THE DIPOLE SOLUTION FOR THE POROUS MEDIUM EQUATION IN SEVERAL SPACE DIMENSIONS

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Dedicated to the memory of P. de Mottoni

ABSTRACT

We establish the existence and uniqueness of the Dipole Solution of the porous medium equation (PME), i.e. the solution of

$$u_t = \Delta(|u|^{m-1}u), \qquad \text{in } Q = \{(x,t) : x = (x_i) \in R^N,\, t > 0\},$$

with $m > 1$ and initial value

$$u(x,0) = M \frac{\partial}{\partial x_1} \delta(x),$$

where δ is Dirac delta function in $\mathbf{R}^N$ and $M \neq 0$. It has the self-similar form $\overline{u}(x,t) = t^{-\alpha} U(xt^{-\beta})$, with exponents α and β determined from dimensional considerations. The profile U has compact support. The proof uses the so-called dual porous medium equation, $z_t = |\Delta z|^{m-1} \Delta z$, and depends on the existence of a certain type of radially symmetric and self-similar solution in dimension $N - 1$.

We also prove that any solution of the PME in a half-space $H = \{x : x_1 > 0\}$ with zero boundary data on ∂H converges as $t \to \infty$ to the Dipole solution with the same moment.

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Introduction

In this paper we address the existence of special solutions $u(x,t)$ of the porous medium equation (PME)

$$u_t = \Delta(|u|^{m-1}u), \quad m > 1, \tag{0.1}$$

posed in $Q = \{(x,t) : x = (x_i) \in R^N,\, t > 0\}$. We work in any space dimension $N \geq 1$ and consider solutions with changing sign. Our main interest is to find a solution to the initial value problem with data

$$u(x,0) = -\frac{\partial}{\partial x_1}\delta(x), \tag{0.2}$$

where δ is Dirac delta function in $\mathbf{R}^N$. This is called a *Dipole Solution* (the minus sign is inserted so as to make the solution positive for $x_1 > 0$. Of course, any rotation in space gives an equivalent Dipole Solution). We prove that such a solution exists and has the self-similar form

$$\overline{u}(x,t) = t^{-\alpha}U(xt^{-\beta}), \tag{0.3}$$

with

$$\alpha = \frac{N+1}{(N+1)m + (1-N)} \quad \text{and} \quad \beta = \frac{1}{(N+1)m + (1-N)}. \tag{0.4}$$

These exponents can be obtained from dimensional considerations: we only have to substitute formula (0.3) into the equation to obtain $(m-1)\alpha + 2\beta = 1$. Enforcing that the moment $\int x_1\, u(x,t)\, dx$ be a conserved quantity implies $\alpha = (N+1)\beta$. So we get (0.4).

The problem can be rephrased as the determination of the profile $U(y)$, which is a nontrivial and nonnegative solution of the nonlinear elliptic equation

$$\Delta(|U|^{m-1}U) + \beta y \cdot \nabla U + \alpha U = 0 \tag{0.5}$$

in $H = \{y \in \mathbf{R}^N : y_1 > 0\}$, with boundary conditions

$$U(0, y_2, \cdots, y_N) = 0. \tag{0.6}$$

We will show that U has compact support. If we write $y \in \mathbf{R}^N$ in the form (y_1, y') with $y' = (y_2, \cdots, y_N)$ we find that U is a function of y_1 and $|y'|$. In particular, it is symmetric in the variables y_j for $j \neq 1$. It can also be extended in an antisymmetric way to the half-space $y_1 < 0$ to form a solution in the whole space with changing sign.

For the Heat Equation, $u_t = \Delta u$, the dipole solution can be obtained as (minus) the derivative with respect to x_1 of the fundamental solution. When $m > 1$ and the space dimension is one an explicit Dipole Solution was found by Barenblatt and Zel'dovich in 1957 [BZ]. One way of obtaining it is as follows: we can intregrate the PME to obtain another evolution equation of degenerate parabolic type, $w_t = (|w_x|^{m-1} w_x)_x$, which admits an explicit solution of point-source type. Its derivative in x provides us with a Dipole Solution for the PME.

This method does not work in dimension N greater than one, because integration in x_1 does not lead to a simple equation. Our results are based on the study of the equation

$$z_t = |\Delta z|^{m-1} \Delta z \tag{0.7}$$

which we shall call the *dual* equation, because it can be obtained from the PME after applying the operator $(-\Delta)^{-1}$, and setting $-\Delta z = u$. A study of this equation in $N = 1$ was done in [BHV]. See Appendix below for further details.

Finally, let us remark that once the existence of a dipole solution of the form (0.3) with initial data (0.2) has been established, it is not difficult to see that there is also a dipole solution of the form (0.3) with initial data given by

$$u(x,0) = -M \frac{\partial}{\partial x_1} \delta(x)\,, \tag{0.8}$$

where M can be any positive constant. This follows from the fact that when U is a solution of (0.5), so is U_λ, defined by

$$U_\lambda(y) = \lambda^{\frac{2}{m-1}} U(y/\lambda)\,. \tag{0.9}$$

Then λ and M are related by $M = \lambda^{\frac{m+1}{m-1}}$.

A second part of our paper concerns the use of the Dipole solution to describe the asymptotic behaviour of the solutions of the initial and boundary problem for the PME

$$(P) \quad \begin{cases} u_t = \Delta(|u|^{m-1}u)\,, \quad \text{in} \quad Q^+ = H \times (0,\infty) \\ u(x,0) = u_0(x) \qquad \text{for} \quad x \in \overline{H} \\ u(x,t) = 0 \quad \text{on} \quad \Sigma = \partial H \times [0,\infty)\,. \end{cases}$$

Of course, we can also think of the problem as the PME in the whole of Q with initial data antisymmetric in x_1, namely $u_0(-x_1, x') = -u_0(x_1, x')$.

There is a conservation law associated to the solutions u of this problem, namely the invariance in time of the first moment

$$\int_H x\, u(x,t)\, dx = \text{constant}. \tag{0.11}$$

This allows us to prove that any solution of problem (P) with nonnegative and integrable initial data with compact support converges as $t \to \infty$ to the Dipole solution having the same moment.

The plan of the paper is as follows: Sections 1 to 6 are devoted to the construction and properties of the Dipole solution. After posing the problem and stating the existence and uniqueness result in Section 1, we prove uniqueness in Section 2 by a Lyapunov-functional argument involving the first moment. The construction of the solution is begun in Section 3 by introducing suitable approximations which form a monotone sequence. The construction is interrupted in Section 4 to study the radially symmetric and self-similar solutions of the PME with one sign change and compact support, for which we prove that the exponent α is anomalous. This extends results obtained in [BHV] in collaboration with Francisco Bernis. It is very curious that such a result can be used to construct a supersolution for our problem, and thanks to this the construction of the Dipole solution is finished in Section 5. Section 6 establishes that its support is compact in space. Section 7 addresses the asymptotic behaviour as $t \to \infty$. Finally, we gather in the Appendix some facts concerning the dual equation.

Let us finally comment on a related question. Can we have radially symmetric self-similar solutions having at $t = 0$ some kind of dipole singularity? It follows from the phase-plane analysis of radially symmetric solutions of (0.1), as done for instance in [H], that no such changing sign solutions exist, even if, in order to avoid the singularity at $r = 0$, we allow ourselves to take out of the domain of definition of U a small neighbourhood of the origin.

1. The Dipole Solution. Preliminaries and statement of results

Possibly the single most important solution of the Porous Medium Equation is the so-called *point-source solution*, which is a nonnegative solution taking as initial data a Dirac mass,

$$u(x,0) = M\delta(x)\,. \tag{1.1}$$

It is fortunate that this solution has an explicit form, [Ba], [ZK], [P], namely

$$u_1(x,t) = t^{-\alpha}\left(C - b|\xi|^2\right)_+^{\frac{1}{m-1}} \tag{1.2}$$

where $\xi = x\,t^{-\beta}$ and

$$\alpha = \frac{N}{N(m-1)+2}\,, \quad \beta = \frac{1}{N(m-1)+2}\,, \quad \text{and} \quad b = \frac{\beta(m-1)}{2m}\,, \tag{1.3}$$

while $C > 0$ is an arbitrary constant which can be explicitly determined in terms of the mass $M = \int u(x,t)\,dx$. The solution is usually known as the Barenblatt solution. It is clear from the formula that u_1 has compact support in x for every $t > 0$. This reflects the basic property of Finite Speed of Propagation, a consequence of the degenerate character of the equation.

The Barenblatt solution plays an important role in the theory of solutions of the Cauchy Problem for the PME with integrable and nonnegative initial data. In particular, the large-time behaviour of any solution in this class is always a Barenblatt profile. In this respect, it plays the same role the fundamental solution plays for the heat equation.

Let us consider now solutions with changing sign. In the case of the heat equation the asymptotic behaviour is still given by the fundamental solution as long as the initial mass $M = \int u_0(x)\,dx$ is not zero. However, when $M = 0$ we have a completely new situation, described by the next term in the asymptotic development. The second term is given by a multiple of the so-called *dipole solution*, which is obtained simply (after a convenient rotation) as the derivative with respect to x_1 of the fundamental solution. It thus reads

$$u_2(x,t) = \frac{x_1}{2t(4\pi t)^{N/2}} \exp(-\frac{x^2}{4t}) \,. \tag{1.4}$$

Further terms of the asymptotic development can be obtained by repeated differentiation of the fundamental solution. All these solutions have exponential decay in x.

It is natural to ask whether a similar situation holds for the Porous Medium Equation. To begin with, if we consider solutions with initial data $u_0 \in L^1(\mathbf{R}^N)$ and mass $\int u_0(x)\,dx > 0$, then the asymptotic behaviour is still given by the Barenblatt solution. Moreover, if the data have compact support then the solution becomes even nonnegative in a finite time, [KV]. The main concern is hence to investigate whether there exist solutions which can represent the behaviour of general solutions with zero mass.

This situation has been investigated recently in one space dimension. The analogon to the sequence of derivatives of the fundamental solutions is here represented by a sequence of self-similar solutions of the the form

$$u(x,t) = t^{-\alpha} U(x/t^{\beta}) \,, \tag{1.5}$$

where U has compact support. In order to satisfy the equation the exponents α and β must be related by

$$2\beta + (m-1)\alpha = 1 \,. \tag{1.6}$$

Introducing as parameter $k = \alpha/\beta$ we may write (1.6) as

$$\alpha = \frac{k}{k(m-1)+2} \quad \text{and} \quad \beta = \frac{1}{k(m-1)+2} \,. \tag{1.7}$$

In the heat equation case $m = 1$ so that $\beta = 1/2$ and $\alpha = k/2$. The sequence of solutions we are considering corresponds to all the positive integer values of k, $k = 1, 2, 3, \cdots$.

In analogy to the requirement of exponential decay for the heat equation, we have here the condition of compact support for the profiles U. Such solutions have been completely classified in [H], where it is shown that there exists a sequence

$$k_1 < k_2 < k_3 < k_4 < k_5 < \ldots. \tag{1.8}$$

going to infinity and such that there exists a compactly supported solution U if and only if $k = k_n$ for some integer $n \geq 1$. The solution $U = U_n$ is unique up to scaling. Moreover, U_n is even (odd) for n odd (even), and has exactly $n-1$ sign changes. The first value of k is again $k_1 = 1$ and U_1 corresponds to the Barenblatt solution.

For the second function we have $k_2 = 2$, U_2 is explicit, and its corresponding similarity solution u_2 has the first order derivative of the Dirac measure as trace at $t = 0$, thus originating the name of *dipole solution*. This solution, found in [BZ], happens to be the derivative in space of the fundamental solution of the equation

$$w_t = (|w_x|^{m-1} w_x)_x\,, \tag{1.9}$$

which is obtained from the PME by integration

$$w(x,t) = \int_{-\infty}^{x} u(s,t)\,ds\,. \tag{1.10}$$

The dipole solution was shown in [KV] to represent the large-time behaviour of solutions of the PME in 1-D with zero mass and nonzero first moment

$$\int x u_0(x)\,dx\,. \tag{1.11}$$

The main goal of this paper is to prove the following N-dimensional result

Theorem A. *There exists a selfsimilar solution $\overline{u}(x,t) = t^{-\alpha}U(xt^{-\beta}) \geq 0$ of the PME with compact support in space which takes initial data of dipole type, namely such that*

$$\lim_{t\to 0} \int \overline{u}(x,t)\phi(x)\,dx = \frac{\partial \phi}{\partial x_1}(0)\,. \tag{1.12}$$

α and β are given by (0.4), i.e. the parameter k is $N+1$. Such a solution is unique if we demand that $U(y) = 0$ for $y_1 = 0$.

In addition to the existence and uniqueness of $\overline{u}$, we also investigate the main properties of our solution and its interface.

Our proof partially relies on the knowledge of a particular positive self-similar solution $z^*(x,t)$ of (0.7) in space dimension $N' = N-1$ having as initial data $z_0^*(x) = |x|^{-N'}$, and which is bounded but not integrable for $t > 0$. In fact the parameters in this similarity solution are such that had the solution been integrable, its initial data would have been a

Dirac mass. It is well known that the heat equation can not handle such initial data which are not locally integrable.

For $N' = 1$ this striking difference between the heat equation and (0.7) is a consequence of what we had proved in [BHV], namely that $k_3 > 3$ for $m > 1$. Thus the third member of the sequence of solutions u_k, u_3, does not correspond to initial data $\delta''(x)$. Consequently z_3 ($z_3'' = -u_3$) does not take δ as initial data. For exponent $k = 3$ we obtain a symmetric solution u^* decaying in x like $O(|x|^{-3})$, which takes on the initial data $|x|^{-3}$ and not $\delta''(x)$ as happens in the heat equation. Taking the second primitive of this solution we obtain a solution z^* of (0.7) with initial data $|x|^{-1}$. It is because of this anomalous behaviour for $N' = 1$ that we can settle the proof of existence of a dipole solution in $N = 2$.

In order to deal with the dimensions $N \geq 3$ we need to generalize the above one-dimensional results for (1.5) to several space dimensions. Then one can look for radial and nonradial solutions U. The compactly supported radial solutions have been also classified in [H]. Basically, the result is the same as for the even solutions in one space dimension, with α and β given in terms of k by (1.7). Then there exists a sequence

$$k_1(N) < k_3(N) < k_5(N) < k_7(N) < \dots, \tag{1.13}$$

such that there exists a compactly supported radial solution $U = U_n(|y|)$ if and only if $k = k_n(N)$ for some odd $n \in N$, the number of sign changes being $(N-1)/2$. Up to scaling U_n is unique. Again the first one corresponds as in the heat equation to $k_1 = N$ and u_1 is the explicit Barenblatt solution.

In Section 4 we generalize the results of [BHV] to show that the second solution u_3 corresponds to an anomalous exponent $k_3 > N+2$ if $m > 1$. Now, for exponent $k = N+2$ we obtain a radially symmetric solution decaying in space as $O(|x|^{-(N+2)})$, with initial data $|x|^{-(N+2)}$ and not $\Delta\delta(x)$ as in the case of the heat equation. Taking the Newton potential of this solution we obtain a self-similar solution of (0.7) with initial data $|x|^{-N}$.

Remark. We record here the curious fact that the dipole solution in one dimension belongs to a family of explicit self-similar solutions containing the dimension N as a parameter. They are given by the formulas

$$\alpha = \frac{1}{m}, \quad \beta = \frac{1}{2m}, \quad \text{and}$$
$$U(y) = |y|^{\frac{2-N}{m}}\Big(C - \frac{m-1}{N(m-1)+2}|y|^{N+\frac{2-N}{m}}\Big)_+^{\frac{1}{m-1}}.$$

While for $N = 1$ this solution is a dipole solution, it is not for $N > 1$. Thus, for $N = 2$ it is the Barenblatt point-source solution. For $N > 3$ it is not even a solution in any neighbourhood of $x = 0$, due to the existence of a singularity in $y = 0$ which continues for all times $t > 0$.

2. Uniqueness of the dipole solution. A conservation law

We begin by stating a basic conservation law valid for solutions of the PME in a half-space.

Lemma 2.1. *For every solution of the PME in $Q^+ = H \times (0, \infty)$ with zero boundary data and compact support we have*

$$\int_H x\, u(x,t)\, dx = constant\,. \tag{2.1}$$

Proof. Integrate by parts. $\square$

The vector quantity $\int x\, u(x,t)\, dx$ is called the (first) moment. Note that for solutions which are symmetric with respect to some coordinate variable x_i the moment has i-th component equal to zero. Therefore, we will only be interested in the first component of the moment, $\int x_1\, u(x,t)\, dx$.

A second observation concerns the requirement of compact support. This can easily be weakened to the requirement of finite integral $\int |x| u(x,t)\, dx$.

This moment conservation law plays an important role in the proof of uniqueness and also in the determination of the asymptotic behaviour of general solutions of the initial and boundary value problem.

The uniqueness of the Dipole Solution is based on the following Lemma.

Lemma 2.2. *Let u_1 and u_2 be two self-similar solutions of the PME of the form (0.3), (0.4), taking on boundary data $u_i(x,t) = 0$ for $x_1 = 0$, $x' \in \mathbf{R}^{N-1}$ and $t \geq 0$, and having compact support. If the moments have the same first component then the solutions coincide.*

Proof. We define $\hat{u} = \max\{u_1, u_2\}$ and $\check{u} = \min\{u_1, u_2\}$, both max and min being defined pointwise. Clearly,

$$\hat{u} - \check{u} = |u_1 - u_2|\,. \tag{2.2}$$

We also have, due to the self-similar form (0.3), (0.4) of u_1 and u_2:

$$\int_H x_1 |u_1(x,t) - u_2(x,t)|\, dx = c, \tag{2.3}$$

where c is positive constant, i.e. it is time-independent.

Assume now that the moments of u_1 and u_2 have the same first component, and do not coincide. Since solutions of the PME eventually cover the whole space domain H, for large times their supports have to overlap at least in part. Because of the selfsimilarity this must then hold for all times. Moreover, there must exist a set Z where both solutions are positive and coincide, otherwise one should be larger then the other, which contradicts the equality of moments.

Consider now the solutions u_3, u_4, which start at time $t = 1$ with value $\hat{u}$, $\check{u}$ respectively. Namely

$$
\begin{aligned}
u_3(x,1) &= \max\{u_1(x,1), u_2(x,1)\}, \\
u_4(x,1) &= \min\{u_1(x,1), u_2(x,1)\}\,.
\end{aligned}
\tag{2.4}
$$

By the maximum principle we have in $H \times (1, \infty)$

$$u_4 \le \check{u} \le u_1,\ u_2 \le \hat{u} \le u_3\,.$$

Moreover, by the Strong Maximum Principle u_3 and u_4 cannot coincide at any point of positivity, contrary to what happens to $\hat{u}$ and $\check{u}$ in Z. Consequently, $u_3 - u_4 \ge \hat{u} - \check{u}$ and both differences are not identical. Hence, for $t > 1$

$$
\begin{aligned}
&\int_H x_1\left(u_3(x,t) - u_4(x,t)\right) dx > \int_H x_1\left(\hat{u}(x,t) - \check{u}(x,t)\right) dx \\
&\quad = \int_H x_1 |u_1(x,t) - u_2(x,t)|\, dx = c\,.
\end{aligned}
$$

However, since both u_3 and u_4 are solutions with compact support we have

$$
\begin{aligned}
&\int_H x_1\left(u_3(x,t) - u_4(x,t)\right) dx = \int_H x_1\left(u_3(x,1) - u_4(x,1)\right) dx \\
&\quad = \int_H x_1\left(\hat{u}(x,1) - \check{u}(x,1)\right) dx = c\,.
\end{aligned}
$$

This contradiction discards the possibility of noncoincidence. $\square$

Remark. As a consequence of the above argument, two different self-similar solutions of the form (0.3)-(0.4) cannot intersect. It follows that the dipole solutions U_λ defined by (0.9) form a strictly monotone family which is increasing in λ. Their supports are also a monotone family of subsets of H.

3. Construction of the solution

In this section we establish the existence of the dipole solution as a limit of solutions obtained from one particular solution by a suitable scaling. We use the fact that a solution of

$$z_t = |\Delta z|^{m-1} \Delta z \tag{3.1}$$

yields a solution of PME if we set $u = -\Delta z$ and vice versa. Let $\Gamma(x)$ be the fundamental solution of Laplace's equation, i.e. $\Delta\Gamma(x) = \delta(x)$. The dipole solution of PME which we seek corresponds to the solution of (3.1) with initial data given by

$$\overline{z}_0(x) = \frac{\partial\Gamma}{\partial x_1} = c_N \frac{x_1}{|x|^N} = c_N \frac{\cos\phi_1}{r^{N-1}}\,, \tag{3.2}$$

where $c_N = 1/(\text{meas}\,\{|x| = 1\})$. The notation $x_1 = r\cos\phi_1, x_2 = r\sin\phi_1\cos\phi_2, \cdots$ stands for the generalized polar coordinates.

Step 1. Approximate problems. We truncate the singularity at $r = 0$ as follows. We consider the sequence of functions

$$z_{0n}(x) = c_N \psi_n(r)\cos\phi_1\,, \quad n = 1, 2, \cdots \tag{3.3}$$

for a certain function $\psi_n(r)$ determined from $\psi_1(r)$ by

$$\psi_n(r) = n^{N-1}\psi_1(nr) \tag{3.4}$$

so that $z_{0n}(x) = n^{N-1} z_{01}(nx)$. We want to take

$$\psi_1(r) = \min(r, \frac{1}{r^{N-1}})\,. \tag{3.5}$$

Clearly in that case

$$\psi_n(r) \uparrow \frac{1}{r^{N-1}} \tag{3.6}$$

for all $r > 0$. Because of the jump in ψ_n'' we have

$$\Delta\psi_n(r)\cos\phi = -N\,n^N \delta(r - \frac{1}{N})\cos\phi\,, \tag{3.7}$$

which means that for any smooth test function

$$< \Delta z_{0n}, \chi >= -c_N\,N\,n^N \int_{r=1/N} \chi\,\cos\phi$$

In order to avoid working with measures we smooth down the function ψ_1 given in (3.5) into a C^∞ function by making a perturbation in a small neighbourhood of the value $r = 1$ in such a way that $\Delta z_{01} \le 0$ and the z_{0n} form a monotone increasing sequence of functions.

Step 2. Approximate solutions. We now solve equation (3.1) with initial data (3.3). In order to do that we set

$$-\Delta z_n = u_n$$

and solve the PME for u_n. See Appendix. We thus obtain unique classical solutions $z_n(x,t)$ by taking the Newtonian potential of the $-u_n$.

Since the initial data are antisymmetric in x_1 and symmmetric in x_i for $i = 2, \cdots, N$, we conclude that the same property holds for the solutions $z_n(\cdot, t)$ at any time $t > 0$. In particular, we will have $z_n(x,t) = 0$ for $x_1 = 0$.

The Maximum Principle holds for equation (3.1) posed in the half-space $H = \{x_1 > 0\}$, therefore $z_n(x,t) \geq 0$ for $x_1 > 0$. We also see that the sequence z_n is increasing in $Q^+ = H \times (0,\infty)$.

Moreover, since $u_{0n} = -\Delta z_{0n} \geq 0$ in H and $u_{0n} = 0$ on the boundary $x_1 = 0$ we have $u_n(x,t) \geq 0$ for every $x \in H$ and every $t > 0$. This means that

$$\frac{\partial}{\partial t} z_n = |\Delta z_n|^{m-1} \Delta z_n \leq 0 \quad \text{in} \quad H\,.$$

Therefore, the functions $z_n(x,t)$ satisfy

$$z_n(x,t) \leq z_{0n}(x) \leq z_0(x)\,. \tag{3.8}$$

We see that the bound does not depend on n.

Step 3. The limit. Self-similarity. In view of these properties we may conclude that the sequence z_n converges to a limit which we denote by z, so

$$z_n(x,t) \uparrow \overline{z}(x,t) \quad \text{in} \quad Q^+ = H \times (0,\infty)\,, \tag{3.9}$$

and $0 \leq \overline{z}(x,t) \leq z_0(x)$ in Q. A number of properties are also easily deduced from the construction. Thus, $\overline{z}(x,t)$ takes on the boundary values $\overline{z}(x,t) = 0$ for $x_1 = 0$. Also $\overline{z}(x,t)$ is symmetric in the variables x_i, $i = 2, \cdots, N$ and

$$\overline{z}_t(x,t) \leq 0 \tag{3.10}$$

at least in distribution sense. We shall see later that this inequality is true everywhere and $\overline{z}_t$ is a continuous function.

A basic property of $\overline{z}$ is selfsimilarity. In order to establish this property we use a similarity-invariance argument, based on the existence of a scaling group which transforms solutions into solutions. In our case we define the transformation

$$(\mathcal{T} z)(x,t) = \lambda^{(N-1)\beta} z(\lambda^\beta x, \lambda t) \tag{3.11}$$

with $\beta = ((N+1)m + (1-N))^{-1}$. The exponents are chosen in such a way that not only $\mathcal{T} z$ is a solution of (3.1) whenever u is, but also the initial data $\overline{z}_0(x)$ are invariant in the transformation. Regarding the approximating sequence $z_n(x)$ we have

$$(\mathcal{T} z_n)(x,0) = z_{n\lambda^\beta}(x,0) \tag{3.12}$$

By uniqueness of bounded solutions of equation (3.1) it follows that

$$z_{n\lambda^\beta}(x,t) = (\mathcal{T} z_n)(x,t) \quad \text{in} \quad H \times (0,\infty)\,. \tag{3.13}$$

and passing to the limit $n \to \infty$ we get

$$\overline{z}(x,t) = (\mathcal{T}\,\overline{z})(x,t) = \lambda^{(N-1)\beta}\overline{z}(\lambda^\beta x, \lambda t)\,,$$

which holds for every (x,t) and every $\lambda > 0$. Fixing x and t and letting $\lambda = 1/t$ we get

$$\overline{z}(x,t) = t^{-(N-1)\beta} Z(xt^{-\beta}) \quad \text{with} \quad Z(x) = \overline{z}(x,1)\,. \tag{3.14}$$

Step 4. The limit is nontrivial. The dipole solution we are looking for will be

$$\overline{u}(x,t) = -\Delta \overline{z}(x,t) \tag{3.15}$$

Of course, $\overline{u}$ will inherit from $\overline{z}$ the self-similar form (0.3)

$$\overline{u}(x,t) = t^{-\alpha} U(xt^{-\beta})\,,$$

with $\alpha = (N+1)\beta$ and β given by (0.4). But before we proceed we have to make sure that $\overline{z}(x,t)$ does not coincide with $\overline{z}_0(x)$, since in this case we would obtain $\overline{u}(x,t) \equiv 0$ in $Q^+ = H \times (0,\infty)$, the trivial solution.

We can exclude this by showing that the function $Z(x)$ is bounded. We shall establish this crucial estimate as a consequence of the results of the next Section. To end the construction we also have to show that $\overline{z}$ is a solution of (3.1) and $\overline{u}$ is a solution of the PME.

4. On radially-symmetric self-similar solutions of the PME

We have explained above that when trying to obtain radial selfsimilar solutions with compact support for the PME we find a sequence of parameters $k_n \to \infty$ for which such solutions exist, [H]. Moreover, it is proved in [BHV] that in one space dimension the sequence begins (as in the corresponding sequence for the heat equation) with $k_1 = 1$, $k_2 = 2$, but the third value is anomalous if $m \neq 1$, and in particular $k_3 > 3$, when $m > 1$. Indeed, it tends to 4 as $m \to \infty$.

Here we extend this result to several space dimensions. Thus, we will show

Theorem 4.1. *The value $k = k_3(N)$ for which we find a compactly supported radial solution of the PME of the form (1.5)-(1.7) with one sign change satisfies for $m > 1$ the inequality*

$$k_3(N) > N + 2\,. \tag{4.1}$$

Proof. (i) We follow the lines of the proof in [BHV] that $k_3 > 3$ in dimension one. Thus, we write $k = k_3(N)$, $U = U_3$, and normalize U by $U(0) = 1$. We also use as independent variable $r = \beta^{1/2}|y|$ since the extra factor $\beta^{1/2}$ simplifies the formulas (4.3)-(4.5) below.

As in [BHV] there exist unique numbers $0 < a < A < \infty$ such that $U > 0$ on $[0, a)$, $U(a) = 0$, $U < 0$ on (a, A), and $U = 0$ on $[A, \infty)$. Let $V(r)$, $W(r)$, and $Z(r)$ be defined by

$$\left\{ \begin{array}{l} V = |U|^{m-1}U\,, \quad W(r) = -\int_r^A s^{N-1}u(s)ds \\ \quad \text{and} \quad Z(r) = \int_r^A s^{1-N}W(s)ds\,. \end{array} \right. \tag{4.2}$$

Note that this means that $U = -\Delta Z$. We have

$$-(r^{N-1}V')' = r^N U' + kr^{N-1}U\,. \tag{4.3}$$

A first integration yields

$$-r^{N-1}V' = rW' + (k - N)W\,. \tag{4.4}$$

In particular, $(k - N)W(0) = 0$, so that in view of $k > N$, $W(0) = 0$. Consequently W is increasing on $(0, a)$ from zero to its maximum value σ in $r = a$ and then decreasing on (a, A) to zero in $r = A$. Hence Z is positive on $[0, A)$. Furthermore $(k - N)Z' = V + r^{2-N}W' = V' + rU$, so that $Z'(0) = 0$. Dividing (4.4) by r^{N-1} and integrating again gives

$$V = rZ' + (k - 2)Z\,, \tag{4.5}$$

implying $Z(0) = 1/(k - 2)$. A final integration leads to

$$\int_0^A r^{N-1}V(r)dr = (k - 2 - N)\int_0^A r^{N-1}Z(r)dr\,, \tag{4.6}$$

which shows that $k > N + 2$ if

$$\int_0^A r^{N-1}V > 0\,. \tag{4.7}$$

(ii) We still have to prove that (4.7) holds. Again we follow the proof for $N = 1$ in [BHV]. Because of the properties of W there exist functions $V_1(W)$ and $V_2(W)$ defined on $[0, \sigma]$ such that

$$V(x) = \left\{ \begin{array}{lll} V_1(W(x)) & \text{for} & x \in [0, a] \\ V_2(W(x)) & \text{for} & x \in [a, A]\,. \end{array} \right. \tag{4.8}$$

Then, because $W'(r) = r^{N-1}U(r) = r^{N-1}|V(r)|^{(1/m)-1}V(r)$,

$$\begin{aligned}\int_0^A r^{N-1}V(r)dr &= \int_0^a r^{N-1}V(r)dr + \int_a^A r^{N-1}V(r)dr \\ &= \int_0^\sigma |V_1(W)|^{1-1/m}dW - \int_0^\sigma |V_2(W)|^{1-1/m}dW\,,\end{aligned} \tag{4.9}$$

which will be larger than zero if we can show that

$$V_1(W) + V_2(W) > 0\,. \tag{4.10}$$

Observe that $V_1(0) + V_2(0) = 1$, and that $V_1(\sigma) + V_2(\sigma) = 0$. Thus if (4.10) is false, there must be a value $W = W^* \in (0, \sigma)$ where $V_1 + V_2$ has a negative minimum. Then there exist unique $0 < r_1 < a < r_2 < A$ with $W^* = W(r_1) = W(r_2)$, so that at $W = W^*$

$$-dV_1/dW = -V'(r_1)/W'(r_1) = r_1^{2-N} + (k-N)r_1^{2(1-N)}W^*|V_1|^{-1/m}\,, \tag{4.11}$$

and similarly

$$-dV_2/dW = -V'(r_2)/W'(r_2) = r_2^{2-N} - (k-N)r_2^{2(1-N)}W^*|V_2|^{-1/m}\,. \tag{4.12}$$

Adding (4.11) to (4.12) we obtain, setting $V^* = V_1(W^*) < |V_2(W^*)|$ and $r_2 > r_1$,

$$\begin{aligned}0 =& r_1^{2-N} + r_2^{2-N} + (k-N)W^*(r_1^{2(1-N)}V_1(W^*)^{-1/m} - r_2^{2(1-N)}|V_2(W^*)|^{-1/m}) \\ >& (k-N)W^*(r_1^{2(1-N)} - r_2^{2(1-N)})V^{*-1/m} > 0\,,\end{aligned} \tag{4.13}$$

contradiction. This completes the proof of (4.1). □

Proposition 4.2. *The radial solutions $\tilde{U}$ of (0.5) with $k = N+2$ have one sign change and behave as $|y| \to \infty$ as*

$$\tilde{U}(y) \sim C|y|^{-(N+2)}\,. \tag{4.14}$$

Proof. In the analysis of [H] radial solutions of (0.5) have precise power decay when k does not belong to the special 'eigenvalue' sequence $k_n(N)$. It is givn by

$$\lim_{r\to\infty} \frac{rU'(r)}{U(r)} = -k\,.$$

Using standard arguments, cf. [GP1-3], this limit can be improved to the form (4.14). □

The decay in time of the similarity solution corresponding to $k = N+2$ is given, cf. (1.7), by

$$\tilde{u}(x,t) \sim t^{-(N+2)/(m(N+2)-N)}\,. \tag{4.15}$$

Taking $\tilde{z}(x,t) = (-\Delta)^{-1}\tilde{u}(x,t)$ we obtain a radially symmetric similarity solution of equation (3.1), which for any fixed $t > 0$ is strictly positive (or negative) and which is asymptotically equal to

$$\tilde{z}(x,t) \sim |x|^{-N} . \tag{4.16}$$

Furthermore the decay rate for $t \to \infty$ of $\tilde{z}$ is given by

$$\tilde{z}(0,t) = C\, t^{-N/(m(N+2)-N)} . \tag{4.17}$$

We now take space dimension $N-1$ with space varibles $x' = (x_2, \cdots, x_N)$ and consider the corresponding solution, i.e. the selfsimilar and radially symmetric solution $\tilde{u} = \tilde{u}(x',t)$ of the PME with $k = N+1$. The decay rate in space is then

$$\tilde{z}(x',t) \sim |x'|^{1-N} , \tag{4.18}$$

while in time we have

$$\tilde{z}(0,t) = C\, t^{-(N-1)/(m(N+1)+1-N)} . \tag{4.19}$$

This latter rate is exactly what we have in (3.14) for $\overline{z}$. We think of $\tilde{z}$ as a function of $(x,t) \in \mathbf{R}^N \times (0,\infty)$ by putting $x = (x_1, x')$. Thus, $\tilde{z}$ is independent of x_1. We will consider this function in the next section as a comparison function in the argument to prove the same decay rate as in (4.19) for the z_n of Section 3.

5. End of the existence proof

We interrupted the flow of the existence proof at the end of Section 3 because of the nagging doubt that maybe our constructed solution of the z-equation gives after all a trivial solution of the PME. We are now in a position to discard this possibility. As we said, this can be done by proving

Lemma5.1. *$Z(x)$ is bounded.*

Proof. We know that

$$z_j(x,1) = \lim_{j\to\infty} j^{N-1} z_1(jx, j^{m(N+1)+1-N})$$

(use for instance (3.13) with $\lambda^\beta = j$ and $n = 1$). Therefore,

$$Z(x) = z(x,1) = \lim_{j\to\infty} z_j(x,1) = \lim_{t\to\infty} t^{(N-1)\beta} z_1(xt^\beta, t) .$$

Thus, the result is true if we can show that

$$z_1(x,t) \le C\, t^{-(N-1)\beta} = C\, t^{-\frac{N-1}{m(N+1)+1-N}} . \tag{5.1}$$

Thus, in order to show that $Z(x)$ is bounded, we only have to obtain a decay rate for z_1

Now, the decay rate in space obtained in (4.18) for the $\tilde{z} = \tilde{z}(x_2, \cdots, x_N, t)$ allows us to rescale it in such a way that

$$|z_{01}(x_1, \cdots, x_N)| < \tilde{z}(x_2, \cdots, x_N, 1)\,, \tag{5.2}$$

because z_{01} is bounded by $|x|^{1-N}$. Therefore, we can use $\tilde{z}(x_2, \cdots, x_N, t)$ as a comparison function for the solution $z_1(x_1, \cdots, x_N, t)$ and thus obtain the decay as needed in (3.14).

By the Maximum Principle, which holds for equation (3.1), we conclude that

$$0 \le z_1(x_1, \cdots, x_N, t) \le \tilde{z}(x_2, \cdots, x_N, t+1)\,, \tag{5.3}$$

for every $(x_1, \cdots, x_N) \in H$ and $t > 0$. This ends the Proof. $\square$

Step 5. Our next task consists in checking that $\overline{z}$ satisfies equation (3.1) in a classical sense and that $\overline{u}$ is a suitable solution of the PME. Both facts are immediate consequences of the limit process once we are certain that convergence takes place in suitable norms. Now, we know that in the domain $Q_\tau^+ = H \times (\tau, \infty)$ the sequence $z_n(x,t)$ is bounded uniformly in x, t and n. The interior estimates derived in the Appendix allow us to prove that in this situation also the $u_n = -\Delta z_n$ form a bounded sequence in the Hölder spaces $C^\alpha_{\rm loc}(S')$ for every set S' compactly contained in $\overline{H} \times (0, \infty)$. Hence, we get in the limit $\overline{z} \in C^{2,1}_{x,t}(S)$ and $\overline{u} \in C(Q^+)$. With suitable antisymmetric extensions equation (3.1) is satisfied in a classical sense in the whole space by $\overline{z}$ when $t > 0$, while $\overline{u}$ satisfies the PME in distribution sense.

Step 6. The initial data. There is no doubt that $\overline{z}$ takes on the initial data $z_0(x)$ continuously away from the origin. This follows from the monotonicity of the sequence z_n and the fact that z_t is nonpositive.

The fact that $\overline{u}$ takes on initial data $\overline{u}(x, 0) = 0$ for $x \ne 0$ needs some extra work. Since the solution is self-similar, we have with $y = xt^{-\beta}$

$$\lim_{t\to 0} \overline{u}(x,t) = |x|^{-(N+1)} \lim_{|y|\to\infty} |y|^{N+1} U(y)\,. \tag{5.4}$$

We are going to prove in the next section that U has compact support. It is then immediate that the limit (5.4) is zero for every $x \ne 0$, and that (1.12) holds. With this the proof of the existence of the dipole solution will be complete.

The properties announced in the introduction are easy to check. $\square$

6. The property of compact support and the coincidence set

It is important to know whether the support of $\overline{u}$ is compact in space for every time $t>0$. Taking $(-\Delta)^{-1}$ this problem transforms into the broader question of the *coincidence set* between $\overline{z}(x,t)$ and $\overline{z}(x,0)$. To begin with, we know that

$$z_n(x,t) \le z_{n+1}(x,t) \le \overline{z}(x,t) \le \overline{z}(x,0)\,. \tag{6.1}$$

In order to continue we need the following result

Lemma 6.1. *Suppose that a bounded continuous function $f(x)$ with compact support satisfies $\int f\,h\,dx = 0$ for every harmonic function defined in a neighbourhood of* $\mathrm{supp}(f)$. *Then the Newtonian potential of f, $N(f)$, which satisfies $\Delta N(f) = f$ in the sense of distributions, has the same support as f.*

Proof. For any $y \notin \mathrm{supp} f$, the fundamental solution $\Gamma(x-y)$ is a harmonic function of x in a neighbourhood of $\mathrm{supp} f$. Thus, by the assumptions of the Lemma, $N_f(y) = \int \Gamma(y-x)f(x)dx = 0$. The other implication is trivial: if $x \notin \mathrm{supp} N(f)$, then $f(x) = \Delta N(f)(x) = 0$. $\square$

Remark. In the application we will restrict h to belong to the set of harmonic functions defined in a neighbourhood of $\mathrm{supp}(f) \cup B$, where B is a ball. The above argument allows then to conclude that the support of $-N(f)$ is contained in $B \cup \mathrm{supp}(f)$.

Let B_n be the ball $B_{1/n}(0)$ which contains the support of u_{0n}. We consider the sets

$$S_n(t) = \mathrm{supp}(u_n(\cdot,t) - u_{0n}) \cup B_n\,, \tag{6.2}$$

$$G_n(t) = \mathrm{supp}(z_n(\cdot,t) - z_{0n}) \cup B_n\,. \tag{6.3}$$

¿From the theory for PME it follows that $S_n(t)$ is a compact set, because the support of u_{0n} is compact, hence so is the support of $u_n(\cdot,t)$. Moreover, the supports of the functions $u_n(\cdot,t)$ are expanding since u_n is a nonnegative solution of a Dirichlet problem for the PME. Therefore the sets $S_n(t)$ are also expanding in time.

Proposition 6.2. *We have $S_n(t) = G_n(t)$.*

Proof. Let $0 < T$. We have

$$\frac{d}{dt}\int u_n\,h\,dx = \int (u_n)_t\,h\,dx = \int \Delta(|u_n|^{m-1}u_n)h\,dx = \int (|u_n|^{m-1}u_n)\Delta h\,dx = 0\,, \tag{6.4}$$

for any harmonic function h defined in a neighbourhood of $S_n(T)$, outside of which $u_n(\cdot,t)$ vanishes for every $0 < t \le T$. Integration in time gives

$$\int u_n(x,t)h(x)\,dx = \int u_{0n}(x)h(x)\,dx = 0\,, \tag{6.5}$$

for this family of h's, so that $f(x) = u_n(x,t) - u_{0n}(x)$ satisfies the assumptions of the Remark to Lemma 6.1. But then we can recover $g(x) = z_n(\cdot,t) - z_{0n}$ by taking $-N(f)$, for these last two functions have the same Laplacian, and both go to zero as $|x| \to \infty$. Thus, they have to be the same, and we can conclude from the Remark to Lemma 6.1 that $u_n(\cdot,t) - u_{0n}$ and $z_n(\cdot,t) - z_{0n}$ have the same support away from B_n. $\square$

Theorem 6.3. *The support of* $\overline{u} = -\Delta\overline{z}$*, where* $\overline{z}$ *is the limit of the sequence* z_n*, is compact. Moreover, the support of* $\overline{u}(\cdot,t)$ *equals the coincidence set of* $\overline{z}(t)$ *and* $\overline{z}(0)$*,*

$$\text{supp}\overline{u}(\cdot,t) = G(t) = \text{supp}(\overline{z}(\cdot,t) - z_0)\,, \tag{6.6}$$

Proof. Since the functions $z_0(x)$ and $z_{0n}(x)$ coincide for $|x| > 1/n$ it follows from (6.1) that $G(t) \subseteq G_n(t)$, thus it is compact. In fact, the sequence of sets $G_n(t)$ shrinks to $G(t)$. By definition the support of $\overline{u}(t)$ is contained in $G(t)$. The equality follows from Lemma 6.1 above since the support of $\overline{u}$ shrinks to $\{0\}$ as $t \to 0$. □

7. Properties of the interface

We study in this section the geometry of $G(t)$, the support of $\overline{u}$ at time $t > 0$, and its boundary $\Gamma(t)$ (the *Free Boundary*). $G(t)$ is the closure in $\overline{H} = \{x \in \mathbf{R}^N : x_1 \geq 0\}$ of the positivity set

$$P(t) = \{x \in H : \overline{u}(x,t) > 0\}\,.$$

To begin with, the self-similarity of $\overline{u}$ immediately implies that

$$G(t) = t^\beta G(1)\,, \quad \Gamma(t) = t^\beta \Gamma(1)\,, \quad P(t) = t^\beta P(1)\,, \tag{7.1}$$

as subsets of H. This reduces the study to time $t = 1$. We have

Property 1. *$G(1)$ possesses radial symmetry in the variables $x' = (x_2, \cdots, x_N)$.*

Proof. It is immediate form the construction. Indeed, a stronger result is true: the function $\overline{u} = \overline{u}(x,x')$ is radially symmetric in the variable x' and nonincreasing in $|x'|$. This monotonicity property can be obtained in different ways, for instance applying the maximum principle to u_{x_2} in an appropriate domain. □

Property 2. *$G(1)$ is starshaped around the origin.*

Proof. This is a consequence of (7.1) and the following well-known property of *retention of positivity*, which holds in particular for nonnegative solutions of the PME with homogeneous Dirichlet data. The property says that if any given point x and time $t_1 > 0$ the solution is positive, then it will be positive at x for all later times $t_2 > t_1$. In other words, the sets $\{P(t)\}$ form an expanding family. A simple proof of this can be obtained by comparison from below with a very small Barenblatt solution (so small that it does not reach the boundary in the time interval $0 \leq t \leq t_1$).

Property 2 is then proved as follows: take a point $(x,t) \in Q^+$ where $\overline{u} > 0$ and let $\tilde{x} = \lambda x$ with $0 < \lambda < 1$. By self-similarity the point $(\tilde{x},\tilde{t})$ corresponds to the positivity set of $\overline{u}$ if $\tilde{t} = \lambda^{1/\beta} t < t$. By the retention property $\overline{u}(\tilde{x},t) > 0$. □

Property 3. *The boundary $\Gamma(1)$ satisfies the cone condition, both external and internal, away from $x_1 = 0$.*

Proof. Let $x = (x_1, \cdots, x_N) \in \Gamma(1)$, $x_1 > 0$. By the symmetry properties we can always assume that we are in $N = 2$ and that $x_2 \geq 0$. There exist points $\tilde{x} \in P(1)$ and points $\hat{x} \notin G(1)$ as close as desired to x. By Property 2 for every for every $\lambda < 1$ we know that $u(\lambda\tilde{x}, 1) > 0$, while for $\lambda > 1$ $\overline{u}(\lambda\hat{x}, 1) = 0$.

On the other hand, the monotonicity of $\overline{u}$ in $|x'|$ produces a similar result when we move from x in any direction ν' perpendicular to the x_1-axis. It follows that the cone generated at x by the directions of the form $\nu = c_1 x + c_2 \nu'$ with c_1 and $c_2 > 0$ is external to $G(1)$ while the opposite cone is internal. Indeed, we can do better by making a symmetry argument around any line $x_2 = ax_1 + \varepsilon$ with $a > 0$ and $\varepsilon > 0$ as small as we like. We conclude at any point x with $x_2 > 0$ the function $\overline{u}(x, 1)$ is monotone in the direction perpendicular to the radius vector joining x to the origin. As a consequence we conclude that $\overline{u}$ is monotone along circles centered at the origin in the (x_1, x_2)-plane.

Combining these results we conclude that there exist complementary interior and exterior cones with angle $\pi/2$ at every point of the interface. $\square$

As a consequence of these results we have

Property 4. *The free boundary $\Gamma(1)$ is a Lipschitz-continuous surface in H. It can defined by the formula*

$$|x'| = f(x_1), \quad \textit{for} \quad 0 \leq x_1 \leq A. \tag{7.2}$$

At the tip of the support in the x_1 axis, $x_1 = A$ we have even more: by symmetry the cones are π radians wide, so this point has a definite tangent plane perpendicular to the x_1-axis.

Finally we address the question of the behaviour of the interface near $x_1 = 0$. It is well-known that in the PME the movement of the free boundary is driven by the pressure gradient, namely *speed* $= -k\nabla v$ with $v = u^{m-1}$ (Darcy's law). This could suggest that the interface does not move along the hyperplane $x_1 = 0$. In fact, this is not so as the following result shows.

Property 5. *There exists the limit*

$$\lim_{x_1 \to 0} f(x_1) = a\,, \tag{7.3}$$

and $a > 0$.

Proof. The existence of the limit is a consequence of the monotonicity implied by the existence of suitable cones. The argument to prove that a is strictly positive is based on comparison with small Barenblatt solutions located near the origin and below our solution. Since it can be seen in detail in [LV], we omit here further details. $\square$

8. Asymptotic behaviour

We take solutions of the PME in a half-space $H = \{x \in \mathbf{R}^N : x_1 > 0\}$ with initial data $u_0 \geq 0$ and data $u = 0$ on the lateral boundary $\Sigma = \{(x,t) : x_1 = 0, \quad t > 0\}$. Under these assumptions we have

Theorem 8.1. *The solution converges to a Dipole Solution.*

Let u be any such solution. We shall show that we can put a dipole solution under u and use it as a subsolution, and similarly, that we can put a dipole solution above u and use it as a supersolution. To do so however we need some properties of ∇u at $x_1 = 0$

Lemma 8.2. *Let u be a solution of the above problem with nonnegative bounded initial data u_0. For every $\tau > 0$, there exists a constant $A > 0$ which depends only on τ and on the supremum of u_0, such that*

$$0 \leq u(x,t) \leq A x_1^{1/m}\,. \tag{8.1}$$

for $t \geq \tau$.

Proof. We use a travelling wave supersolution as a barrier function. Let

$$\tilde{u}(x,t) = P(x_1 - ct + b)\,. \tag{8.2}$$

Substitution into the PME gives $cP' + (P^m)'' = 0$, which can be integrated to give

$$cP + (P^m)' = D\,,$$

where we take D a positive constant. Choosing $P(0) = 0$, we obtain a solution $P(s)$ defined for positive s with $(P^m)(s) \leq Ds$, which runs from zero to D/c as s runs through R^+. Choose $b/c = \tau$, and let D be so large that $\tilde{u}(x,0) = P(x_1 + b) \geq u(x,0)$. Then by the maximum principle, $\tilde{u}(x,\tau) = P(x_1) \geq u(x,\tau)$, so that $u(x,\tau) \leq (Dx_1)^{1/m}$. Since the right-hand side of this last estimate is an exact solution of the PME, it follows again from the maximum principle that the same estimate holds for all $t \geq \tau$. $\square$

Lemma 8.3. *Let u be a solution of the above problem with nontrivial nonnegative bounded initial data u_0. For every $x_0' \in R^{N-1}$ there exists a $\tau > 0$ which depends only on the initial data and x', such that for every $t_0 > \tau$ we have*

$$u(x,t) \geq C x_1^{1/m}$$

for (x,t) in a neigbourhood of $(0, x_0', t_0)$ whenever $x_1 > 0$. The constant $C > 0$ depends on the point $(0, x_0', t_0)$ and on the neighbourhood.

Proof. Consider any Barenblatt solution centered in H, and with support contained in H at $t = 0$. It was shown in [LV], that if we solve the problem in the halfspace with these initial data, that when the support of this solution hits $x_1 = 0$, the set $\sigma(t) = \{x' \in R^{N-1} : (0,x') \in \operatorname{supp} u(t)\}$ is an expanding ball, with $u(x,t) \geq C x_1^{1/m} > 0$ locally for $x_1 > 0$ and x' near the center of the ball $\sigma(t)$. Using this kind of solutions as subsolutions, the Lemma follows. $\square$

Lemma 8.4. *Let $\overline{u}$ be the dipole solution as constructed in the previous sections, and let U be its similarity profile. Then in a certain neighbourhood of the origin in H we have*

$$U(y) \geq C y_1^{1/m} > 0 \quad \textit{for} \quad y_1 > 0\,. \tag{8.3}$$

Proof. This follows from Lemma 8.2 and the similarity form of the dipole solution. $\square$

Lemma 8.5. *Every compactly supported solution lies for t large between two appropriately scaled dipole solutions. Writing*

$$\overline{u}_\lambda(x,t) = t^{-\alpha} U_\lambda(x t^{-\beta}) = \lambda^{\frac{2}{m-1}} \overline{u}(x/\lambda, t)\,, \tag{8.4}$$

where U_λ is defined by (0.9), and α and β are given by (0.4), there exist positive constants $\lambda_1(t)$ and $\lambda_2(t)$ such that if t is large enough

$$\overline{u}_{\lambda_1(t)}(x,t) \leq u(x,t) \leq \overline{u}_{\lambda_2(t)}(x,t)\,. \tag{8.5}$$

Proof. The previous lemmas allow us to put any solution $u(x,t)$ between two appropriately scaled dipole solutions for some fixed positive $t = t_0$. By the maximum principle this remains true for all $t > t_0$. $\square$

To complete the proof of Theorem 8.1 we use a standard Lyapunov functional argument. Let $\overline{u}$ be the dipole solution whose first moment has the same first component as the solution u under consideration. Define

$$J(u,\overline{u})(t) = \int_H |u(x,t) - \overline{u}(x,t)| x_1 \, dx\,. \tag{8.6}$$

Lemma 8.6. *Unless $u \equiv \overline{u}$, J is strictly decreasing in time.*

Proof. This follows from the same argument as in the uniqueness proof for the dipole solution in Section 2. $\square$

Next we scale the solution by

$$u_\lambda(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t)\,, \tag{8.7}$$

where α and β are given by (0.4). Observe that

$$J(u_\lambda, \overline{u})(t) = J(u, \overline{u})(\lambda t),. \tag{8.8}$$

By standard a priori estimates we can pass to the limit along a subsequence $\lambda_j \to \infty$. Denote the limit of u_{λ_j} by $\tilde{u}$.

Lemma 8.7. *We have $J(\tilde{u},\overline{u})(t) \equiv c$, for some $c \geq 0$.*

Proof.

$$J(\tilde{u},\overline{u})(t) = \lim_{j\to\infty} J(u_{\lambda_j},\overline{u})(t) = \lim_{\lambda_j\to\infty} J(u,\overline{u})(\lambda t) = \lim_{t\to\infty} J(u,\overline{u})(t) = c \qquad \square$$

Lemma 8.8. $c = 0$.

Proof. By Lemma 8.7, $J(\tilde{u},\overline{u})(t) \equiv c$, so it is not strictly decreasing. Hence Lemma 8.6 implies that $\tilde{u} \equiv \overline{u}$. $\square$

This completes the proof of Theorem 8.1.

APPENDIX

Some results about the theory of the Dual Equation

We collect here some results on the theory of the dual equation (0.7). We begin by results which are obtained by taking the Newtonian potential of solutions of the PME, as explained above.

Let X be the set of twice continuously differentiable functions f in H such that $f(x) = 0$ on $\partial H = \{x : x_1 = 0\}$, $f(x) \to 0$ as $|x| \to \infty$, and Δf has compact support (we use the notations H, Q, Q^+ and Σ, as defined in the Introduction). For those functions as initial data we have a classical solution of the Z equation.

Lemma A.1. *Let z_0 belong to X. Then there exists a unique function $z(x,t) \in C([0,\infty) : X)$ which satisfies equation (0.7) in Q^+ and such that $\Delta z(\cdot,t)$ has compact support for every $t > 0$.*

Proof. Solve the PME with initial data $u_0(x) = -\Delta z_0$. It is known that under such circumstances the solution $u(x,t)$ is Hölder continuous with a modulus that depends only on the sup norm of the initial data if we are away from $t = 0$. Take then the Newtonian potential of the solution. Uniqueness follows from the next Lemma. $\square$

Lemma A.2. *The Maximum Principle applies to the above solutions. Moreover, we have a semigroup of contractions in X with respect to the sup norm.*

Proof. This property is straightforward to check, our solutions being classical. If z_1 and z_2 are solutions, there can be no point where the function $(z_1(x,t) - z_2(x,t))e^{at}$, $a > 0$ can have an interior maximum. $\square$

In fact, this is the basic property used to obtain a theory of generalized solutions for equation (0.7) with merely continuous initial data. One way is to approximate z_0 with smooth initial data in X, solve and then pass to the limit in the sequence solutions which will be a converging sequence in the sup norm. The construction of a solution in the framework of semigroup theory is done in [BH], [Ha] and [Ko] for instance. See [C] for general reference to generation of nonlinear semigroups with accretive operators.

Lemma A.3. *If $z_0 \le 0$ we have*

$$z_t \le -\frac{z}{(m-1)t}\,. \tag{A.1}$$

Proof. This estimate, as explained in [BC], can be obtained for quite general evolution equations if (i) they are homogeneous (i.e. have power-type scaling), and (ii) they satisfy the Maximum Principle. The proof consists in noting that if z is a solution of (0.7), then the function

$$z_\lambda(x,t) = \lambda z(x, \lambda^{m-1} t)$$

is also a solution of (0.7). Moreover, for $\lambda > 1$ it has larger initial data. Therefore

$$\left.\frac{dz_\lambda}{d\lambda}\right|_{\lambda=1} \ge 0\,.$$

This gives $z + (m-1)tz_t \le 0$. i.e. (A.1). $\square$

In our analysis above we have only considered solutions with nonpositive Laplacian.

Lemma A.4. *If $\Delta z_0(x) \le 0$ then $\Delta z(x,t) \le 0$ and $z_t \le 0$. (Similarly, resp. $\Delta z_0(x) \le 0$ implies $\Delta z(x,t) \ge 0$ and $z_t \ge 0$).*

Proof. It follows from is the Maximum Principle for the PME. $\square$

Lemmas A.3 and A.4 imply a control for Δz in sup-norm.

Corollary A.5. *If $\Delta z_0(x) \le 0$ we have $\Delta z(x,t)$ bounded in $S_\tau = H \times (\tau, \infty)$ for every $\tau > 0$.*

We also have gradient bounds

Lemma A.6. *We get a local estimate for ∇z in L^2 in terms of local bounds for z_0 in $L^{(m+1)/m}$.*

Proof. Write the equation in the form $G(z_t) = \Delta z$, with G defined by $G(s) = |s|^{1/m}\mathrm{sign}\,(s)$. Take a cutoff function ζ. Multiplying the equation by $z\zeta^2$ and integrating by parts in H we get

$$\int |\nabla z|^2 \zeta^2\, dx = -\int G(z_t) z\zeta^2\, dx - 2\int z\zeta \nabla z \cdot \nabla\zeta\, dx\,.$$

Use now (A.1) to obtain, after some manipulations

$$\frac{1}{2}\int |\nabla z|^2\zeta^2\, dx \le \frac{c_m}{t^{1/m}} \int z^{(m+1)/m}\zeta^2\, dx + 2\int z^2 |\nabla\zeta|^2\, dx \quad \square \tag{A.2}$$

Lemma A.7. *Let $\Delta z \le 0$. There exist local C^α bounds in S_τ for $u = -\Delta z$ in terms of any L^p-norm of $u_0 = -\Delta z_0$.*

Proof. This is standard PME theory. Proofs can be found in [CF], [dB] or [S]. $\square$

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