THE SEMILINEAR HEAT EQUATION WITH A HEAVISIDE SOURCE TERM

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Abstract. We consider the initial value problem for the equation $u_t = u_{xx} + H(u)$, where H is the Heaviside graph, on a bounded interval with Dirichlet boundary conditions, and discuss existence, regularity and uniqueness of solutions and interfaces.

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1. Introduction

In this paper we deal with the equation

$$u_t = u_{xx} + H(u)\,, \tag{1.1}$$

where H is the Heaviside graph,

$$H(s) = 0 \text{ for } s < 0\,, H(0) = [0,1]\,, H(s) = 1 \text{ for } s > 0\,.$$

Thus (1.1) is a semilinear parabolic equation with a discontinuous source term. Equation (1.1) arises in [NS2] as the limit for $\epsilon \downarrow 0$ of a simple model equation for combustion in porous media. In [NS1] equilibria of a slightly more general equation with $H(u)$ replaced by $H(u-1)f(u)$ are studied in the context of highly exothermic combustion. We also mention [L], where the reaction term jumps at a certain critical value of u from a small number to a large number, and [RK], which deals with travelling wave solutions of a system describing nerve conduction. This system consists of one ordinary and one partial differential equation, and contains a reaction term similar to the one in [L]. For more references on the physical background we refer to [NS1,NS2]. See also [FT,GM1,GM2].

The essential feature of (1.1) is that the reaction term is switched on at a certain fixed value of the temperature u. From this point of view solutions of (1.1) with changing sign are of particular interest. We are interested in the uniqueness of solutions for initial boundary value problems as well as for the Cauchy problem for (1.1). Since $H(u)$ is not (Lipschitz) continuous in $u = 0$, nonuniqueness may occur. If one takes zero initial values for the Cauchy problem both zero itself as well as the function $u(x,t) = t$ are solutions. A similar counterexample exists if one replaces the Heaviside function by u_+^p with $0 < p < 1$. For that case a detailed analysis of nonnegative solutions can be found in [AE], the result being that, roughly speaking, only zero initial data admit two solutions.

In this paper we consider solutions which are allowed to take both negative and positive values. Our study was partially motivated by the question as to whether the reaction can be switched on if at $t = 0$ the initial values of u are everywhere negative (i.e. the source term is zero), except at one single point, where the initial value is zero. To get an idea, we consider similarity solutions of the form

$$u(x,t) = tf\left(\frac{x}{\sqrt{t}}\right), \tag{1.2}$$

with

$$f(0) = \alpha \in (0,1) \text{ and } f'(0) = 0. \tag{1.3}$$

In the appendix we shall show that (1.1-1.3) uniquely determine f as a function of the similarity variable $\eta = x/\sqrt{t}$, and that $f(\eta)$ is decreasing for $\eta > 0$. Moreover,

$$c(\alpha) = -\lim_{\eta\uparrow\infty} \frac{f(\eta)}{\eta^2} > 0 \tag{1.4}$$

exists, so that u satisfies

$$u(x,0) = \lim_{t\downarrow 0} u(x,t) = -c(\alpha)x^2. \tag{1.5}$$

Here c depends smoothly on α, and

$$c \to 0 \text{ as } \alpha \downarrow 0 \text{ or } \alpha \uparrow 1. \tag{1.6}$$

Thus, besides $u(x,t) = -c(x^2 + 2t)$, which is always a solution of (1.1), we find that for small $c > 0$ there are at least two other (similarity) solutions with initial data $u(x,0) = -cx^2$, which both become positive in $x = 0$ for $t > 0$. Apparently, there are two (or more) ways for the reaction to be switched on. As far as we know, this rather surprising and unexpected phenomenon has not been observed before, and is not properly understood, neither by physicists, nor by mathematicians. We note that a similar example exists in any space dimension, as well as for the equation

$$u_t = u_{xx} + u_+^p ,$$

if $0 < p < 1$. Again one has nonpositive initial data of power type admitting, besides a strictly negative solution, at least two different similarity solutions which become positive in $x = 0$ for $t > 0$.

Next we ask ourselves, whether there is uniqeness if the reaction has already been switched on in part of the domain, and whether the burning zone is bounded by a sharp front. We shall investigate these questions for the model problem

$$(P)\begin{cases} u_t = u_{xx} + H(u) & \text{in } Q_T = (0,1) \times (0,T]\,; \\ -u(0,t) = 1 = u(1,t) & \text{for } 0 < t \leq T\,; \\ u(x,0) = u_0(x) & \text{for } 0 \leq x \leq 1\,, \end{cases}$$

and show that essentially the answer is yes. In establishing this result the example above plays an important role.

For the sake of simplicity we have taken the lateral boundary conditions to be time independent, although this is not really essential. We assume that the function u_0 satisfies the following hypothesis:

H1. $u_0 \in C([0,1])$ and $-u_0(0) = 1 = u_0(1)$.

Definition 1.1. *$u \in C(\bar{Q}_T)$ is called a solution of (P) if u satisfies the boundary conditions of (P) and if*

$$\phi = u_t - u_{xx},$$

defined in the distributional sense, is a bounded measurable function and satisfies $\phi \in H(u)$ a.e. in Q_T.

We introduce two more hypotheses on the initial data.

H2. There exist $\zeta_0^-, \zeta_0^+ \in (0,1)$ such that:

$$\begin{cases} u_0(x) < 0 & \text{for } 0 \leq x < \zeta_0^-\,; \\ u_0(x) = 0 & \text{for } \zeta_0^- \leq x \leq \zeta_0^+\,; \\ u_0(x) > 0 & \text{for } \zeta_0^+ < x \leq 1\,. \end{cases}$$

H3. $u_0 \in C^1([0,1])\,, \zeta_0^- = \zeta_0^+ = \zeta_0\,,$ and $u_0'(\zeta_0) > 0$.

Thus we concentrate on the effect of one sign change in the initial data.

Theorem 1.1. *Suppose u_0 satisfies H1. Then (P) has a solution in the sense of Definition 1.1. If H1-3 hold, this solution is unique. Moreover, if only H1-2 hold, then any solution $u(\cdot,t)$ of (P) satisfies H1-3 for any positive value of t. In other words, nonuniqueness can only occur at $t=0$.*

The existence part of this theorem is almost immediate from [LSU]. We conjecture that Theorem 1.1 is also valid without assumption H3. For the uniqueness part the following result on the existence of a sharp reaction front is important.

Theorem 1.2. *Suppose H1-2 hold, and that u is a solution of (P). Then there exists $\zeta \in C([0,T])$ with $\zeta(0)=\zeta_0^-$ such that for all $0<t\le T$*

$$u(x,t)<0 \ \text{ for } 0\le x<\zeta(t) \ \text{ and } u(x,t)>0 \ \text{ for } \zeta(t)<x\le 1\,,$$

and

$$u_x(\zeta(t),t)>0\,.$$

If also H3 holds then $0<t\le T$ can be replaced by $0\le t\le T$.

We shall refer to this function ζ as the *interface*. The proof of this theorem, which is of independent interest, depends on level-curve arguments for both u and u_x, as well as the use of the similarity solutions mentioned above as barrier functions.

Finally we note that the strict positivity of u_x improves the regularity of the interface. To be more precise, it is shown in [GM1], that it causes the interface to be Hölder continuous, and then standard bootstrap arguments yield that $\zeta(t)$ is smooth.

Theorem 1.3. *In Theorem 1.2 the interface satisfies $\zeta \in C^\infty((0,T])$.*

This last theorem is not of importance as far as the uniqueness question is concerned, but is stated here for the sake of completeness.

The organisation of this paper is as follows. In Section 2 we establish some general properties of solutions, and in particular of the interface. Then in Section 3 we prove the uniqueness result, and briefly discuss the existence of a solution. The smoothness of the interface is touched upon in Section 4. Finally, in the appendix, we compute the similarity solution introduced above in detail.

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2. Properties of solutions

In this section we formulate and prove a number of propositions for general solutions of (P), which imply Theorem 1.2.

2a. Preliminaries

The following results are standard. See e.g. [F] or [LSU].

Proposition 2.1. *Let H1 be satisfied. Suppose u is a solution of (P) and let ϕ be as in Definition 1.1. Then u can be written as $u_1 + u_2$, where u_1 is the solution of*

$$(P_1)\begin{cases} u_t = u_{xx} & \text{in } Q_T\,; \\ -u(0,t) = 1 = u(1,t) & \text{for } 0 < t \le T\,; \\ u(x,0) = u_0(x) & \text{for } 0 \le x \le 1\,, \end{cases}$$

and u_2 is the solution of

$$(P_2)\begin{cases} u_t = u_{xx} + \phi & \text{in } Q_T\,; \\ u(0,t) = u(1,t) = 0 & \text{for } 0 < t \le T\,; \\ u(x,0) = 0 & \text{for } 0 \le x \le 1\,, \end{cases}$$

Proposition 2.2. *Let u_0 satisfy H1. Then problem (P_1) has a unique classical solution $u_1 \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$. Moreover $u_1 \in C^\infty([0,1]) \times (0,T])$, and if u_0 satisfies $u_0 \in C^1([0,1])$ then $u_{1x} \in C(\bar{Q}_T)$.*

Proposition 2.3. *Let $\phi \in L^\infty(Q_T)$. Then there exists a unique $u_2 \in C(\bar{Q}_T)$ such that u_2 satisfies the boundary conditions of (P_2) , and $u_{2t} - u_{2xx} = \phi$ in the distributional sense. Moreover*

$$u_2\ ,\ u_{2x} \in C^{\alpha,\alpha/2}(\bar{Q}_T) \quad \forall \alpha \in (0,1)$$

and

$$u_{2t} \in L^p(Q_T) \quad \forall p \in [1,\infty).$$

If $G(x,\zeta,t,\tau)$ is the Green's function of the first kind then

$$u_2(x,t) = \int_0^t d\tau \int_0^1 G(x,\zeta,t,\tau)\phi(\zeta,\tau)\, d\zeta\,.$$

2b. Existence and continuity of the interface

Definition 2.1. *Let u be a solution of (P). Define*

$$\zeta^-, \zeta^+ : [0,T] \to (0,1)$$

by

$$\begin{aligned} \zeta^+(t) &= \inf\{\zeta \in (0,1)\,, u(\cdot,t) > 0 \text{ on } (\zeta,1]\} \\ \zeta^-(t) &= \sup\{\zeta \in (0,1)\,, u(\cdot,t) < 0 \text{ on } [0,\zeta)\}\,. \end{aligned}$$

Proposition 2.4. *Let u be a solution of (P) and suppose H1-2 hold. Let $\zeta^-(t)$, $\zeta^+(t)$ be defined by Definition 2.1. Then for all $t \in [0,T]$*

$$u(x,t) = 0 \quad \forall\, x \in [\zeta^-(t), \zeta^+(t)]$$

Proof. (i) Suppose for some t_0 there exists $a \in (\zeta^-(t_0), \zeta^+(t_0))$ with $u(a,t_0) < 0$. Let $S \subset Q_{t_0} = (0,1) \times (0,t_0]$ be the component of $\{u < 0\}$ containing (a,t_0) and let $\tilde{S}$ be the component of $\{u < 0\}$ in Q_{t_0} containing the segment $[0,\zeta^-(t_0)] \times \{t_0\}$. We claim that $\tilde{S} \cap S = \emptyset$, for otherwise we would have $S = \tilde{S}$, allowing us to join (a,t_0) with a point (b,t_0), $b < \zeta^-(t_0)$ by means of a smooth curve $\gamma \subset S$. Let $\mathcal{D}$ be the region bounded by γ and the segment $(b,a) \times \{t_0\}$, and define t_1 to be the maximal t for which $u < 0$ on $\mathcal{D} \cap ((0,1) \times (0,t)) = \mathcal{D}_t$. Since u satisfies $u_t = u_{xx}$ in $\mathcal{D}_{t_1}$, the strong maximum principle implies $t_1 = t_0$, but then, again by the strong maximum principle, $u < 0$ on $[b,a] \times \{t_0\}$, contradiction. So indeed $\tilde{S} \cap S = \emptyset$. Hence S cannot touch the parabolic boundary, but on S we have $u_t = u_{xx}$, and on the parabolic boundary of S $u = 0$. This is again a contradiction. We conclude that $u(x,t) \geq 0$ for all $x > \zeta^-(t)$, for all $t \in (0,T]$.

(ii) Suppose for some $a < \zeta^+(t_0)$ we have $u(a,t_0) > 0$. Then by continuity of u, $\limsup_{t \to t_0} \zeta^-(t) < a$, so that, by part (i) of this proof, for $\tau > 0$ sufficiently small, $u \geq 0$ on $(a,1) \times (t_0 - \tau, t_0] = \hat{Q}$. Let $\tilde{u}$ be the solution of the heat equation on $\hat{Q}$ with $u = \tilde{u}$ on the parabolic boundary of $\hat{Q}$. Then by the weak comparison principle, since $u_t - u_{xx}$ defined in the distributional sense is a nonnegative bounded measurable function, we have $u \geq \tilde{u}$ on $\hat{Q}$. Since $u(a,t) > 0$ on $[t_0 - \tau, t_0]$, the strong maximum principle gives $\tilde{u} > 0$ on $\hat{Q}$, implying $u(\zeta^+(t_0), t_0) \geq \tilde{u}(\zeta^+(t_0), t_0) > 0$, which is again a contradiction.

This completes the proof. □

Next we shall establish that $\zeta^+ = \zeta^-$ for positive t, and that $\zeta = \zeta^+ = \zeta^-$ is continuous. To do this we use the example from Section 1.

Proposition 2.5. *There exists a constant $c^* > 0$ such that for all $0 < c \leq c^*$ there exists a function $f\colon [0,\infty) \to \mathbf{R}$, $f(0) > 0$, $f' < 0$ on $(0,\infty)$, $f'(0) = 0$, $f(s) \downarrow -\infty$ as $s \uparrow \infty$, such that*

$$u(x,t) = \begin{cases} t\,f(|x|/\sqrt{t}) & \text{for } x \in \mathbf{R}\,, t > 0 \\ -cx^2 & \text{for } x \in \mathbf{R}\,, t = 0 \end{cases}$$

satisfies $u \in C(\mathbf{R} \times [0,\infty))$ and $\phi = u_t - u_{xx}$ defined in the distributional sense, is a bounded measurable function with $\phi \in H(u)$ a.e. in $\mathbf{R} \times [0,\infty)$. In particular $u(0,t) = f(0)\,t > 0$ for $t > 0$.

Proof. See Appendix. $\square$

In order to use this solution as a barrier function we need a comparison principle.

Proposition 2.6. *Let $R = (a,b) \times (t_1,t_2]$ and let $\partial_p R$ be the parabolic boundary of R. Suppose u_1, $u_2 \in C(\bar{R})$ are two solutions of $u_t - u_{xx} = H(u)$. Then*

$$u_1 < u_2 \text{ on } \partial_p R \;\Rightarrow\; u_1 < u_2 \text{ on } R$$

Proof. Let $\delta = \min(u_2 - u_1)$ on $\partial_p R$, then $\delta > 0$, and there exists $\tau \in (t_1,t_2)$ such that $u_2 - u_1 > \delta/2$ on $[a,b] \times [t,\tau]$. Let $\phi_i = u_{it} - u_{ixx}$, $(i = 1,2)$. Clearly $\phi_2 \geq \phi_1$ on $[a,b] \times [t,\tau]$, so that $(u_1 - u_2)_t \leq (u_1 - u_2)_{xx}$ in distributional sense. Hence by the weak comparison principle for the heat equation, $u_1 - u_2 \leq -\delta$ on $[a,b] \times [t_1,\tau]$. Now let τ be the maximal value for which this hold. Then clearly $\tau = t_2$. $\square$

The following proposition is in the spirit of [H].

Proposition 2.7. *Suppose H1 holds and let u be a solution of (P). For all $\epsilon > 0$ there exists $\delta > 0$ such that*

$$t_0 < t < t_0 + \delta \;\Rightarrow\; \zeta^+(t) < \zeta^+(t_0) + \epsilon\,,$$

where ζ^+ is as in Definition 2.1.

Proof. Let $\epsilon > 0$. Consider the region $R_\delta = (\zeta^+(t_0), \zeta^+(t_0) + 2\epsilon) \times (t_0, t_0 + \delta)$, where $\delta > 0$.

Let

$$\underline{u}(x,t) = \begin{cases} (t - t_0) f\left((x - \zeta^+(t_0) - \epsilon)/\sqrt{t - t_0}\right) & t > t_0 \\ -c(x - \zeta(t_0) - \epsilon)^2 & t = t_0 \end{cases}$$

where f is as in Proposition 2.5. Since both u and $\underline{u}$ are uniformly continuous, there exists $\delta > 0$ such that $u > \underline{u}$ on $\partial_p R_\delta$. Note that δ depends only on ϵ and not on t_0, because $u \in C(\bar{Q}_T)$.

By Proposition 2.6, $u > \underline{u}$ on R_δ, and this completes the proof since $\underline{u}(\zeta^+(t_0) + \epsilon, t) = (t - t_0) f(0) > 0$. $\square$

Proposition 2.8. *Let u be a solution of (P) and suppose H1-2 hold. Let ζ^+ be as in Definition 2.1. Then for all $t_0 \in (0,T)$ we have*

$$\lim_{t \uparrow t_0} \zeta^+(t) = \zeta^+(t_0) \geq \lim_{t \downarrow t_0} \zeta^+(t)\,,$$

and also

$$\lim_{t \uparrow T} \zeta^+(t) = \zeta^+(T) \ \ \textit{and} \ \lim_{t \downarrow 0} \zeta^+(t) \leq \zeta_0^+\,.$$

In particular, all these limits exist.

Proof. The existence the limits is immediate from Proposition 2.7. The equalities follow from the fact that u is a supersolution for the heat equation. $\square$

Proposition 2.9. *Let u be a solution of (P). Then $\phi = u_t - u_{xx} = 0$ a.e in the set $A = \{(x,t) \in Q_T\colon u(x,t) = 0\}$.*

Proof. First observe that $u \in H^1_{loc}(Q_T)$; hence $u_x = u_t = 0$ a.e. on A. Also $u_{xx} \in L^2_{loc}(Q_T)$ so that for almost all t, $u_{xx}(\cdot, t) = 0$ a.e. on $\{x \in (0,1)\colon u_x(x,t) = 0\}$. Thus $u_{xx} = 0$ a.e. on A, so that the same holds for $\phi = u_t - u_{xx}$. $\square$

Proposition 2.10. *Let u be a solution of (P) and suppose H1-2 hold. Let ζ^-, ζ^+ be as in Definition 2.1. Then*

$$\zeta^- \equiv \zeta^+ \in C([0,T])\,,$$

and, writing $\zeta \equiv \zeta^+ \equiv \zeta^-$,

$$\zeta(0) = \zeta_0^-\,.$$

Proof. First we show that $\zeta^+ \in C((0,T])$. Suppose not, then by Proposition 2.8, there exists $t_0 \in (0,T]$ such that

$$\lim_{t \downarrow t_0} \zeta^+(t) < a < b < \lim_{t \uparrow t_0} \zeta^+(t)\,,$$

for some $a, b \in (0,1)$.

Thus, by Proposition 2.4, there exists $\tau < t_0$ such that $u \le 0$ on $R = (a,b) \times (\tau, t_0]$, and by Proposition 2.9, $u_t = u_{xx}$ a.e. in R. Hence $u \in C^\infty(R)$, and either $u < 0$ on R or $u \equiv 0$ on R. Clearly $u < 0$ on R contradicts the definition of $\zeta^+(t_0)$, so that $u \equiv 0$ on R. But u is also a solution of $u_t = u_{xx}$ on $(0,b) \times (\tau, t_0)$, and since solutions of the heat equation are analytic in x, $u \equiv 0$ on $(0,b) \times (\tau, t_0)$, contradiction.

So ζ^+ is continuous, and by the same argument as above $u < 0$ for $x < \zeta^+(t)$, which implies $\zeta^- \equiv \zeta^+$.

Finally we show that $\zeta(t) \to \zeta_0^-$ as $t \downarrow 0$. By Proposition 2.7, $\zeta(0) = \lim_{t \downarrow 0} \zeta(t)$ exists. Suppose $\zeta(0) > \zeta_0^-$. Then we can choose $\zeta_0^- < b^- < b < \zeta(0) < b^+ \le 1$, such that $b - b^- = b^+ - b$. Now consider the solution $\underline{u}(x,t)$ defined on the whole space of the equation

$$u_t = u_{xx} + H(x - b^+)\,,$$

with initial data

$$\underline{u}(x,0) = -\frac{1}{2}(b^- - x)_+^2\,.$$

Clearly, because $H(u(x,t)) \ge H(x - b^+)$ for small t, we have, using continuity of u and the strong comparison principle for the heat equation, that $\underline{u}(x,t) < u(x,t)$ for all $t > 0$ small and $\zeta_0^- \le x \le 1$. This gives a contradiction if we can show that $\underline{u}(b,t) = 0$. But this is easy, namely we write

$$\underline{u}(x,t) = v(x,t) - \frac{1}{2}(x - b^+)_+^2\,,$$

so that v solves the heat equation with initial data which are anti-symmetric around $x = b$. Hence $\underline{u}(b,t) = v(b,t) = 0$. $\square$

2c. Properties of the solution at the interface

Now that the continuity of the interface between the sets where u is positive and negative has been established, our next goal is to show that for $t > 0$, u_x is positive at the interface. This will complete the proof of Theorem 1.2. The proofs of the following propositions all rely on level curve arguments, and in particular on Sard's theorem [GP]. Note that we do not assume that initially u_x has finitely many sign changes. Similar arguments can be used to improve the results in [H] and [BH], where such an assumption was needed on the initial data. The idea that this assumption can be relaxed is due to M. Bertsch, and appeared for the first time in [HH].

Proposition 2.11. *Let H1-2 be satisfied, and u be a solution of (P), and let $\zeta(t)$ be as in Proposition 2.10. Then the interior of the set $\{t \in (0,T] \colon u_x(\zeta(t),t) = 0\}$ is empty.*

Proof. Suppose not, then for some $0 < t_1 < t_2$, $u_x(\zeta(t),t) = 0$ for $t_1 \le t \le t_2$.

Let $x_\delta = \min\{x \in (\zeta(t_2), 1] \colon u_x(x,t_2) = \delta\}$. For $\delta > 0$ sufficiently small x_δ is well defined, and by Sard's theorem, for almost all such δ, the level curve of $u_x = \delta$ in $\{(x,t) \colon \zeta(t) < x < 1\,,\, t > t_1\}$ containing (x_δ, t_2) is well defined.

We can parametrize this curve by

$$s \to \gamma(s) = (x(s), t(s))\,,\; s \in (-1,1)\,, \tag{2.1}$$

where $\gamma(0) = (x_\delta, t_2)$, and $t(s) < t_2$ for $s < 0$, the latter because of the fact that tracing the level curve backwards in t, it cannot turn around because of the maximum principle for u_x, which satisfies the heat equation where $u > 0$. More precisely, $t(s)$ is increasing for $s \in (-1,0)$. Also $u_{xx}(\gamma(s)) \ge 0$ for $s < 0$. Since $u_x(\zeta(t),t) = 0$ for $t \in [t_1,t_2]$ we either have

$$\lim_{s\downarrow -1} \gamma(s) = (\tilde{x}_\delta, t_1) \;\text{ with } \tilde{x}_\delta > \zeta(t_1)\,, \tag{2.2}$$

or

$$\lim_{s\downarrow -1} \gamma(s) = (1, t^*) \;\text{ with } t^* \in [t_1, t_2)\,. \tag{2.3}$$

We have

$$\frac{d}{dt} u(\gamma(s)) = u_x \dot{x} + u_t \dot{t} = \delta \dot{x} + (u_{xx} + 1)\, \dot{t} \ge \delta \dot{x} + \dot{t}\,. \tag{2.4}$$

In the first case (2.2) we have

$$u(x_\delta, t_2) - u(\tilde{x}_\delta, t_1) \ge \delta(x_\delta - \tilde{x}_\delta) + (t_2 - t_1)\,. \tag{2.5}$$

Clearly (2.5) implies

$$u(\zeta(t_2), t_2) = \lim_{\delta\downarrow 0} u(x_\delta, t_2) \ge t_2 - t_1\,, \tag{2.6}$$

contradiction. The second case (2.3) is left to the reader. $\square$

Proposition 2.12. *Let H1-2 be satisfied, and u be a solution of (P), and let $\zeta(t)$ be as in Proposition 2.10. For every $t_0 \in (0,T]$ there exists an open neighborhood N of $(\zeta(t_0), t_0)$ such that $u_x \ge 0$ in N.*

Proof. First we show that near the interface on the left side u_x is positive. By Proposition 2.11 there exists t_1 arbitrarily small, such that $u_x(\zeta(t_1),t_1)>0$. Consider the region $Q^-=\{(x,t)\colon 0<x<\zeta(t)\,,\,t_1<t\le T\}$. For $\delta>0$ sufficiently small there exists a unique $x_\delta\in(0,\zeta(t_1))$ such that $u(x_\delta,t_1)=-\delta$. We may assume that $u_x(x,t_1)>0$ on $[x_\delta,\zeta(t_1)]$. Now let $y_\delta=\max\{x\in(0,\zeta(T))\,:\,u(x,T)=-\delta\}$. Again for almost all δ the level curve of u through (y_δ,T) in $\{x<\zeta(t)\,,\,t_1<t\le T\}$ is smooth, and by the maximum principle, it must be of the form $s\to\gamma(s)=(x(s),t(s))$, $s\in(-1,0]$ with $\gamma(0)=(y_\delta,T)$ and $t(s)<T$ and increasing. Clearly $\gamma(-1)=(x_\delta,t_1)$. Then, because $u_x\ge 0$ on the interface, by the (strong) maximum principle, $u_x>0$ in $\{(x,t)\colon t_1\le t\le T\colon x(s(t))<x<\zeta(t)\}$, where $s(t)$ is the inverse of $t(s)$.

Next we consider the region $Q^+=\{(x,t)\colon \zeta(t)<x<1\,,\,t_1<t\le T\}$ and show that u_x is nonnegative near the interface. This will take a little more effort. Since the maximum principle for u is not valid in Q^+, we do not consider level curves of u but of u_x, as in the proof of Propostion 2.11. We shall show that for every $t\ge t_1$ there exists a neighbourhood N of $(\zeta(t),t)$ such that $u_x\ge 0$ on N.

Again, by the continuity of u_x and Proposition 2.11, this is true for some arbitrarily small $t=t_1$. Suppose it fails for some $t_0>t_1$. Without loss of generality we may assume that $t_0<T$. Then there exists a sequence $(\bar x_n,\bar t_n)$ in Q^+, with $(\bar x_n,\bar t_n)\to(x_0,t_0)$, and $u_x(\bar x_n,\bar t_n)<0$. Writing $\delta_n=-u_x(\bar x_n,\bar t_n)$, we may assume that $\delta_n\downarrow 0$ and that the level curve γ_n of u_x going through $(\bar x_n,\bar t_n)$ are smooth, and also that $u_x(x,\bar t_n)>-\delta_n$ for $x\in[\zeta(\bar t_n),\bar x_n)$ (otherwise we just choose a smaller $\bar x_n$).

What can happen to γ_n? It cannot intersect $x=\zeta(t)$. Hence u_x locally solves the heat equation on γ_n, so tracing γ_n backwards in time, it cannot turn around. To be precise, parametrizing γ_n by

$$s\to\gamma_n(s)=(x_n(s),t_n(s))\,,\;s\in(-1,1)\,,\tag{2.7}$$

we have, possibly after a change of parameters, two cases to consider. Either

$$t(s)\text{ is increasing on }(-1,0)\text{ with }t(0)>t_0\,,\tag{2.8}$$

or

$$t(s)\text{ is increasing on }(-1,0),\text{ decreasing on }(0,1),\text{ with }t(0)\le t_0\,.\tag{2.9}$$

First we exclude (2.9). We would have $\gamma_n(s_n)=(\bar x_n,\bar t_n)$ for some $s_n\in(-1,0)$ and $\gamma_n(1)=(x_n(1),t_n(1))$ with

$$x_n(1)=1,\;t_n(1)\ge t_1\;\text{ or }x_n(1)<1,\;t_n(1)=t_1\,,\tag{2.10}$$

so that certainly $u(\gamma_n(1))>0$. But then

$$u(\gamma_n(1)-u(\bar x_n,\bar t_n)=\int_{s_n}^1\big(u_x(\gamma_n(s))\dot x_n(s)+u_t(\gamma_n(s))\dot t_n(s)\big)\,ds=\tag{2.11}$$

(2.12)
$$\int_{s_n}^{1} \left(-\delta_n \dot{x}_n(s) + u_{xx}(\gamma_n(s))\dot{t}_n(s) + \dot{t}_n(s)\right) ds \le -\delta_n(x_n(1) - \bar{x}_n) + t_n(1) - \bar{t}_n\,,$$

because $u_{xx}(\gamma_n(s))\dot{t}_n(s) \le 0$.

Suppose $t_n(1) = t_1$. Then we have from (2.11-12)

$$u(\bar{x}_n, \bar{t}_n) \ge u(\gamma_n(1)) + \delta_n(x_n(1) - \bar{x}_n) + \bar{t}_n - t_n(1)\,. \tag{2.13}$$

Clearly for n large the right-hand side of (2.13) is bounded away from zero, which contradicts $u(\bar{x}_n, \bar{t}_n) \to 0$. On the other hand, if $t_n(1) > t_1$ and $x_n(1) = 1$, we have

$$u(\bar{x}_n, \bar{t}_n) \ge 1 + \delta_n(x_n(1) - \bar{x}_n) + \bar{t}_n - t_n(1)\,. \tag{2.14}$$

Again for n large the right-hand side is bounded away from zero, because $t_n \to t_0 > t_n(1)$. Thus (2.9) is impossible.

It remains to derive a contradiction if (2.8) holds for all large n. So suppose that all γ_n intersect $\{t = t_0\}$, say γ_n is parametrized by $s \in (-1, 1)$ with $t_n(0) = t_0$, $t_n(s)$ increasing for $s \in (-1, 0)$. Equivalently, we can parametrize $\gamma_n(s)$ for $s \in (-1, 0)$ by $x = \psi_n(t)$, where $t \in [t_n^{min}, t_0]$ with $t_n^{min} \ge t_1$.

Because different γ_n cannot intersect, and because $(\bar{x}_n, \bar{t}_n) \to (x_0, t_0)$, we can choose a subsequence of the sequence of curves (γ_n) which we again denote by (γ_n), such that $\{\psi_n\}$ is a decreasing sequence of functions.

We claim that $\psi_n(t_0) \downarrow \zeta(t_0)$. If not, then $\psi_n(t_0) \downarrow \tilde{x}_0 > \zeta(t_0)$, but $(\bar{x}_n, \bar{t}_n) = (\psi_n(\bar{t}_n), \bar{t}_n) \to (\zeta(t_0), t_0)$ as $n \to \infty$. But then, since $u_x(\psi_n(t), t) = -\delta_n \to 0$ as $n \to \infty$, we must have $u_x(x, t_0) = 0$ for $\zeta(t_0) \le x \le \tilde{x}_0$, contradiction. So $\psi_n(t_0) \downarrow \zeta(t_0)$. Clearly the definition of $\zeta(t_0)$ implies that for infinitely many n, $u_x(\cdot, t_0)$ must attain positive values between $\psi_n(t_0)$ and $\psi_{n+1}(t_0)$. This yields, by Sard's theorem again, the existence of infinitely smooth level curves $\tilde{\gamma}_n$ of u_x with $\tilde{\gamma}_n$ between γ_n and γ_{n+1}, and $u_x(\tilde{\gamma}_n(s)) > 0$.

The part of $\tilde{\gamma}_n$ going backwards from $t = t_0$ is again parametrized by t, i.e. $x = \tilde{\psi}_n(t)$. Choose any $t^* \in (t_1, t_0)$ such that $u_x(\zeta(t^*), t^*) > 0$ and $t^* > t_1^{min}$. Then $u_x(\cdot, t^*)$ must have infinitely many zeros in the interval

$$\left(\lim_{n\uparrow\infty} \psi_n(t^*), \psi_1(t^*)\right) \subset\subset (\zeta(t^*), 1)\,.$$

This contradicts the analyticity of u with respect to x, and so (2.8) is also impossible. $\square$

Proposition 2.13. *In Proposition 2.12, $u_x \ge 0$ in N can be replaced by $u_x > 0$ in N.*

Proof. By Proposition 2.12, for every $t > 0$ there exists an open rectangular neighbourhood N of $(\zeta(t), t)$ such that $u_x \ge 0$ in N. Clearly $u_x > 0$ in some point in N. In N, $p = u_x$ satisfies (in the distributional sense) $p_t \ge p_{xx}$. Since $u_x \not\equiv 0$ on N this implies $u_x > 0$ on N. $\square$

3. Existence and uniqueness

Proposition 3.1. *Let H1-3 hold. Then (P) has unique solution.*

Proof. (i) Uniqueness: Clearly by Propositions 2.1-3 we have $u_x \in C(\bar{Q}_T)$. Now suppose we have two solutions u_1, u_2. It is sufficient to show that $u_1 = u_2$ on Q_{t^*} for some $t^* > 0$. Let ζ_1, ζ_2 be the corresponding interfaces, and let $w = u_1 - u_2$. Then $\zeta_1(0) = \zeta_2(0) = \zeta_0$ and

$$w(x,t) = \int_0^t d\tau \int_{\zeta_1(\tau)}^{\zeta_2(\tau)} G(x,\xi,t,\tau)\, d\xi\,, \tag{3.1}$$

so that

$$|w(x,t)| \le \int_0^t |\zeta_2(\tau) - \zeta_1(\tau)| \frac{K}{\sqrt{t-\tau}}\, d\tau\,. \tag{3.2}$$

For t^* small, we have $u_{ix} \ge \delta > 0$ near ζ_0, hence

$$|\zeta_1(t) - \zeta_2(t)| \le \frac{|u_2(\zeta_1(t),t)|}{\delta} = \frac{|w(\zeta_1(t),t)|}{\delta} \le \frac{1}{\delta}||w||_{L^\infty(Q_t)}\,. \tag{3.3}$$

Let $y(t) = ||w||_{L^\infty(Q_t)}$. Then

$$y(t) \le \frac{K}{\delta}\, y(t) \int_0^t \frac{1}{\sqrt{t-\tau}} = \frac{2K}{\delta} y(t)\sqrt{t}\,, \tag{3.4}$$

implying $y(t) = 0$ for $t^2 < \frac{\delta}{2K}$. This completes the proof of uniqueness.

(ii) Existence: This follows from results in [LSU], but it can be derived easily by approximating H by a sequence of smooth functions H_n. $\square$

4. Smoothness of the interface

In this section we indicate the proof of Theorem 1.3. The strict positivity of u_x on the interface allows one to locally reconstruct the solution in a rectangular neighbourhood R of any point at the interface as the limit of a sequence of solutions u_n of the equations

$$u_t = u_{xx} + H_n(u)\,,$$

where H_n is a sequence of smooth functions approximating H. This can be done in such a way that u_{nx} is bounded away from zero uniformly in n on R. Then the following proposition follows from results in [GM1] which establish the Hölder continuity of u_t on rectangles strictly contained in R.

Proposition 4.1. *Let H1-2 be satisfied, and let u be a solution of (P). Then its interface satisfies $\zeta(t) \in C^{1+\alpha}((0,T])$ for some $\alpha \in (0,1)$.*

Proof of Theorem 1.3. We put $y = x/\zeta(t)$ and $z(y,t) \equiv u(\zeta(t)y,t)$. Then z solves

$$\begin{cases} z_t = z_{yy}/\zeta^2 + yz_y\dot{\zeta}/\zeta & \text{in } [0,1]\times[0,T] \\ z(y,0) = u_0(y\zeta(0)) & \text{in } [0,1] \\ z(0,t) = -1 & t \in [0,T] \\ z(1,t) = 0 & t \in [0,T] \end{cases}$$

The coefficients in this problem are smooth in x and as smooth in t as $\zeta'(t)$, while the boundary conditions are time independent. Standard bootstrap arguments allow one to increase the regularity of z and $\zeta(t)$ step by step for $t > 0$ so that $\zeta(t)$ is smooth. $\square$

Appendix: Similarity Solutions

We consider similarity solutions of (1.1) of the form (1.2). Writing $\eta = x/\sqrt{t}$ we find

$$f - \frac{1}{2}\eta f' - f'' = H(f)\,. \tag{A1}$$

Since we are interested in symmetric solutions which change sign, we restrict ourselves to solutions $f(\eta)$ satisfying (1.3), which implies $H(f) = 1$ near $\eta = 0$. Thus, as long as $f(\eta) > 0$, we have

$$f(\eta) = \alpha - \frac{1-\alpha}{2}\eta^2\,, \tag{A2}$$

which can be verified by direct substitution. Let

$$\lambda = \sqrt{\frac{2\alpha}{1-\alpha}}\,. \tag{A3}$$

For $\eta > \lambda$, $f(\eta)$ becomes negative, and, as long as it remains so, we have

$$f'' + \frac{1}{2}\eta f' - f = 0\,. \tag{A4}$$

We set

$$w(\eta) = \frac{-f(\eta)}{(\eta^2+2)\sqrt{2\alpha(1-\alpha)}}\,. \tag{A5}$$

Then $w(\eta)$ satisfies

$$(\eta^2+2)w''+(4\eta+\frac{1}{2}\eta(\eta^2+2))w'=0 \text{ for } \eta>\lambda\,, \tag{A6}$$

while

$$w(\lambda)=0 \text{ and } w'(\lambda)=1/(\lambda^2+2)\,, \tag{A7}$$

so that

$$w(\eta)=(\lambda^2+2)e^{\lambda^2/4}\int_\lambda^\eta (s^2+2)^{-2}e^{-s^2/4}\,ds\,. \tag{A8}$$

Clearly

$$w(\infty)=\lim_{\eta\uparrow\infty} w(\eta)=(\lambda^2+2)e^{\lambda^2/4}\int_\lambda^\infty (s^2+2)^{-2}e^{-s^2/4}\,ds \tag{A9}$$

exists so that

$$\lim_{\eta\uparrow\infty}\frac{f(\eta)}{\eta^2}=-c(\lambda)\,, \tag{A10}$$

where, using (A3) again,

$$c(\lambda)=2\lambda e^{\lambda^2/4}\int_\lambda^\infty (s^2+2)^{-2}e^{-s^2/4}\,ds\,. \tag{A11}$$

As α runs from zero to one, λ runs from zero to $+\infty$. Clearly c tends to zero as λ tends to zero or infinity, so that for small $c>0$ we have at least two different values of λ and α. For the similarity solution we have

$$\begin{aligned} u(x,t)&>0 \text{ for } |x|<\lambda\sqrt{t}\\ u(x,t)&=0 \text{ for } |x|=\lambda\sqrt{t}\\ u(x,t)&<0 \text{ for } |x|>\lambda\sqrt{t} \end{aligned}$$

while

$$\lim_{t\downarrow 0} u(x,t)=\lim_{t\downarrow 0} tf\left(\frac{x}{\sqrt{t}}\right)=-c(\lambda)x^2\,, \tag{A12}$$

so that $u\in C(\mathbf{R}\times[0,\infty))$.

Conclusion

The results of this paper indicate that solutions of the semilinear heat equation with a Heaviside source term are generally well behaved, except for initial profiles which either are identically zero, or touch zero from below. In the latter case nonuniqueness may occur in a way which is not yet understood, and needs a closer investigation. We note that it is very well possible that solutions of this equation with more general boundary data develop such profiles. Local uniqueness certainly holds if all sign changes of the initial data are transversal.

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