eta)$ with $\eta = xt^{-\beta}$, and letting as usual $V = |U|^{m-1}U$. Note that though (7.2) looks more general than (2.6), they are in fact equivalent.

The novelty in posing the problem lies in the fact that for $m < 1$ there are no nontrivial solutions of (7.2) with compact support, so that the definition of eigenfunction has to be changed into "solution of (7.2) with rapid decay as $|\eta| \to \infty$". It will turn out from the analysis performed in this section that for $0 < \alpha < \infty$. i.e.

$$0 < k < \frac{2}{1-m} \tag{7.3}$$

every nontrivial solution of (7.2) defined for positive y has a discrete set of zeroes. Moreover, if this set is finite, the solution eventually decays to zero in a monotone way, the decay rate either being

$$|\eta|^{km}V(\eta) \to C_1, \tag{7.4}$$

or

$$|\eta|^{\frac{2m}{1-m}}V(\eta) \to C_2, \tag{7.5}$$

as $\eta \to \infty$ or $\eta \to -\infty$; C_i denotes a nonzero constant. Because km is smaller than $2m/(1-m)$ in the range (7.3), the decay rate $V(\eta) = O(\eta^{2m/1-m})$ is optimal for such solutions. Therefore we will define as eigenfunction for (7.2) a global solution of the equation in $\mathbf{R}$ such that it has a finite number of zeroes and the decay rate (7.5) as $\eta \to \pm\infty$. It will be also proved that global solutions of (7.2) having fast decay as $y \to \pm\infty$ are necessarily symmetric or antisymmetric.

As in the case $m > 1$ we find two eigenvalues $k_1 = 1$ and $k_2 = 2$ and corresponding eigenfunctions given still by formulas (2.9) and (2.11). Observe that U_1 is positive and even while U_2 is positive in $\mathbf{R}_+$ and odd, and both have correct decay rates. Again as in Sections 2, 3 we look for a third eigenfunction in even form (i.e. with $V'(0) = 0$) which will have two zeros in $\mathbf{R}$.

Theorem 7.1. *Let $0 < m < 1$. Then there exists a unique number $k_3 \in (2, \frac{2}{1-m})$, such that equation (7.2) has eigenfunctions with exactly two zeroes. Moreover, such solutions are necessarily symmetric and hence can be obtained from a single one by rescaling and/or change of sign.*

Proof. Step 1. The phase plane. For $0 < m < 1$ local existence and uniqueness of a solution of (7.2), given any real α, β and values of V and V' at some finite y-value, follows immediately from the classical theory of ordinary differential equations. Moreover, solutions cannot blow up as $|\eta|$ increases thanks to the monotonicity of the function Φ_1 in Lemma 4.5. Hence, all solutions defined in a neighbourhood of $\eta = 0$ exist globally.

Since (7.2) is scale-invariant there is a standard change of variables to transform the equation into an autonomous equation, cf. [BO]. Assuming $\eta > 0$ we set

$$t = \log \eta \qquad \text{and} \qquad f(t) = \eta^{\frac{2m}{1-m}} V(\eta) = e^{\frac{2m}{1-m}t} V(e^t), \tag{7.6}$$

which transforms (7.2) into the autonomous equation

$$\left(\frac{d}{dt} - \frac{1+m}{1-m}\right)\left(\frac{d}{dt} - \frac{2m}{1-m}\right) f + \left(\beta \frac{d}{dt} - \frac{2\beta}{1-m} + \alpha\right)\left(|f|^{\frac{1}{m}-1} f\right) = 0. \tag{7.7}$$

Next we introduce the new variable

$$g = \left(\frac{d}{dt} - \frac{2m}{1-m}\right) f = e^{\frac{1+m}{1-m}t} V'(e^t). \tag{7.8}$$

This allows to write the system

$$(S) \begin{cases} g' = & \frac{1+m}{1-m} g - \alpha |f|^{\frac{1}{m}-1} f - \frac{\beta}{m} |f|^{\frac{1}{m}-1} g \\ f' = & \frac{2m}{1-m} f + g\,. \end{cases}$$

Note that if $V(0)$ and $V'(0)$ are well defined then $(g(t), f(t)) \to (0,0)$ (which is a critical point for (S)) as $t \downarrow -\infty$. Thus the only orbits of interest to us are the orbits coming out of the origin. If $V(0) \neq 0$ then as $t \to -\infty$

$$\frac{g(t)}{|f(t)|^{\frac{1+m}{2m}} \operatorname{sign} f(t)} = \frac{V'(e^t)}{|V(e^t)|^{\frac{1+m}{2m}} \operatorname{sign} V(e^t)} \to \frac{V'(0)}{|V(0)|^{\frac{1+m}{2m}} \operatorname{sign} V(0)},$$

whereas if $V(0) = 0$ and $V'(0) \neq 0$,

$$\frac{g(t)}{f(t)} = \frac{\eta V'(\eta)}{V(\eta)} \to 1\,.$$

Apart from the origin there are two critical points, P and $-P$, given by

$$\pm P = c_1\left(-\frac{2m}{1-m}, 1\right), \qquad \text{with} \qquad c_1 = \left(\frac{2m(1+m)}{(1-m)}\right)^{\frac{m}{1-m}}. \tag{7.9}$$

(Recall that $\alpha(m-1)+2\beta=1$. The fact that P *does not depend on* α will be used below in the uniqueness part and motivates the introduction of the variable η). These two points contain the explicit solutions

$$V(\eta)=\pm c_1\eta^{-\frac{2m}{1-m}}. \tag{7.10}$$

As in (2.8), if $U(\eta)$ is a solution of (7.2) then $\hat{U}(\eta)=\lambda^{-2}U(\lambda^{m-1}\eta)$ is also a solution of (7.2) for each $\lambda>0$. Applying (7.6) and (7.8) we get

$$\hat{g}(t)=g(t-(1-m)\log\lambda), \qquad \hat{f}(t)=f(t-(1-m)\log\lambda).$$

This means that the scaling invariance of equation (7.2) corresponds to the translation invariance of the autonomous system (S). In particular, the symmetric solutions of (7.2) with $V(0)>0$ are mapped into a unique orbit of (S) which we denote by γ_s. The same happens with the antisymmetric solutions with $V'(0)>0$ which go into an orbit γ_a. Moreover, change of sign in the solutions corresponds to reflection around the origin in the (g,f)-plane. Note that for $k=1$ ($k=2$) γ_s (γ_a) connects the origin to P and does not intersect the g-axis. Intersections of an orbit with the g-axis correspond to zeroes of the solutions of (7.2) mapped into it.

In order to describe the global behaviour of γ_s we sketch the phase plane for (S) in Fig.1. It is invariant under the transformation $(g,f)\to(-g,-f)$.

Fig. 1. The phase plane for (S).

What can happen in forward t-time to an orbit γ coming out of the origin? From Fig.1 we see that only three possibilities may occur: (i) it can go into P or $-P$ after a finite number of rotations around the origin, (ii) it can escape to infinity, also after a finite number of rotations, and (iii) it can turn around forever.

If case (iii) occurs then the solutions contained in γ certainly do not satisfy (7.4) or (7.5). Anyway, they would correspond to the orbit being bounded and uniformly away from the origin for all large t, which means that the solution $|U(\eta)|$ would have maxima of the order of $|\eta|^{-2/(1-m)}$.

Let us examine the possibility (ii) for orbits to escape to infinity. To do so we compactify the phase plane by setting

$$g = \big(\frac{r}{1-r}\big)^{\frac{m}{1-m}} \cos\phi, \quad f = \big(\frac{r}{1-r}\big)^{\frac{m}{1-m}} \sin\phi, \tag{7.11}$$

which transforms (S) into

$$(\mathrm{S}^*)\begin{cases} (1-r)r' = & \frac{1-m}{m} r(1-r)^2\big(\frac{1+m}{1-m} + \sin\phi(\cos\phi - \sin\phi)\big) \\ & -\frac{1-m}{m} r^2(1-r)|\sin\phi|^{\frac{1}{m}-1}\cos\phi(\alpha\sin\phi + \frac{\beta}{m}\cos\phi) \\ (1-r)\phi' = & r|\sin\phi|^{\frac{1}{m}-1}\sin\phi(\alpha\sin\phi + \frac{\beta}{m}\cos\phi) \\ & +(1-r)\cos\phi(\cos\phi - \sin\phi), \end{cases}$$

where the factor $(1-r)$ on the left hand sides is irrelevant for the shape of the orbits and therefore ignored. Obviously, if an orbit γ of (S) escapes to infinity, its image γ^* has to go to one of the critical points at $r = 1$, which are

$$\phi = 0(\bmod \pi) \quad \text{and} \quad \phi = -\arctan\frac{\beta}{m\alpha}(\bmod \pi).$$

Suppose that γ^* goes into $(r, \phi) = (1, 0\,(\mathrm{mod}\ \pi))$. From Fig. 1 we then conclude that γ goes to infinity with f remaining bounded. Computing df/dg we find that it is bounded away from zero for large $t-$values. But then γ^* cannot go to $(r, \phi) = (1, 0(\mathrm{mod}\pi))$, contradiction.

The above argument shows that γ can only go to infinity if its image γ^* goes to $(r, \phi) = (1, -\arctan\frac{\beta}{m\alpha}(\mathrm{mod}\ \pi))$. Therefore solutions V of (7.2) contained in such an orbit satisfy

$$\frac{\eta V'(\eta)}{V(\eta)} = \frac{g(t)}{f(t)} \to -\frac{m\alpha}{\beta} = -mk,$$

which implies (7.4) after some standard work on the equation. We skip the details because the solutions with decay (7.4) are not essential in our considerations.

We are left with the orbits of (S) going into P or $-P$. Because of (7.6) the corresponding solution of (7.2), V, has decay rate (7.5). This decay rate is optimal in view of the above discussion. Since we are looking for the symmetric solutions of (7.2) which have two zeroes as well as the optimal decay rate, and, say, $V(0) > 0$, this corresponds to looking for the orbit γ_s connecting the origin and $-P$ intersecting the (negative) g-axis exactly once. We shall show in the sequel that this occurs for exactly one value $k = k_3$.

Step 2. Existence. To prove that such an orbit exists for some k we begin by computing the determinant of the linearization of (S) around P (and $-P$). We find that it is equal to $-2\frac{1+m}{1-m}$, which is negative. Thus P and $-P$ are saddles and then it is obvious from Fig.1 that one of the orbits going into P $(-P)$ is contained in the second (fourth) quadrant and comes from infinity, and is therefore not relevant here.

Next we observe that the eigenfunctions of (7.2) have to be either odd or even. To prove it, note first that every global solution U of (7.2) is mapped into two orbits, one obtained from applying the transformation (7.6) to $U(\eta)$, the other from applying it to $U(-\eta)$. Since only one orbit can go into the saddle P (resp. $-P$) coming from the origin, the two orbits corresponding to $U(\eta)$ and $U(-\eta)$ must be either the same or identical after reflection in the origin. Hence the conclusion.

Next, we introduce the sets

$$A = \{2 < k < \frac{2}{1-m} : \gamma_s \text{ intersects the positive } g\text{-axis at least once}\};$$

and

$$B = \{2 < k < \frac{2}{1-m} : k \notin A \text{ and } \gamma_s \text{ escapes to infinity through the fourth quadrant}\}.$$

By standard continuity arguments A and B are open. We show that these sets are nonempty. For $k = 2$ we know that $-\gamma_a$ connects the origin and $-P$ without intersecting the g-axis. From the phase plane (see Fig.1) and the discussion of the critical points at infinity before, it then follows that γ_s has no choice but to escape to infinity through the fourth quadrant, without intersecting the positive g-axis. By continuity arguments the same thing must happen for k sufficiently close to 2. Therefore B is not the empty set.

Let us prove that A is nonempty. Let $k = \frac{2}{1-m}$. Then (2.4) loses its meaning and we have to go back to equation (2.6) instead of (7.2). Introducing the functions f and g as in (7.6) with η replaced by y we obtain in this case

$$f'' + \frac{2m(1+m)}{(1-m)^2} f = (\frac{1+3m}{1-m} - \frac{1}{m}|f|^{\frac{1-m}{m}})f' \qquad (7.12)$$

Without the first order term on the right hand side (7.12) describes a harmonic oscillator. This first order term can be seen as a (nonlinear) friction term for large $|f|$, and as "negative" friction for small $|f|$. It follows from Liénard's theorem for such type of equations that every solution of (7.12) oscillates around zero as t increases to infinity. Clearly by continuity this implies that for k close to $\frac{2}{1-m}$ the symmetric solution of (7.2) must have an arbitrarily large number of zeroes. Thus A is not the empty set.

Because A and B are both nonempty open subsets of $(2, \frac{2}{1-m})$ there exists a number $k_3 \in (2, \frac{2}{1-m})$ not contained in the union of A and B. Since $k_3 > 1$ the symmetric solutions must change sign (see [Hu] for similar arguments when $m > 1$), so that for $k = k_3$ γ_s intersects the negative g-axis. This can also be seen from (7.13) below and the reasoning thereafter. But then it is clear from the phase plane analysis and the analysis of the critical points at infinity above, that γ_s must enter the fourth quadrant, stay there, and remain bounded. This can only happen if γ_s goes into $-P$. The existence of k_3 is established.

Step 3. Uniqueness. The proof depends on a monotonicity property of systems (S) and (S*) with respect to α (considering β as depending on α by (2.4)). It is here that we use equation (7.2) instead of (2.6) as there is no equivalent property in terms of k. Since this argument is worked out in detail for the slow diffusion case in [Hu], we only sketch it here. Rewriting (S^*) as

$$(S^*)\begin{cases} \frac{m}{1-m}\frac{r'}{r} = F(r,\phi,m,\alpha) \\ (1-r)\phi' = G(r,\phi,m,\alpha), \end{cases}$$

we find that

$$FG_\alpha - F_\alpha G = \frac{2m}{1-m} r(1-r)|\sin\phi|^{\frac{1-m}{m}}(\sin\phi + \frac{1-m}{2m}\cos\phi)^2 \geq 0, \tag{7.13}$$

in other words, the arrows in the phase plane turn monotonically in one direction.

The orbit γ_s^* containing the symmetric profiles is the only one to come out of the critical point $(r,\phi) = (0, \frac{\pi}{2})$ along the eigenvector $(1 \;\; \alpha)$ corresponding to the largest eigenvalue 2 of the linearization of (S^*) around $(r,\phi) = (0, \frac{\pi}{2})$.

Now suppose that there are two values of α for which γ_s connects the origin and $-P$ without crossing the positive g-axis. Then γ_s^* goes into the image of $-P$ which has ϕ-coordinate $\phi = -\arctan\frac{1-m}{2m} \in (\frac{3\pi}{2}, 2\pi)$. Note that in choosing this branch of the arctangent we disregard the connections which do cross the positive g-axis. Because of (7.13) these two connections can not intersect, and since $-P$ is independent of α, they form the boundary of a region which is negatively invariant for one of the two values of α and positively invariant for the other. In the latter case this implies that all orbits entering it have to go into $-P$, which is impossible because $-P$ is a saddle. This completes the proof of the uniqueness and thereby of the theorem. #

We recall for later use the information obtained above about the behaviour of the eigenfunctions.

Corollary 7.2. *The eigenfunction U_3 of (7.2) has the asymptotic behaviour*

$$U_3(\eta) \approx \pm c_1 \eta^{-\sigma} \qquad \text{as} \quad \eta \to \infty \tag{7.14}$$

with $c_1 = \left(\frac{2m(m+1)}{1-m}\right)^{\frac{m}{1-m}}$ *and* $\sigma = 2/(1-m)$. *Moreover,*

$$|U_3(\eta)| \le c_1 |\eta|^{-\sigma} \qquad \text{for all} \quad \eta \in \mathbf{R}. \tag{7.15}$$

The result is a consequence of formula (7.6) and the phase-plane picture which easily allows to deduce that $|f| \le c_1$. The plus or minus sign in (7.14) can be chosen by changing a given U_3 into $-U_3$. It is worth noticing that this behaviour (including the exact constant c_1) is not altered under the natural rescaling $\hat{U}(\eta) = \lambda^{-2} U(\lambda^{m-1}\eta)$.

The argument used in the above proof can be easily extended to show that there there exists an infinite sequence of eigenvalues as in the cases $m \ge 1$.

Theorem 7.3. *There exists a strictly increasing sequence*

$$0 < k_1 = 1 < k_2 = 2 < k_3 < k_4 < k_5 < \dots . < \frac{2}{1-m}, \tag{7.16}$$

such that the symmetric (anti-symmetric) solutions of (7.2) have optimal decay if and only $k = k_n$ with n odd (even), the number of zeroes being $n-1$. The asymptotic behaviour (7.14) is always true.

We do not know whether $k_i \to 2/(1-m)$ as $i \to \infty$, though we conjecture that it does so. On the other side, the behaviour of the k_i, $i \ge 3$, as $m \to 0$ follows immediately from (7.16).

Corollary 7.4. *As $m \to 0$ we have $k_i \to 2$ for $i = 3, 4, \cdots$.*

We are now ready to show as in Section 2 for $m \ge 1$ that $k_3 \ne 3$. In our case the result is

Theorem 7.5. *For $0 < m < 1$ we have $k_3 < 3$.*

The proof is done exactly as in Section 3, with same definitions for V, W and Z, except that none of these functions has compact support and consequently A is replaced by ∞. Formula (3.8) still holds, $Z_3(x)$ continues to be nonpositive, but now we have instead of (3.9)

$$\int_0^\infty V_3(x)\, dx < 0\,. \tag{7.17}$$

as a consequence of applying the inequality (3.12): $V_1 > |V_2|$ to formula (3.11) and taking into account that now the exponent $1 - \frac{1}{m}$ is negative. It follows that $k_3 < 3$. #

Finally, as in the case $m > 1$ we can show

Theorem 7.6. *The function $k_3(m)$ is continuous in $0 < m \leq 1$.*

Proof. It is very similar to the one performed in Section 5 for $m > 1$. We give next a sketch of the parts which differ for the reader's convenience. We begin by considering the eigenfunctions of equation (2.6) rather than (7.2). As in Theorem 5.1 we have $m_j \to \bar{m}$, $k_j \to \bar{k}$ and $\bar{U}(y)$ has a unique zero in $(0, \infty)$. Furthermore, $0 < \bar{m} \leq 1$ and

$$\bar{k} \leq \min\{3, \frac{2}{1-\bar{m}}\}.$$

If $\bar{m} = 1$ we apply Lemma 5.2 and the Remark of Section 5. Since $2/(1-m_j) \to \infty$, we may take λ arbitrarily large in Lemma 5.2. Hence, $\bar{U}(y)$ is an eigenfunction of the linear equation and $\bar{k} = k_3(1) = 3$.

If $0 < \bar{m} < 1$, it follows that $\bar{k} < 2/(1-\bar{m})$, because when $\bar{k} = 2/(1-\bar{m})$ the function $\bar{U}(y)$ would have infinitely many zeroes (see the paragraph including formula (7.12)). Thus, $\bar{\alpha} < \infty$, where $\bar{\alpha}$ is the value of α corresponding to $k = \bar{k}$. Going now to equation (7.2), we deduce from (7.15) that $\eta^{\sigma_j} U_j(\eta)$ is bounded uniformly in η and j. Hence, $\bar{U}(\eta)$ is an eigenfunction of (7.2) with $\bar{k} = k_3(\bar{m})$.

8. Asymptotic behaviour for (E) with $0 < m < 1$

We turn now to the application of the results of the preceding section to equation (E). As in Section 6 for $m > 1$ we begin by establishing basic existence and uniqueness theory by reducing the study of (E) to a repeated integration of (PME).

Thus, it is known [HP] that for every $u_0 \in L^1_{loc}(\mathbf{R})$ there exists a solution $u = u(x,t) \in C(Q) \cap C([0,\infty) : L^1_{loc}(\mathbf{R}))$. Moreover, if u_0 is integrable so is $u(\cdot, t)$ for every $t > 0$ and the L^1-norm is conserved in time. Also, the solution is unique under the condition that u_t be a locally integrable function in Q. Finally, if $u_0 \geq 0$, then u is positive and smooth everywhere in Q. Integrating these results we obtain for every initial data z_0, say as in (6.1), a unique solution $z(x,t) \in C^{2;1}_{x,t}(Q)$ of (E). The comparison Lemmas of Section 6 are still true.

On the other hand, we can construct as in the case $m > 1$ a solution

$$z_3(x,t) = -t^{-\gamma} Z_3(\eta) \quad \text{with} \quad \eta = xt^{-\beta}, \tag{8.1}$$

where $\gamma = \alpha - 2\beta$, α and β are given in terms of $k = k_3$ by (2.5) and Z_3 is obtained by integration of the eigenfunction $U_3(\eta)$ constructed in the last Section. After the change of sign z_3 becomes nonnegative in Q. We observe that since $k_3 < \sigma = 2/(1-m)$ and recalling the asymptotic formula (7.14), we have for every $x \neq 0$ and $t \approx 0$ (which implies $\eta \to \infty$)

$$z_3(x,t) \approx t^{-\gamma} \eta^{2-\sigma} = t^{(\sigma - k_3)\beta} x^{2-\sigma}, \tag{8.2}$$

hence $z_3(x,t) \to 0$ as $t \to 0$ if $x \neq 0$. However, at $x = 0$ we have

$$\lim_{t\to 0} z_3(0,t) = \infty\,. \tag{8.3}$$

This means that z_3 has an isolated singularity at $(0,0)$. This singularity can also be seen in some L^p-norms, $p < \infty$. Thus, $Z(\eta)$ is L^p-integrable in $\mathbf{R}$ if $p > p_0 = (1-m)/(2m)$ (since $2-\sigma = 2m/(1-m)$) and then we have formula (2.22) for the L^p-norm, hence there exists

$$p_* = 1/(k_3 - 2) > 1 \tag{8.4}$$

such that as $t \to 0$ $\|z_3(\cdot,t)\|_p$ tends to ∞ if $p > p_*$, to 0 if $p_0 < p < p_*$, and is constant in time if p is precisely p_*. Note that $k_3 < 2/(1-m)$ implies that $p_0 < p_*$. For $m > 1/3$ we have $p_0 < 1$ so that namely

$$\lim_{t\to 0} \int z_3(x,t) \to 0\,, \tag{8.5}$$

i.e. the singularity cannot be seen in $L^1(\mathbf{R})$. On the other hand, for $0 < m \leq 1/3$ we have $p_0 \geq 1$ and then $\int_{\mathbf{R}} Z(\eta)\,d\eta = \infty$. However, if $I = (-r,r)$, $r > 0$ and $t \approx 0$ we have

$$\int_I z_3(x,t)\,dx \approx Ct^{\beta(\sigma-k_3)} \to 0\,; \tag{8.6}$$

we conclude that the singularity disappears in $L^1_{\rm loc}$. A last observation about the solution z_3 comes from estimate (7.15), which implies that $Z_3(\eta) \leq C\eta^{2-\sigma}$, hence

$$z_3(x,t) \leq Ct^{(\sigma-k_3)\beta} x^{-2m/(1-m)}\,. \tag{8.7}$$

for some constant $C > 0$.

We may also apply to Z_3 the rescaling (2.8) to obtain a whole one-parameter family of selfsimilar solutions of (E) $\hat{z}(x,t) = \hat{z}_3(x,t;\lambda)$ given in terms of z_3 by the formula

$$\hat{z}(x,t) = \lambda^{-2m} z_3(\lambda^{m-1}x, t)\,. \tag{8.8}$$

Of course, the properties previously mentioned for z_3 apply also to the functions $\hat{z}$. In particular, (8.7) is true with the same constant.

The study of the asymptotic behaviour of general solutions of (E) proceeds now by means of comparison with these selfsimilar solutions. In this way we can obtain an analogon of Lemma 6.3.

Theorem 8.1. *For every solution of (E) with initial data $z_0(x) \in C(\mathbf{R})$ such that*

$$z_0(x) = O(|x|^{-2m/(1-m)}) \quad \text{as} \quad |x| \to \infty \tag{8.9}$$

there exists a constant $C > 0$ such that

$$|z(x,t)| \leq C(t+1)^{-\gamma} \quad \text{with} \quad \gamma = (k_3 - 2)\beta_3\,. \tag{8.10}$$

On the other hand, for nonnegative initial data our solutions will be positive in Q, and we obtain bounds both from above and below.

Theorem 8.2. *If moreover $z_0 \geq 0$ then for every $\tau > 0$ we have a positive selfsimilar solution $\hat{z}_3(x,t;\lambda)$ such that $z(x,t) \geq \hat{z}_3(x,t;\lambda)$ if $t \geq \tau$. Hence,*

$$z(x,t) \geq C_1(x)t^{-\gamma}\,. \tag{8.11}$$

Proof of Theorem 8.1. Step 1. Suppose that we prove that there exists a $\tau > 0$ and a rescaling $\hat{z}$ of the solution z_3 such that $z_0(x) \geq \hat{z}(x,\tau)$ for every $x \in \mathbf{R}$. Then by the maximum principle we will conclude that

$$z(x,t) \leq \hat{z}(x,t+\tau) \leq \hat{z}(0,t+\tau) = C(t+\tau)^{-\gamma} \tag{8.12}$$

which is equivalent to (8.10).

Step 2. To prove the previous claim let $C > 0$ be such that $z_0(x) \leq C|x|^{-\mu}$, where $\mu = \sigma - 2 = 2m/(1-m) > 0$ and let $M = \sup\{z_0(x) : -R \leq x \leq r\}$. Twice integrating estimate (7.14) for U_3 we deduce that z_3 satisfies for fixed $\tau > 0$

$$\lim_{x\to\infty} z_3(x,t)|x|^{\mu} = c_2\tau^{\alpha}\,, \tag{8.13}$$

where $c_2 = c_1/(1-\sigma)(2-\sigma)$. Applying rescaling (8.2) and taking $\lambda < 1$ small enough, we may assume that $\hat{z}(x,\tau)|x|^m$ uniformly approaches the limit $c_2 > 0$ for $\varepsilon < |x| < \infty$. Choosing τ large we then obtain $\hat{z}(x,\tau) \geq z_0(x)$ in that region. On the other hand, for λ small we have also $\hat{z}(0,\tau) = z_3(0,\tau)/\lambda^{2m} >> M$, hence $\hat{z}(x,\tau) \geq z_0(x)$ for $|x| \leq \varepsilon$. Summing up, $\hat{z}(\cdot,\tau) \geq z_0(\cdot)$ in $\mathbf{R}$. #

proof of Theorem 8.2. As in the previous theorem it is enough to prove that $z(x,\tau) \geq \hat{z}(x,\tau)$ for all $x \in \mathbf{R}$ and some small τ. Observe to begin with that since $z_0 \not\equiv 0$ we may assume that $z_0(0) > 0$, hence $z(x,t) \geq c > 0$ for $|x| \leq r$ and $0 \leq t \leq \tau$ if r and τ are small enough.

We first compare z and $\hat{z}$ in the region $D = \{(x,t) : x > r, 0 < t < \tau\}$. Let us check the parabolic boundary of D. For $t = 0$, $x \geq r$ we have $z_0(x) \geq 0 = \hat{z}(x)$. On the other hand, for $0 \leq t \leq \tau$ we get $z(r,t) \geq c$, while $\hat{z}(r,t) \leq c$ if τ is small enough by estimate (8.7), which is uniform in λ. Hence, $z(x,t) \geq \hat{z}(x,t)$ in D. A similar comparison can be done in $D^* = (-\infty, r) \times (0,\tau)$ after possibly taking a smaller τ.

We are thus left with the comparison for $|x| \leq r$ and $t = \tau$. But for λ large enough and $|x| \leq r$

$$\hat{z}(x,\tau) \approx z_3(0,\tau)\lambda^{-2m} < c\,,$$

which can be made arbitrarily small by choosing a large λ. #

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