POSITIVE SOLUTIONS OF THE EQUATION $\Delta u + u^p = 0$ ON A BOUNDED DOMAIN

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These are are the notes of a series of lectures given in the seminar "Topics in Nonlinear Analysis" at the "Center for Nonlinear Partial Differential Equations (Delft-Leiden)", in autumn of 1991. In these course notes we present the basic theorems, and the necessary background as a selfcontained resumé on the problem

$$(P)\begin{cases}\Delta u + u^p = 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega.\end{cases}$$

We restrict ourselves mainly to the subcritical case $(1 < p < (n+2)/(n-2))$, and we only consider positive solutions.

Our choice of topics is largely determined by one of the goals of the seminar, which is to present recent results in a self-contained form, with details filled in as much as possible. We chose to do S.C. Lin's recent uniqueness theorem for positive solutions of (P) with Morse index 1 on convex domains in dimension 2 [L].

Thus, besides a careful proof of the standard existence results in terms of weak solutions and mountain passes, we also treat applications of the implicit function theorem, which require knowledge of the linearized problem. Moreover we discuss the nature of the numerous integral identities that have been around (see e.g. [P,PS,vdV1]) for solutions of (P) and related problems. This goes back to E. Noether [N].

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1. Introduction.

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$, where $n > 1$. For $p > 1$ fixed we consider the following problem:

$$(P)\begin{cases} \Delta u + u^p = 0 & \text{in} \quad \Omega, & (1.1)\\ u > 0 & \text{in} \quad \Omega, & (1.2)\\ u = 0 & \text{in} \quad \partial\Omega. & (1.3)\end{cases}$$

Here $\Delta = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_n^2}$, and $u = u(x) = u(x_1, ..., x_n)$. A *classical solution* of (P) is a function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ which satisfies (1.1)-(1.3) pointwise. It will be more convenient here to work with so-called *weak* solutions, see [Br] as a general reference. We recall that [A,Br] the Sobolev space $H^1(\Omega)$ is defined by

$$H^1(\Omega) = \{u \in L^2(\Omega) : \frac{\partial u}{\partial x_i} \in L^2(\Omega), i = 1, ..., n\}$$

where $\frac{\partial u}{\partial x_i}$ are the distributional first order derivatives of u. The space $H^1(\Omega)$ is a Hilbert space with inner product

$$(u, v)_{H^1} = \int_\Omega uv + \sum_{j=1}^n \int_\Omega \frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_j} \tag{1.4}$$

The space $H_0^1(\Omega)$ is defined to be the closure in $H^1(\Omega)$ with respect to (1.4) of the subspace consisting of all smooth functions with compact support strictly contained in Ω. Because of the Poincaré inequality, we can replace the inner product (1.4) on $H_0^1(\Omega)$ by the equivalent inner product

$$(u, v)_{H_0^1} = \sum_{j=1}^n \int_\Omega \frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_j} = \int_\Omega \nabla u \nabla v. \tag{1.5}$$

We are now ready to define weak solutions of (P).

Definition 1.1. *Let $1 < p \leq \frac{n+2}{n-2}$ and $n > 2$ or $1 < p < \infty$ and $n = 2$. Then $u \in H_0^1(\Omega)$ is called a weak solution of (P) if*
(i) $u \geq 0$ a.e. on $\Omega, u > 0$ on a subset of Ω with positive measure;
(ii) for all $\varphi \in H_0^1(\Omega)$ the following identity holds

$$\int_\Omega \nabla\varphi\nabla u = \int_\Omega \varphi u^p. \tag{1.6}$$

We have to check that (1.6) makes sense, i.e. the right hand side must exist for all u and φ in $H_0^1(\Omega)$. This depends on the Sobolev Embedding Theorem (see Section 4).

Theorem 1.2. *For all $1 \le q \le \frac{2n}{n-2}$ $(1 \le q < \infty$ if $n = 2)$ the embedding*

$$H_0^1(\Omega) \hookrightarrow L^q(\Omega)$$

holds and is continuous.

As a direct consequence we have that (1.6) makes sense, namely, if $n = 2$ then φ and u are in any $L^q(\Omega)$ with $1 \le q < \infty$, and hence the same goes for φu^p, while for $n > 2$, we have

$$\varphi \in L^{q_1}(\Omega) \quad \text{with} \quad q_1 = \frac{2n}{n-2};$$

$$u^p \in L^{q_2}(\Omega) \quad \text{with} \quad q_2 = q_1/p, \quad \text{so that}$$

$$\varphi u^p \in L^1(\Omega) \quad \text{if} \quad 1 \ge \frac{1}{q_1} + \frac{1}{q_2} = \frac{p+1}{q_1} = \frac{n-2}{2n}(p+1),$$

$$\text{i.e.} \quad p+1 \le \frac{2n}{n-2} \quad \text{or} \quad p \le \frac{2n}{n-2} - 1 = \frac{n+2}{n-2} = p_c \tag{1.7}$$

This number p_c is called the *critical exponent*, and turns out to be a borderline between existence and non-existence of solutions of (P), on star-shaped domains. Let us indicate what is going on here for the case that Ω is a ball, i.e.

$$\Omega = B_R = \{x \in \mathbf{R}^n : \quad x_1^2 + ... + x_n^2 < R^2\}.$$

First we state the celebrated Gidas-Ni-Nirenberg Theorem [GNN], see also Section 7.

Theorem 1.3. *Let Ω be a ball, and let u be a classical solution of (P). Then u is radially symmetric.*

In view of Theorem 1.3, (P) is equivalent to

$$(P_r)\begin{cases} U''(r) + \dfrac{n-1}{r}U'(r) + U(r)^p = 0, & 0 \le r < R, \qquad (1.8)\\ U(r) > 0, & 0 \le r < R, \qquad (1.9)\\ U'(0) = 0, & \qquad (1.10)\\ U(R) = 0, & \qquad (1.11)\end{cases}$$

which can be solved using the techniques from the theory of ordinary differential equations.

Exercise 1.4
(i) Show that the transformation

$$X = \frac{rU'(r)}{U(r)}; \quad Y = r^2 U(r)^{p-1}; \quad t = \log r \tag{1.12}$$

transforms (1.8) into the quadratic system

$$(Q)\begin{cases} X' = \dfrac{dX}{dt} = X(2-n-X) - Y, & (1.14)\\ Y' = \dfrac{dY}{dt} = Y(2+(p-1)X). & (1.15)\end{cases}$$

(ii) Show that solutions of (1.8) satisfying (1.9) and (1.10) correspond to orbits of (Q) in the upper half plane coming out of $(0,0)$.
(iii) Show that there is only one such orbit γ.
(iv) Show that for $p = \frac{n+2}{n-2}$ all orbits of (Q) are symmetric in the line $X = \frac{2-n}{2}$.
(v) Show that γ escapes to infinity in finite time through the second quadrant along the negative X-axis, if and only if $p < \frac{n+2}{n-2}$ $(p < \infty$ if $n = 2)$.
(vi) Show that (P_r) has a solution if and only if $p < \frac{n+2}{n-2}$ $(p < \infty$ if $n = 2)$.
(vii) Show that this solution is unique.
(viii) Show that (P) has a radial solution for every annulus if $n > 2$ and $p = \frac{n+2}{n-2}$.

The result to bear in mind here is:

Theorem 1.5. *Let Ω be a ball and $p > 1$. Then (P) has a (classical) solution if and only if $p < \frac{n+2}{n-2}$ $(p < \infty$ if $n = 2)$. This solution is unique.*

2. Weak solutions as critical points.

We introduce the convention that

$$u^p = \begin{cases} u^p & \text{for} \quad u \geq 0 \\ -(-u)^p & \text{for} \quad u < 0. \end{cases} \tag{2.1}$$

Definition 2.1. *For $u \in H_0^1(\Omega)$ and $1 \leq p \leq \frac{n+2}{n-2}$ $(p < \infty$ if $n = 2)$ let*

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1}. \tag{2.2}$$

Note that (2.2) makes sense because of Theorem 1.2, which implies $u \in L^{p+1}(\Omega)$.

Theorem 2.2. *Let $1 \leq p \leq \frac{n+2}{n-2}$ $(p < \infty$ if $n = 2)$. Then $J : H_0^1(\Omega) \to \mathbf{R}$ is a continuously differentiable function, and $u \in H_0^1(\Omega)$ is a critical point of J if and only if u satisfies (1.6) for all $\varphi \in H_0^1(\Omega)$.*

Proof. We have to check that there exists a continuous map dJ of $H_0^1(\Omega)$ into the dual space of $H_0^1(\Omega)$, such that

$$J(u+\varphi) = J(u) + \langle dJ(u), \varphi\rangle + R(u,\varphi) \tag{2.3}$$

where

$$\frac{R(u,\varphi)}{\|\varphi\|_{H_0^1}} \to 0 \quad \text{as} \quad \varphi \to 0 \quad \text{in} \quad H_0^1(\Omega). \tag{2.4}$$

Here $\|\cdot\|_{H_0^1}$ is the norm corresponding to (1.5).

We write $J(u)$ as $J(u) = J_1(u) - J_2(u)$ with $J_1(u) = \frac{1}{2}\|u\|_{H_0^1}^2$ and $J_2(u) = \frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1}$, where $\|\cdot\|_{L^{p+1}}$ is the norm in $L^{p+1}(\Omega)$.
Clearly

$$J_1(u+\varphi) = \frac{1}{2}(u+\varphi, u+\varphi)_{H_0^1} = J_1(u) + (u,\varphi)_{H_0^1} + \frac{1}{2}\|\varphi\|_{H_0^1}^2, \tag{2.5}$$

so that J_1 is continuously differentiable.

For J_2 we first need the infinite dimensional version of the fact that continuous differentiability follows from continuity of the first order partial derivatives.

Proposition 2.3. *To show that $J : H \to \mathbf{R}$ is continuously differentiable, it is sufficient to show that for all $\varphi \in H$,*

$$\lim_{t\downarrow 0} \frac{J(u+t\varphi) - J(u)}{t} = \langle dJ(u), \varphi\rangle, \tag{2.6}$$

with $dJ : H \to H^*$ continuous.

We compute (2.6) for J_2:

$$\lim_{t\downarrow 0} \frac{J_2(u+t\varphi) - J_2(u)}{t} = \frac{d}{dt} J_2(u+t\varphi)\Big|_{t=0} = \frac{d}{dt}\int_\Omega \frac{1}{p+1}|u+t\varphi|^{p+1}\Big|_{t=0} =$$

$$\int_\Omega \frac{\partial}{\partial t}\frac{1}{p+1}|u+t\varphi|^{p+1}\Big|_{t=0} = \int_\Omega (u+t\varphi)^p\varphi\Big|_{t=0} = \int_\Omega u^p\varphi. \tag{2.7}$$

The differentiation under the integral in (2.7) is allowed if for t close to zero $(u+t\varphi)^p\varphi$ can be dominated by one fixed function $g \in L^1(\Omega)$ which does not depend on t. Indeed we have, say if $|t| < 1$, that

$$|(u+t\varphi)^p| \le (|u| + |\varphi|)^p \in L^{\frac{p+1}{p}},$$

so that

$$|(u+t\varphi)^p\varphi| \le (|u| + |\varphi|)^p|\varphi|,$$

which is in $L^1(\Omega)$ because $\varphi \in L^{p+1}(\Omega)$ and

$$\left(\frac{p+1}{p}\right)^{-1} + (p+1)^{-1} = 1.$$

Thus by Proposition 2.3

$$\langle dJ_2(u), \varphi\rangle = \int_\Omega u^p\varphi \tag{2.8}$$

if the map
$$u \in H_0^1(\Omega) \to (u^p, \cdot) \in H_0^1(\Omega)^*, \tag{2.9}$$
where
$$(u^p, \varphi) = \int_\Omega u^p \varphi, \tag{2.10}$$
can be shown to be continuous.

Since $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$, this follows from the duality of $L^{\frac{p+1}{p}}(\Omega)$ and $L^{p+1}(\Omega)$ and the next lemma with $q = p$.

Lemma 2.4. *Let $p < q \le p+1$. Then the maps*
$$u \in L^{p+1}(\Omega) \longrightarrow u^q \in L^{\frac{p+1}{q}} \tag{2.11}$$
and
$$u \in L^{p+1}(\Omega) \longrightarrow |u|^q \in L^{\frac{p+1}{q}} \tag{2.12}$$
are continuous.

Proof. Clearly it suffices to show (2.11) is continuous, because $u \in L^p \to |u| \in L^p$ is continuous for any p. First we consider the case $q \ge 1$. Then
$$\|u^q - v^q\|_{L^{\frac{p+1}{q}}} = \left\{\int_\Omega |u^q - v^q|^{\frac{p+1}{q}}\right\}^{\frac{q}{p+1}} \le q\left\{\int_\Omega \left(|u-v| \max(|u|,|v|)^{q-1}\right)^{\frac{p+1}{q}}\right\}^{\frac{q}{p+1}} \le$$
$$(\text{since } \left(\frac{q}{p+1}\right)^{-1} = (p+1)^{-1} + \left(\frac{p+1}{q-1}\right)^{-1})$$
$$q\|u-v\|_{L^{p+1}}\left\{\int_\Omega (\max|u|,|v|)^{p+1}\right\}^{\frac{q-1}{p+1}} = q\|u-v\|_{L^{p+1}} \|\max(|u|,|v|)\|_{L^{p+1}}^{q-1}. \tag{2.13}$$
Hence (2.11) is continuous for $q \ge 1$. For $0 < q < 1$ we have
$$\|u^q - v^q\|_{L^{\frac{p+1}{q}}} \le \|2^{1-q}|u-v|^q\|_{L^{\frac{p+1}{q}}} = 2^{1-q}\left\{\int_\Omega |u-v|^{p+1}\right\}^{\frac{q}{p+1}} = 2^{1-q}\|u-v\|_{L^{p+1}}^q, \tag{2.14}$$
so that also for $0 < q < 1$, (2.11) is continuous. This completes the proof of Lemma 2.4.

We conclude that also J_2 is continuously differentiable, and hence, by (2.5) and (2.8),
$$\langle dJ(u), \varphi\rangle = (u, \varphi)_{H_0^1} - (u^p, \varphi) = \int_\Omega \nabla u \nabla \varphi - \int_\Omega u^p \varphi, \tag{2.15}$$
so that $dJ(u) = 0$ if and only if (1.6) holds. This completes the proof of Theorem 2.2.

3. Construction of the mountain pass solution.

In this section we obtain a weak solution as a mountain pass of the function

$$J(u) = \frac{1}{2}\|u\|^2_{H^1_0} - \frac{1}{p+1}\|u\|^{p+1}_{L^{p+1}}. \tag{3.1}$$

Because of the explicit form of the nonlinearity, this can be done directly, without invoking any general critical point theorems as in e.g. [AR,F,R].

For every $\varphi \in H^1_0(\Omega)$ fixed, let

$$f(t) = J(t\varphi) = \frac{t^2}{2}\|\varphi\|^2_{H^1_0} - \frac{t^{p+1}}{p+1}\|\varphi\|^{p+1}_{L^{p+1}}. \tag{3.2}$$

We have

$$f'(t) = t\|\varphi\|^2_{H^1_0} - t^p\|\varphi\|^{p+1}_{L^{p+1}}, \tag{3.3}$$

which is zero for

$$\bar{t} = \left\{ \frac{\|\varphi\|^2_{H^1_0}}{\|\varphi\|^{p+1}_{L^{p+1}}} \right\}^{\frac{1}{p-1}} = \bar{t}(\varphi), \tag{3.4}$$

so that

$$f(\bar{t}) = \frac{p-1}{2(p+1)}\left\{ \frac{\|\varphi\|_{H^1_0}}{\|\varphi\|_{L^{p+1}}} \right\}^{2\frac{p+1}{p-1}}. \tag{3.5}$$

Clearly, $J(u)$ has a local minimum 0 in $u = 0$. If we walk away from zero in any direction $\varphi \in H^1_0(\Omega)$, at first $J(u)$ increases to the maximum given by (3.5), and then decreases to $-\infty$. If we can find a $\overline{\varphi}$ such that (3.5) is minimal for $\varphi = \overline{\varphi}$, then intuitively, J should have a saddle point in $\overline{u} = \bar{t}(\overline{\varphi})\overline{\varphi}$. If we think of the graph of J as the surface of a mountain in $H^1_0(\Omega) \times \mathbf{R}$, the point $(\overline{u}, J(\overline{u})_{H^1_0}$ would be a mountain pass. Thus we want to minimize (3.5) which is equivalent to showing that

$$\min_{\varphi \in H^1_0(\Omega)\cap S_{p+1}} \|\varphi\|_{H^1_0} \tag{3.6}$$

exists, where S_{p+1} is the unit sphere

$$S_{p+1} = \left\{ u \in L^{p+1}(\Omega) : \|u\|_{L^{p+1}} = 1 \right\}$$

in L^{p+1}. Clearly $\|\varphi\|_{H^1_0}$ is bounded away from zero on $H^1_0(\Omega) \cap S_{p+1}$, because of the continuous Sobolev embedding (see Section 4)

$$H^1_0(\Omega) \hookrightarrow L^{p+1}(\Omega).$$

In fact (3.6) is the reciprocal of the operator norm of this embedding.

Summarizing we have

Proposition 3.1. *Suppose that there exists a $\overline{\varphi} \in H_0^1(\Omega) \cap S_{p+1}$ such that*

$$\|\overline{\varphi}\|_{H_0^1} = \min_{\varphi \in H_0^1(\Omega) \cap S_{p+1}} \|\varphi\|_{H_0^1} = m_p. \tag{3.7}$$

Then

$$\overline{u} = m_p^{\frac{2}{p-1}} \overline{\varphi} \tag{3.8}$$

is a critical point of J, i.e. $dJ(\overline{u}) = 0$.

Proof. Observe that $\overline{u} = \overline{t}(\overline{\varphi})\overline{\varphi}$. We have to show that indeed $dJ(\overline{u}) = 0$, so suppose that there exists a $\psi \in H_0^1(\Omega)$ such that

$$\langle dJ(\overline{u}), \psi \rangle < 0. \tag{3.9}$$

It is no restriction to assume that

$$(\overline{u}, \psi)_{H_0^1} = 0; \quad (\psi, \psi)_{H_0^1} = 1, \tag{3.10}$$

because J has a maximum in $\overline{u}$ along the line through 0 and $\overline{u}$.

Now consider the 2-dimensional subspace L spanned by $\overline{u}$ and ψ. Let

$$D_\varepsilon = \{u \in L;\ (u - \overline{u}, \psi)_{H_0^1} > 0,\ \|u - \overline{u}\|_{H_0^1} < \varepsilon\}.$$

Then (3.9) implies that for all ε sufficiently small, $J < J(\overline{u})\ inD_\varepsilon$. On the other hand, because of continuity, $J(t(\overline{u}+\varepsilon\psi))$ must have its maximum for $\overline{t}_\varepsilon$ such that $\overline{t}_\varepsilon(\overline{u}+\varepsilon\psi) \in D_\varepsilon$. This shows that $J(\overline{u})$ cannot have been the smallest maximum and completes the proof.

Our next step is to show that for $p < \frac{n+2}{n-2}$ ($p < \infty$ if $n = 2$) the minimum in (3.6) is indeed achieved. This depends on the following result, see Section 4.

Theorem 3.2. *In Theorem 1.2 the embedding is compact for all $1 \le q < \frac{2n}{n-2}$ ($1 \le q < \infty$ if $n = 2$).*

Proposition 3.3. *Let $1 < p < \frac{n+2}{n-2}$ ($1 < p < \infty$ if $n = 2$). Then there exists a minimizer $\overline{\varphi}$ of (3.6), i.e. (3.7) holds.*

Proof. Consider m_p defined by

$$m_p = \inf_{\varphi \in H_0^1(\Omega) \cap S_{p+1}} \|\varphi\|_{H_0^1}. \tag{3.10}$$

Clearly, $m_p > 0$, and by definition, there exists a sequence $\varphi_n \in H_0^1(\Omega)$ such that

$$\|\varphi_n\|_{L^{p+1}} = 1 \qquad \forall n\ \in \mathbf{N} \tag{3.11}$$

and

$$\|\varphi_n\|_{H_0^1} \to m_p \quad \text{as} \quad n \to \infty. \tag{3.12}$$

Hence, because bounded subsets of Hilbert spaces are precompact in the weak topology, there exists a subsequence, again denoted by φ_n, such that for some $\overline{\varphi} \in H_0^1(\Omega)$ $\varphi_n \rightharpoonup \overline{\varphi}$ (weakly) in $H_0^1(\Omega)$ as $n \to \infty$, and also $\varphi_n \to \overline{\varphi}$ (strongly) in $L^{p+1}(\Omega)$, because the compactness of the embedding of $H_0^1 \hookrightarrow L^{p+1}$ (Theorem 3.2) assures the precompactness of $\{\varphi_n\}$ in the strong L^{p+1} topology. Clearly $\overline{\varphi} \in H_0^1(\Omega) \cap S_{p+1}$. Since

$$m_p \le \|\overline{\varphi}\|_{H_0^1} \le \liminf_{n\to\infty} \|\varphi_n\|_{H_0^1} = m_p, \tag{3.13}$$

$\overline{\varphi}$ is a minimizer for (3.10). This completes the proof.

Theorem 3.4. *Let* $1 < p < \frac{n+2}{n-2}$ $(1 < p < \infty$ if $n = 2)$. *Then* (P) *has a weak solution* $\overline{u}$ *which minimizes*

$$\min_{\substack{\varphi\in H_0^1(\Omega)\\ \|\varphi\|_{L^{p+1}}=1}} \|\varphi\|_{H_0^1}. \tag{3.14}$$

Proof. Clearly the minimizer $\overline{\varphi}$ in Proposition 3.3 can be taken to be non-negative. If it is not, simply replace $\overline{\varphi}$ by $|\overline{\varphi}|$. Let $\overline{u}$ be given by (3.8). Then $\overline{u}$ also minimizes (3.14) while by Proposition 3.1, $dJ(\overline{u}) = 0$ so that by Theorem 2.2, $\overline{u}$ is a weak solution of (P).

4. Sobolev inequalities.

The space $H_0^1(\Omega)$ can be defined in two ways, which are equivalent if $\Omega \subset \mathbf{R}^n$ is bounded, namely

$$H_0^1(\Omega) = \mathrm{cp}\big(C_0^\infty(\Omega), \|\cdot\|_{H_0^1}\big), \tag{4.1}$$

and

$$H_0^1(\Omega) = \mathrm{cl}\big(C_0^\infty(\Omega), H^1(\Omega)\big), \tag{4.2}$$

where H^1 was defined in Section 1, $\|\cdot\|_{H_0^1} = (\int_\Omega |\nabla\cdot|^2)^{1/2}$, $'\mathrm{cp}'$ denotes completion, and $'\mathrm{cl}'$ denotes closure.

For the class $C_0^\infty(\Omega)$ one has the so-called Poincaré inequality ,

$$\int_\Omega u^2 dx \le C_\Omega \int_\Omega |\nabla u|^2 dx, \quad \forall u \in C_0^\infty(\Omega). \tag{4.3}$$

Using (4.3) one obtains the equivalence of the norms $\|\cdot\|_{H^1}$ and $\|\cdot\|_{H_0^1}$, and consequently the equivalence of (4.1) and (4.2) on $H_0^1(\Omega)$. By a density argument, expression (4.3) is valid for all u in $H_0^1(\Omega)$.

The constant C_Ω in (4.3) is domain-dependent. By a similar argument as in the proof of Proposition 3.3, one can show that

$$\min_{\varphi\in H_0^1} \frac{\|\varphi\|_{H_0^1}^2}{\|\varphi\|_{L^2}^2} = \lambda_1 > 0 \tag{4.4}$$

has a minimizer in $H_0^1(\Omega)$. Clearly the smallest constant possible in (4.3) is then $C_\Omega = \lambda_1^{-1}$ (N.B. λ_1 is the first eigenvalue of $-\Delta$ with Dirichlet boundary data).

Question. Is there a Poincaré inequality for $\Omega = \mathbf{R}^n$?

Answer. Assume there is an inequality of type (4.3), say:

$$\int_{\mathbf{R}^n} u^2 dx \le C \int_{\mathbf{R}^n} |\nabla u|^2 dx, \quad \forall u \in C_0^\infty(\mathbf{R}^n). \tag{4.5}$$

Using the dilatation-invariance of $\mathbf{R}^n$ we define

$$u_\epsilon(x) = \epsilon^{\frac{2-n}{2}} u(\frac{x}{\epsilon}), \quad \epsilon > 0, \quad n > 2, \quad y = \frac{x}{\epsilon}. \tag{4.6}$$

Take any function $u(x) \in C_0^\infty(\mathbf{R}^n)$, and let

$$\int_{\mathbf{R}^n} u^2 dx = C_1, \qquad \int_{\mathbf{R}^n} |\nabla u|^2 dx = C_2.$$

For u_ϵ we have

$$\int_{\mathbf{R}^n} |\nabla u_\epsilon|^2 dx = \int_{\mathbf{R}^n} |\nabla_y u|^2 dy = C_2, \tag{4.7}$$

and

$$\int_{\mathbf{R}^n} u_\epsilon^2 dx = \epsilon^2 \int_{\mathbf{R}^n} u^2 dy = \epsilon^2 C_1, \tag{4.8}$$

so that by (4.5)

$$\epsilon^2 C_1 \le C_0 \cdot C_2, \tag{4.9}$$

for all $\epsilon > 0$, contradiction. Thus the answer to our question is negative.

The scaling (4.6) we performed seems to be crucial in order to reject the inequality (4.5). The next logical step now is to try and make the result of the scaling the same on both sides, by changing the inequality. Let us therefore try

$$\left(\int_{\mathbf{R}^n} |u|^p dx \right)^{2/p} \quad \text{instead of} \quad \int_{\mathbf{R}^n} u^2 dx.$$

Scaling by (4.6) again, (4.8) now reads

$$\left(\int_{\mathbf{R}^n} |u_\epsilon|^p dx \right)^{2/p} = \epsilon^{2-n+\frac{2N}{p}} \left(\int_{\mathbf{R}^n} |u|^p dy \right)^{2/p} = \epsilon^{2-n+\frac{2N}{p}} C_1. \tag{4.10}$$

If we choose $2 - n + \frac{2n}{p} = 0$, i.e. $p = \frac{2n}{n-2}$, we cannot argue as before in order to reject the inequality

$$\left(\int_{\mathbf{R}^n} |u|^p dx \right)^{2/p} \le C_0 \int_{\mathbf{R}^n} |\nabla u|^2 dx, \quad \forall u \in C_0^\infty(\mathbf{R}^N), \tag{4.11}$$

so if $p = \frac{2n}{n-2}$ there is a chance that (4.11) holds.

Theorem 4.1. (Sobolev-inequality) *Let $n > 2$ and $p = \frac{2n}{n-2}$, then*

$$\|u\|_{L^p} \le C_0 \|\nabla u\|_{L^2}, \quad \forall u \in C_0^\infty(\mathbf{R}^n). \tag{4.12}$$

Proof. Following the book of Adams [A], we shall sketch a proof due to Gagliardo [G]. Let $\omega_i(x)$ be a straight line through x parallel to the x_i-axis, and let

$$\gamma = 2\frac{n-1}{n-2}. \tag{4.13}$$

Observe that if $u \in C_0^\infty(\mathbf{R}^n)$,

$$\int_0^\infty \frac{d}{dt} \Big| u(x + te_i) \Big|^\gamma dt = -\Big| u(x) \Big|^\gamma. \tag{4.14}$$

Hence

$$\sup_{y\in\omega_i(x)} |u(y)|^\gamma \le \gamma \int_{-\infty}^{\infty} |u(x)|^{\gamma-1}|D_i u(x)|dx_i. \tag{4.15}$$

Let $\hat{x}_i = (x_i, .., x_{i-1}, x_{i+1}, ..., x_n)$, and write

$$F_i(\hat{x}_i) = \sup_{y\in\omega_i(x)} |u(y)|^{\gamma/n-1}. \tag{4.16}$$

By (4.15)

$$|F_i(\hat{x}_i)|^{n-1} \le \gamma \int_{-\infty}^{\infty} |u(x)|^{\gamma-1}|D_i u(x)|dx_i,$$

so

$$\int |F_i(\hat{x}_i)|^{n-1} d\hat{x}_i \le \gamma \int_{\mathbf{R}^N} |u(x)|^{\gamma-1}|D_i u(x)|dx, \tag{4.17}$$

whence, by Hölder's inequality,

$$\begin{aligned}\int_{\Omega_i} |F_i(\hat{x}_i)|^{n-1} d\hat{x}_i &\le \gamma \Big(\int_{\mathbf{R}^n} |D_i u|^2\Big)^{1/2} \Big(\int_{\mathbf{R}^n} |u|^{(\gamma-1)2}\Big)^{1/2} \\ &\le \gamma \Big(\int_{\mathbf{R}^n} |\nabla u|^2\Big)^{1/2} \|u\|^{\gamma-1}_{L^{(\gamma-1)2}},\end{aligned} \tag{4.18}$$

where $\gamma - 1 = \frac{n}{n-2}$.

By [A,Lemma 5.9] we have

$$\int \prod_{i=1}^{n} F_i(\hat{x}_i)dx \le \prod_{i=1}^{n} \Big(\int |F_i(\hat{x}_i)|^{n-1} d\hat{x}_i\Big)^{1/n-1}, \tag{4.19}$$

so that, using $\gamma/(n-1) = \frac{2}{n-2}$,

$$\begin{aligned}\|u\|^{\frac{2n}{n-2}}_{L^{\frac{2n}{n-2}}} &\le \int \prod_{i=1}^{n} F_i(\hat{x}_i)dx \\ &\le \prod_{i=1}^{n} \Big(\int |F_i(\hat{x}_i)|^{n-1} d\hat{x}_i\Big)^{\frac{1}{n-1}} \\ &\le \gamma^{n/n-1} \Big(\int_{\mathbf{R}^n} |\nabla u|^2\Big)^{\frac{n}{2(n-1)}} \|u\|^{\frac{n}{n-1}(\gamma-1)}_{L^{(\gamma-1)2}}.\end{aligned} \tag{4.20}$$

Thus

$$\begin{aligned}\|u\|^{2\frac{n-1}{n-2}}_{L^{\frac{2n}{n-2}}} &\le \gamma \Big(\int_{\mathbf{R}^n} |\nabla u|^2\Big)^{1/2} \|u\|^{\gamma-1}_{L^{\frac{2n}{n-2}}} \\ &= \gamma \|\nabla u\|_{L^2} \|u\|^{\frac{n}{n-2}}_{L^{\frac{2n}{n-2}}},\end{aligned}$$

which gives, dividing both sides by $\|u\|_{L^{\frac{2n}{n-2}}}^{\frac{n}{n-2}}$, that

$$\|u\|_{L^{\frac{2n}{n-2}}} \leq \gamma \|\nabla u\|_{L^2} \quad \forall u \in C_0^\infty(\mathbf{R}^n).$$

This completes the proof.

Remark. It is possible to find the optimal constant in (4.11) for which the inequality is true. This can be found in Gilbarg & Trudinger [GT] or Talenti [T].

Observe that $\|\nabla u\|_{L^2}$ is norm on $C_0^\infty(\mathbf{R}^n)$. As before (4.11) extends to all functions

$$u \in \mathcal{D}^{1,2}(\mathbf{R}^n) = cp(C_0^\infty(\mathbf{R}^n), \|\cdot\|_{\mathcal{D}^{1,2}}), \tag{4.21}$$

where

$$\|\cdot\|_{\mathcal{D}^{1,2}} = \|\nabla \cdot\|_{L^2}. \tag{4.22}$$

Corollary 4.2. *For $n > 2$ we have*

$$\mathcal{D}^{1,2}(\mathbf{R}^n) \hookrightarrow L^{\frac{2n}{n-2}}(\mathbf{R}^n), \tag{4.23}$$

and

$$\|u\|_{L^{\frac{2n}{n-2}}} \leq C\|u\|_{\mathcal{D}_{1,2}} \tag{4.24}$$

Remark. For bounded domains $\Omega \subset \mathbf{R}^n$, we have $L^{p_1}(\Omega) \subset L^{p_2}(\Omega)$ if $p_1 > p_2$. This is not true for $\Omega = \mathbf{R}^n$, so $\mathcal{D}^{1,2}(\mathbf{R}^n) \not\hookrightarrow L^p(\mathbf{R}^n)$, $p < \frac{2n}{n-2}$, as the counter example

$$u(x) = \frac{1}{(1+|x|^2)^{\frac{n-2}{2}}}$$

shows.

Now we employ the interpolation inequalities for L^p-spaces (see e.g. Brezis [Br]), which read

$$\|u\|_{L^r} \leq \|u\|_{L^{p_1}}^\alpha \|u\|_{L^{p_2}}^{1-\alpha} \quad \text{for} \quad p_1 \leq r \leq p_2, \tag{4.25}$$

where

$$\frac{1}{r} = \frac{\alpha}{p_1} + \frac{1-\alpha}{p_2}. \tag{4.26}$$

Theorem 4.3. *Let $n > 2$ and let $\Omega \subset \mathbf{R}^n$ be a bounded domain. Then*

$$H_0^1(\Omega) \hookrightarrow L^p(\Omega), \quad 2 \leq p \leq \frac{2n}{n-2} \tag{4.27}$$

and

$$\|u\|_{L^p} \leq C(p)\|u\|_{H_0^1} = C(p)\|\nabla u\|_{L^2}. \tag{4.28}$$

Proof. Exercise, use Theorem 4.1 combined with the interpolation-inequalities above.

The embedding (4.27) is sometimes denoted by $i : H_0^1 \hookrightarrow L^p$. Inequality (4.28) shows that i is continuous. Next we examine for which p it is compact. Without a proof we state here the Rellich-Knodrachov embedding theorem, see [A].

Theorem 4.4. *Let $n > 2$ and $\Omega \subset \mathbf{R}^n$ bounded. Then*

$$i : H_0^1 \hookrightarrow L^2(\Omega) \quad \text{is compact.}$$

Theorem 4.5. *Let $n > 2$ and let $\Omega \subset \mathbf{R}^n$ be bounded. Then*

$$i : H_0^1 \hookrightarrow L^p(\Omega) \quad \text{is compact}$$

if

$$1 \leq p < \frac{2n}{n-2}.$$

Proof. Consider a bounded sequence $\{u_m\} \subset H_0^1(\Omega)$, then along a subsequence we have $u_m \rightharpoonup u$. Using the L^p-interpolation again we obtain

$$\begin{aligned} \|u_m - u\|_{L^p} &\leq \|u_m - u\|_{L^2}^{\alpha} \|u_m - u\|_{L^{\frac{2n}{n-2}}}^{1-\alpha} \\ &\leq \|u_m - u\|_{L^2}^{\alpha} \|u_m - u\|_{H_0^1}^{1-\alpha} \leq C\|u_m - u\|_{L^2}^{\alpha} \to 0, \end{aligned}$$

as $m \to \infty$, using the compactness of the embedding in L^2 (Theorem 4.4). Clearly the latter implies strong convergence of $u_m \to u$ in $L^p(\Omega)$ if $2 \leq p < \frac{2n}{n-2}$. This proves the theorem.

We note that compactness is lost when $p = \frac{2n}{n-2}$. This can be seen again from the scaling (4.6): as $\varepsilon \to 0$, the functions u_ε cannot converge in L^p, because this space does not contain the δ-function.

5. The implicit function theorem.

Let us first recall the implicit function theorem [HG].

Theorem 5.1. *(The Implicit Function Theorem) Let X, Y be Banach spaces, and let $\Theta \subset \mathbf{R} \times X$ be open. Suppose $f : \Theta \to Y$ is continuously differentiable, let $(\lambda_0, x_0) \in \Theta$ and $y_0 = f(\lambda_0, x_0)$, and suppose, writing*

$$\langle df(\lambda_0, x_0), (\lambda - \lambda_0, x - x_0)\rangle =$$

$$\langle d_\lambda f(\lambda_0, x_0), \lambda - \lambda_0\rangle + \langle d_x f(\lambda_0, x_0), x - x_0\rangle,$$

that

$$d_x f(\lambda_0, x_0) : \quad X \to Y$$

has a bounded inverse. Then there exists a continuously differentiable function $\xi(\lambda)$ with values in X defined in a neighbourhood of $\lambda = \lambda_0$, such that $\xi(\lambda_0) = x_0$, and $f(\lambda, \xi(\lambda)) = y_0$. The curve $\lambda \to (\lambda, \xi(\lambda))$ parameterizes the level set of y_0 in a neighbourhood of (λ_0, x_0).

We are going to apply the implicit function theorem to the function

$$f(p, u) = \Delta u + u^p, \tag{5.1}$$

where we stress once more the convention (2.1). For the domain of f we take

$$Dom(f) = [1, \frac{n+2}{n-2}) \times H_0^1(\Omega), \tag{5.2}$$

which for $n = 2$ should be read as $[1, \infty) \times H_0^1(\Omega)$.

What is then the range of f? Let us recall that u is a weak solution of

$$(D) \begin{cases} -\Delta u = g & \text{in } \Omega \\ \quad u \;\; = 0 & \text{on } \partial\Omega \end{cases}$$

for a given right hand side g, if $u \in H_0^1(\Omega)$ and

$$\int_\Omega \nabla u \nabla \varphi \quad = \quad \int_\Omega g\varphi \quad \forall \varphi \in H_0^1(\Omega). \tag{5.3}$$

Thus for every g for which the right hand side is a well defined bounded linear functional on $H_0^1(\Omega)$, there exists a unique $u \in H_0^1(\Omega)$ representing the functional

$$\varphi \;\rightarrow\; \int_\Omega g\varphi \tag{5.4}$$

as its Riesz representation

$$\varphi \;\rightarrow\; (u, \varphi)_{H_0^1}. \tag{5.5}$$

Observing that (5.4) defines a distribution on Ω for any $g \in L^1_{\text{loc}}(\Omega)$, we let $H^{-1}(\Omega)$ be the class of all distributions T on Ω such that the map

$$\varphi \in H_0^1(\Omega) \quad \rightarrow \quad \langle T, \varphi \rangle \tag{5.6}$$

is bounded with respect to the $H_0^1(\Omega)$-norm. Certainly $H^{-1}(\Omega)$ contains the functions in $L^2(\Omega)$, but it also contains their first order partial derivatives (in the distributional sense), because

$$\varphi \rightarrow \langle \frac{\partial g}{\partial x_i}, \varphi \rangle = \langle -g, \frac{\partial \varphi}{\partial x_i} \rangle.$$

It can be shown that this is all, so that $H^{-1}(\Omega)$ is spanned by L^2 functions and their first order derivatives. With the norm

$$||T||_{H^{-1}} = \sup_{\substack{\varphi \in H_0^1(\Omega) \\ ||\varphi|| = 1}} \langle T, \varphi \rangle, \tag{5.7}$$

the map

$$(-\Delta)^{-1} : \quad H^{-1}(\Omega) \rightarrow H_0^1(\Omega)$$

is an isometrical isomorphism. The space $H^{-1}(\Omega)$ is commonly called the dual space of $H_0^1(\Omega)$ which has to be interpreted in the sense of (5.6), and does not contradict the fact that Hilbert spaces are isomorphic to their own duals. Thus we have that also the map

$$u \in H_0^1(\Omega) \to -\Delta u \in H^{-1}(\Omega) \tag{5.8}$$

is an isomorphism, and therefore it is automatically continuously differentiable, its derivative being given by the map $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ itself [A].

Next we consider for fixed p the map

$$u \in H_0^1(\Omega) \quad \to \quad u^p. \tag{5.9}$$

First observe that for $1 \leq p \leq \frac{n+2}{n-2}$ $(1 \leq p < \infty$ if $n = 2)$, the range of (5.9) is contained in $H^{-1}(\Omega)$, because $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$, so that $u^p \in L^{\frac{p+1}{p}}(\Omega)$ and hence

$$H_0^1(\Omega) \ni \varphi \quad \to \quad \int_\Omega u^p \varphi \tag{5.10}$$

defines a bounded linear functional on $H_0^1(\Omega)$ because $(\frac{p+1}{p})^{-1} + (p+1)^{-1} = 1$. In fact (5.9) is a composition of three maps:

$$u \in H_0^1(\Omega) \hookrightarrow u \in L^{p+1}(\Omega) \to u^p \in L^{\frac{p+1}{p}}(\Omega) \hookrightarrow H^{-1}(\Omega), \tag{5.11}$$

and its continuity follows directly from Lemma 2.4.

Finally we show that (5.9) is continuously differentiable. It suffices again to show this for

$$N : u \in L^{p+1}(\Omega) \to u^p \in L^{\frac{p+1}{p}}(\Omega). \tag{5.12}$$

Common sense suggests that

$$dN(u) : \varphi \in L^{p+1}(\Omega) \to p|u|^{p-1}\varphi \in L^{\frac{p+1}{p}}, \tag{5.13}$$

and

$$dN : u \in L^{p+1}(\Omega) \to p|u|^{p-1} \in L^{\frac{p+1}{p-1}}(\Omega). \tag{5.14}$$

The continuity of dN follows from Lemma 2.4 with $q = p - 1$. However, we still have to show that dN is really the derivative of N.

We have (pointwise for every $x \in \Omega$)

$$\begin{aligned}(u+\varphi)^p - u^p &= \int_0^1 \frac{d}{dt}(u+t\varphi)^p dt = \int_0^1 p|u+t\varphi|^{p-1}\varphi dt \\ &= \int_0^1 p|u|^{p-1}\varphi dt + \int_0^1 p(|u+t\varphi|^{p-1} - |u|^{p-1})\varphi dt \\ &= p|u|^{p-1}\varphi + pR(u,\varphi)\varphi, \end{aligned} \tag{5.15}$$

where

$$R(u,\varphi) = \int_0^1 (|u+t\varphi|^{p-1} - |u|^{p-1})dt. \tag{5.16}$$

In order to show that (5.13) defines the (continuous) derivative of N, it suffices to show that for fixed u,

$$\|R(u,\varphi)\|_{L^{\frac{p+1}{p-1}}} \to 0 \quad \text{as} \quad \|\varphi\|_{L^{p+1}} \to 0. \tag{5.17}$$

Indeed

$$\begin{aligned}
\|R(u,\varphi)\|_{L^{\frac{p+1}{p-1}}} &= \left\{\int_\Omega \left\{\int_0^1 (|u+t\varphi|^{p-1} - |u|^{p-1})dt\right\}^{\frac{p+1}{p-1}} dx\right\}^{\frac{p-1}{p+1}} \\
&\le \left\{\int_\Omega \int_0^1 \left||u+t\varphi|^{p-1} - |u|^{p-1}\right|^{\frac{p+1}{p-1}} dt dx\right\}^{\frac{p-1}{p+1}} \\
&= \left\{\int_0^1 \int_\Omega \left||u+t\varphi|^{p-1} - |u|^{p-1}\right|^{\frac{p+1}{p-1}} dx dt\right\}^{\frac{p-1}{p+1}} \\
&\le \int_0^1 \left\||u+t\varphi|^{p-1} - |u|^{p-1}\right\|_{L^{\frac{p+1}{p-1}}} dt \to 0
\end{aligned}$$

as $\|\varphi\|_{L^{p+1}} \to 0$, again because of Lemma 2.4. Thus we have proved

Proposition 5.2. *Let $1 \le p \le \frac{n+2}{n-2}$ $(1 \le p < \infty$ if $n = 2)$. Then*

$$\tilde{f} : H_0^1(\Omega) \to H^{-1}(\Omega),$$

defined by

$$\tilde{f}(u) = \Delta u + u^p, \tag{5.18}$$

is continuously differentiable with

$$d\tilde{f} : H_0^1(\Omega) \to B(H_0^1(\Omega), H^{-1}(\Omega))$$

given by

$$d\tilde{f} : \ u \to \Delta + p|u|^{p-1}. \tag{5.19}$$

This means that for all $\varphi \in H_0^1(\Omega)$

$$\langle d\tilde{f}(u), \varphi\rangle = \Delta\varphi + p|u|^{p-1}\varphi \in H^{-1}(\Omega),$$

i.e. for all $\psi \in H_0^1(\Omega)$

$$\langle\langle d\tilde{f}(u), \varphi\rangle, \psi\rangle = -\int_\Omega \nabla\varphi\nabla\psi + p\int_\Omega |u|^{p-1}\varphi\psi. \tag{5.20}$$

We recall that

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1},$$

and

$$\langle dJ(u), \varphi\rangle = \int_\Omega \nabla u \nabla\varphi - \int_\Omega u^p\varphi = \langle \Delta u + u^p, \varphi\rangle. \tag{5.21}$$

Hence

$$\tilde{f}(u) = -dJ(u), \tag{5.22}$$

so that J is twice continuously differentiable with

$$\langle d^2J(u), (\varphi,\psi)\rangle = -\langle\langle d\tilde{f}(u), \varphi\rangle, \psi\rangle = \int_\Omega \nabla\varphi\nabla\psi - p\int_\Omega |u|^{p-1}\varphi\psi \tag{5.23}$$

Next we consider $\Delta u + u^p$ as the function $f(p,u)$ in (5.1). Then for p fixed, $f(p,u)$ is partially differentiable with respect to u, and $d_u f(p,u)$ is given by $d\tilde{f}(u)$ above. Clearly

$$d_u f(p,u) = \Delta + p|u|^{p-1} \tag{5.24}$$

is continuous in u, and also continuous in $p \in [1, \frac{n+2}{n-2}]$, because the map $p \to p|u|^{p-1}$ is continuous from $\mathbf{R}$ to $L^{\frac{n}{2}}(\Omega)$. The latter space can be considered as a subspace of multiplication operators in $B(L^{\frac{2n}{n-2}}(\Omega), L^{\frac{2n}{n+2}}(\Omega))$, in view of

$$\left(\frac{2n}{n+2}\right)^{-1} = \left(\frac{n}{2}\right)^{-1} + \left(\frac{2n}{n-2}\right)^{-1},$$

and hence embeds continuously in $B(H_0^1(\Omega), H^{-1}(\Omega))$. We conclude

Proposition 5.3. *The mapping*

$$d_u f : [1, \frac{n+2}{n-2}] \times H_0^1(\Omega) \to B(H_0^1(\Omega), H^{-1}(\Omega))$$

is continuous. (For $n = 2$ take $Dom(f) = [1,\infty) \times H_0^1(\Omega)$.)

Next we consider the partial derivatives of f with respect to p, i.e. the derivative of

$$p \in [1, \frac{n+2}{n-2}) \to u^p \in H^{-1}(\Omega).$$

Proposition 5.4. *The mapping*

$$f : [1, \frac{n+2}{n-2}) \times H_0^1(\Omega) \to H^{-1}(\Omega)$$

is continuous partially differentiable with respect to p, with

$$d_p f(p,u) = u^p \log|u|,$$

meaning that for all $\varphi \in H_0^1(\Omega)$,

$$\langle d_p f(p,u), \varphi\rangle = \int_\Omega \varphi u^p \log|u|.$$

Proof. This is left to the reader as an exercise. Note that $p = \frac{n+2}{n-2}$ is not allowed here because of the $\log|u|$ in the derivative.

As a result of what we have done so far we state

Theorem 5.5. *The function*

$$f : [1, \frac{n+2}{n-2}) \times H_0^1(\Omega) \to H^{-1}(\Omega)$$

defined by $f(p,u) = \Delta u + u^p$ is continuously differentiable.

Now we can try to apply the implicit function theorem: suppose we have $\overline{p} \in (1, \frac{n+2}{n-2})$ and $\overline{u} \in H_0^1(\Omega)$, such that $f(\overline{p}, \overline{u}) = 0$. Does there exist then a curve $p \to u(p)$, defined for p close to $\overline{p}$, such that $u(\overline{p}) = \overline{u}$ and $f(p, u(p)) = 0$? To answer this question we need to know whether

$$d_u f(\overline{p}, \overline{u}) = \Delta + \overline{p}|\overline{u}|^{\overline{p}-1}$$

is invertible and has a bounded inverse. In other words, omitting the overbars:

Question. Given any $f \in H^{-1}(\Omega)$, does there exist a $\omega \in H_0^1(\Omega)$, such that

$$-\Delta\omega - p|u|^{p-1}\omega = f, \tag{5.25}$$

and, moreover, such that the map $f \in H^{-1}(\Omega) \to \omega \in H_0^1(\Omega)$ is bounded? (We have put in the minus for notational convenience later on.)

Note that (5.25) is equivalent to

$$a(\omega, \varphi) = \int_\Omega \nabla\omega\nabla\varphi - p\int_\Omega |u|^{p-1}\omega\varphi = \langle f, \varphi\rangle \quad \forall \varphi \in H_0^1(\Omega). \tag{5.26}$$

Proposition 5.6. *The bilinear form $a(\omega, \varphi)$, defined on $H_0^1(\Omega) \times H_0^1(\Omega)$ by (5.26), is bounded and symmetric.*

Proof. Symmetry is obvious. Boundedness follows from

$$\begin{aligned} |a(\omega,\varphi)| &\le \int_{|\Omega} \nabla\omega\nabla\varphi| + p\int_\Omega |u|^{p-1}|\omega||\varphi| \\ &\le \|\omega\|_{H_0^1}\|\varphi\|_{H_0^1} + p\|u^{p-1}\|_{L^{\frac{n}{2}}}\|\omega\|_{L^{\frac{2n}{n-2}}}\|\varphi\|_{L^{\frac{2n}{n-2}}} \\ &\le (1 + pC_0)\|\omega\|_{H_0^1}\|\varphi\|_{H_0^1}, \end{aligned} \tag{5.27}$$

where C_0 is the Sobolev constant for the embedding $H_0^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$ (Theorem 4.1).

Clearly the bilinear form $a(\omega, \varphi)$ is not coercive because we do not have $a(\omega, \omega) \ge \delta\|\omega\|^2$ for all $\omega \in H_0^1(\Omega)$ and some fixed $\delta > 0$, but we can make it coercive by replacing it by

$$\tilde{a}(\omega, \varphi) = \int_\Omega \nabla\omega\nabla\varphi + (A - p|u|^{p-1})\omega\varphi. \tag{5.28}$$

Then solving

$$\tilde{a}(\omega, \varphi) = \langle f, \varphi\rangle \quad \forall \varphi \in H_0^1(\Omega), \tag{5.29}$$

corresponds to solving

$$-\Delta\omega + (A - p|u|^{p-1})\omega = f. \tag{5.30}$$

Proposition 5.7. *For A sufficiently large $\tilde{a}$ is a symmetric bounded coercive bilinear form on $H_0^1(\Omega)\times H_0^1(\Omega)$, Hence it defines an equivalent innerproduct.*

Proof. We only have to show coerciveness. We have

$$\begin{aligned}\tilde{a}(\omega,\omega) &= \int_\Omega |\nabla\omega|^2 + \int_\Omega (A-p|u|^{p-1})\omega^2 \\ &\geq \|\omega\|^2_{H_0^1} - \int_\Omega (p|u|^{p-1}-A)_+\omega^2 \\ &\geq \|\omega\|^2_{H_0^1} - \|(p|u|^{p-1}-A)_+\|_{L^{\frac{n}{2}}}\|\omega\|_{L^{\frac{2n}{n-2}}}\|\omega\|_{L^{\frac{2n}{n-2}}} \\ &\geq \left(1 - C\|(p|u|^{p-1}-A)_+\|_{L^{\frac{n}{2}}}\right)\|\omega\|^2_{H_0^1} \geq \delta\|\omega\|^2_{H_0^1},\end{aligned}$$

for A large enough and some fixed $\delta>0$.

Now by the Riesz representation theorem, the map $\tilde{T}: f\in H^{-1}(\Omega)\to\omega\in H_0^1(\Omega)$ is well defined by (5.29). Moreover, considered as a map from $H_0^1(\Omega)$ with inner product $\tilde{a}(\cdot,\cdot)$, it is compact and symmetric. Symmetry follows because if $f\to\omega$ and $g\to z$, then

$$\tilde{a}(\omega,g)=\langle f,g\rangle=\langle g,f\rangle=\tilde{a}(z,f)=\tilde{a}(f,z),$$

while compactness follows from the compactness of the imbedding $H_0^1(\Omega)\hookrightarrow H^{-1}(\Omega)$.

Applying the spectral theorem for bounded linear symmetric compact operators, we find that $H_0^1(\Omega)$ has an orthonormal basis $\{\varphi_1,\varphi_2,...\}$ of eigenvectors ($\tilde{a}(\varphi_i,\varphi_j)=\delta_{ij}$) of $\tilde{T}$ corresponding to eigenvalues $\tilde{\mu}_1,\tilde{\mu}_2,\tilde{\mu}_3,...$, which are all positive in view of

$$\tilde{\mu}_i\tilde{a}(\varphi_i,\varphi_i)=\tilde{a}(\tilde{\mu}_i\varphi_i,\varphi_i)=\langle\varphi_i,\varphi_i\rangle.$$

Also the following characterizations hold

$$\begin{aligned}\tilde{\mu}_1 &= \max_{0\neq\varphi\in H_0^1(\Omega)}\frac{\tilde{a}(\tilde{T}\varphi,\varphi)}{\tilde{a}(\varphi,\varphi)} = \max_{0\neq\varphi\in H_0^1(\Omega)}\frac{\langle\varphi,\varphi\rangle}{\tilde{a}(\varphi,\varphi)} \\ &= \max_{0\neq\varphi\in H_0^1(\Omega)}\frac{\int\varphi^2}{\int|\nabla\varphi|^2+(A-p|u|^{p-1})\varphi^2},\end{aligned}$$

the maximum being achieved in φ_1, where φ_1 can be taken nonnegative.

$$\tilde{\mu}_2 = \max_{\substack{\tilde{a}(\varphi,\varphi_1)=0\\ \varphi\neq0}}\frac{\langle\varphi,\varphi\rangle}{\tilde{a}(\varphi,\varphi)} = \max_{\substack{\varphi\neq0\\ \int\varphi\varphi_1=0}}\frac{\int\varphi^2}{\tilde{a}(\varphi,\varphi)},$$

the maximum being achieved in φ_2,

$$\tilde{\mu}_3 = \max_{\substack{\varphi\neq0\\ \int\varphi\varphi_1=\int\varphi\varphi_2=0}}\frac{\int\varphi^2}{\tilde{a}(\varphi,\varphi)},$$

etc., with $\mu_1 \geq \mu_2 \geq \mu_3 \geq ... \downarrow 0$ because of compactness. Thus the map $\tilde{T}^{-1}$ considered from $H_0^1(\Omega)$ to $H_0^1(\Omega)$ has eigenvectors $\{\varphi_1, \varphi_2, ...\}$ which still form an orthonormal basis in $H_0^1(\Omega)$, with eigenvalues $\tilde{\lambda}_i$ corresponding to φ_i, and of course $\tilde{\lambda}_i = \mu_i^{-1}$, $0 < \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq ... \uparrow +\infty$, and

$$\tilde{\lambda}_1 = \min_{0 \neq \varphi \in H_0^1(\Omega)} \frac{\tilde{a}(\varphi, \varphi)}{\int \varphi^2}, \quad \tilde{\lambda}_2 = \min_{\substack{0 \neq \varphi \in H_0^1(\Omega) \\ \int \varphi\varphi_1 = 0}} \frac{\tilde{a}(\varphi, \varphi)}{\int \varphi^2}, \quad \text{etc.}$$

Thus $d_u f(p, u) = \Delta + p|u|^{p-1}$ has eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ... \uparrow +\infty$ with

$$\lambda_1 = \tilde{\lambda}_1 - A = \min_{0 \neq \varphi \in H_0^1(\Omega)} \frac{\int |\nabla \varphi|^2 - p|u|^{p-1}\varphi^2}{\int \varphi^2}, \tag{5.31}$$

$$\lambda_2 = \min_{\substack{0 \neq \varphi \in H_0^1(\Omega) \\ \int \varphi\varphi_1 = 0}} \int_\Omega \frac{|\nabla \varphi^2 - p|u|^{p-1}\varphi^2}{\int_\Omega \varphi^2}, \quad \text{etc.} \tag{5.32}$$

Clearly it follows that

Theorem 5.8. *The mapping $d_u f(p, u) : H_0^1(\Omega) \to H^{-1}(\Omega)$ has a bounded inverse $\Longleftrightarrow$ all $\lambda_i \neq 0 \Longleftrightarrow$ there is no nonzero solution of*

$$\Delta \omega + p|u|^{p-1}\omega = 0. \tag{5.33}$$

Thus we can conclude that the implicit function theorem can be applied around any solution for which (5.33) has only the trivial solution. In the next section we shall show that for the mountain pass solution

$$\lambda_1 < 0 \leq \lambda_2 \leq \lambda_3 \leq ... \uparrow \infty,$$

so that the only thing to exclude is that the second eigenvalue is zero.

6. The eigenvalues of the mountain pass.

We have seen that $f : [1, \frac{n+2}{n-2}] \times H_0^1(\Omega) \to H^{-1}(\Omega)$, defined by $f(p, u) = \Delta u + u^p$, has a continuous partial derivative $d_u f(p, u) : H_0^1(\Omega) \to H^{-1}(\Omega)$, given by $d_u f(p, u) = \Delta + p|u|^{p-1}$. The eigenvalues of $-d_u f(p, u)$, considered as $-d_u f(p, u) : H_0^1(\Omega) \to H_0^1(\Omega)$, are $\lambda_1 \leq \lambda_2 \leq \ldots \uparrow \infty$ corresponding to a basis of eigenvectors $\{\varphi_1, \varphi_2, \varphi_3, \ldots\}$. We have also seen that $-\Delta - p|u|^{p-1}$ is the second derivative of $J : H_0^1(\Omega) \to \mathbf{R}$ defined by

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{p+1} \int_\Omega |u|^{p+1}.$$

Now let us consider the mountain pass $\overline{u}$ in Section 3 which gave us our positive (weak) solution of $\Delta u + u^p = 0, u \in H_0^1(\Omega)$. This $\overline{u}$ was obtained by minimizing the maxima of J along half lines starting at the origin. Our first observation now is that

Proposition 6.1. *Let λ_1 be the first eigenvalue of $-\Delta - p|u|^{p-1}$, where u is any weak (positive) solution of (P). Then $\lambda_1 < 0$.*

Proof. We have, for all $\varphi \in H_0^1(\Omega)$,

$$-\int \nabla\varphi\nabla u + \int \varphi u^p = 0, \tag{6.1}$$

and

$$-\int \nabla\varphi\nabla\varphi_1 + \int p|u|^{p-1}\varphi\varphi_1 = -\int \lambda_1\varphi_1\varphi. \tag{6.2}$$

Inserting $\varphi = \varphi_1$ in (6.1) and $\varphi = u$ in (6.2) we find that, subtracting (6.1) from (6.2),

$$(p-1)\int u^p\varphi_1 = -\lambda_1\int \varphi_1^2$$

implying $\lambda_1 < 0$ because u and φ are positive.

Theorem 6.2. *Let $u > 0$ be the mountain pass obtained in Section 3. Then $\lambda_2 \geq 0$.*

Proof. Suppose that $\lambda_2 < 0$. We shall establish a contradiction by showing that there must then exist a two dimensional subspace $L \subset H_0^1(\Omega)$, such that $u \in L$, and the restriction of J to L has a strict local maximum in u, implying that u cannot be the mountain pass. There are two cases to consider.

Case (1). u is a linear combination of the eigenfunctions φ_1 and φ_2 corresponding to $\lambda_1 < 0$ and $\lambda_2 < 0$. Note that

$$\langle d^2J(u), (\varphi_1, \varphi_1)\rangle = \lambda_1\int \varphi_1^2 < 0; \quad \langle d^2J(u), (\varphi_2, \varphi_2)\rangle = \lambda_2\int \varphi_2^2 < 0;$$

$$\langle d^2J(u), (\varphi_1, \varphi_2)\rangle = \langle d^2J(u), (\varphi_2, \varphi_1)\rangle = 0,$$

so that writing $g(\xi, \eta) = J(u + \xi\varphi_1 + \eta\varphi_2)$, we have $g \in C^2(\mathbf{R}^2)$, and, after a suitable renormalization $dg(0,0) = 0$ and

$$d^2g(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hence g has a strict local maximum in $(0,0)$, so we can take for L the subspace spanned by φ_1 and φ_2.

Case (2). u is not a linear combination of φ_1, φ_2. Let M be the subspace spanned by u, φ_1 and φ_2, and define $g(\xi, \eta, \zeta) = J(u + \xi\varphi_1 + \eta\varphi_2 + \zeta u)$. Then also $dg(0,0,0) = 0$, and, since J has a maximum in u along the line spanned by u,

$$d^2g(0,0,0) = \begin{pmatrix} -1 & 0 & b \\ 0 & -1 & c \\ b & c & -A \end{pmatrix},$$

for some real numbers $A < 0$, b, c. In other words, the second order terms in the Taylor series of g are

$$-(\xi^2 + \eta^2 + A\zeta^2) + 2(b\xi + c\eta)\zeta.$$

Now let

$$L = \{u + \xi\varphi_1 + \eta\varphi_2 + \zeta u; \quad \xi, \eta, \zeta \in \mathbf{R}; \quad b\xi + c\eta = 0\}.$$

Then clearly the second terms in the Taylor series of the restriction of J to L are negative definite, implying that the restriction of J to L has a strict local maximum in u. This completes the proof.

Thus in order have the implicit function theorem applicable to the mountain pass, we must show that $\lambda_2 \neq 0$.

Theorem 6.3. *Suppose u is a solution in $H_0^1(\Omega)$ of $\Delta u + u^p = 0$ and suppose φ is a solution in $H_0^1(\Omega)$ of $\Delta\varphi + p|u|^{p-1}\varphi = 0$. Then both u and φ are smooth up to the boundary. Moreover the following identities hold:*

$$\frac{1}{2}\oint_{\partial\Omega} (\frac{\partial u}{\partial \nu})^2 (x, \nu) = \left(\frac{n}{p+1} - \frac{n-2}{2}\right) \int_\Omega |u|^{p+1},$$

and

$$\oint_{\partial\Omega} \frac{\partial u}{\partial \nu}\frac{\partial \varphi}{\partial \nu}(x, \nu) = 0.$$

Proof. See Section 10 and the appendix.

The essence of Lin's proof is

Theorem 6.4. *In the situation of the previous theorem suppose that $u > 0$ in Ω, and that φ_2 is the second eigenfunction. Then, if $n = 2$, and Ω is convex,*

$$\oint_{\partial\Omega} \frac{\partial u}{\partial \nu}\frac{\partial \varphi_2}{\partial \nu}(x, \nu) \neq 0.$$

Sketch of the proof. Since φ_2 satisfies $\int_\Omega \varphi_1\varphi_2 = 0$ and since φ_1 is nonnegative (in fact $\varphi_1 > 0$ in Ω because of the strong maximum principle [PW]), it follows that φ_2 must change sign in Ω, and the following proposition [CH] says in what way this must happen.

Proposition 6.5. *Let*

$$\Omega_+ = \{x \in \Omega : \varphi_2(x) > 0\},$$

and

$$\Omega_- = \{x \in \Omega : \varphi_2(x) < 0\}.$$

Then Ω_+ and Ω_- are connected.

We postpone the proof of Proposition 6.5 and continue with the sketch of the proof of Theorem 6.4. Roughly speaking, Proposition 6.5 says that in two dimensions there are only two possibilities. Either (I) we have, up to a sign change, that φ is positive everywhere near the boundary, or (II) there are exactly two points P_1 and P_2 on the boundary, which lie in the closure of both Ω_+ and Ω_-.

Now observe that everything we did so far is translation invariant. Thus, we may assume that, from the point of view of Ω, the origin of $\mathbf{R}^2$ is whereever we like.

Case I. Choose the origin in Ω. Then $(x,\nu) \geq 0$ on $\partial\Omega$, and not identically zero. By the boundary point lemma, both $\frac{\partial u}{\partial \nu}$ and $\frac{\partial \varphi_2}{\partial \nu}$ are strictly negative on $\partial\Omega$, so that the integral in Theorem 6.4 is strictly positive.

$$\oint_{\partial\Omega} \frac{\partial u}{\partial \nu}\frac{\partial \varphi_2}{\partial \nu}(x,\nu) > 0$$

. **Case II.** Draw two lines L_i $(i = 1, 2)$ tangential to $\partial\Omega$, touching Ω in P_i, and let the origin be the intersection of these lines. Since both (x,ν) and $\frac{\partial \varphi_2}{\partial \nu}$ change sign on the boundary only in the points P_1 and P_2, again the positivity of the integral in Theorem 6.4 follows. We shall check the details of this proof later on. (Are P_1 and P_2 well defined ? What happens if the two lines do not intersect ?)

Proof of Proposition 6.5. We use the eigenvalue formulas

$$\lambda_1 = \min_{0 \neq \varphi \in H_0^1(\Omega)} \frac{\int_\Omega \{|\nabla\varphi|^2 - p|u|^{p-1}\varphi^2\}}{\int_\Omega \varphi^2},$$

achieved by $\varphi_1 > 0$, and

$$\lambda_2 = \min_{\substack{0 \neq \varphi \in H_0^1(\Omega) \int_\Omega \\ \varphi\varphi_1 = 0}} \frac{\int_\Omega \{|\nabla\varphi|^2 - p|u|^{p-1}\varphi^2\}}{\int_\Omega \varphi^2}.$$

Observe that every minimizer in the second formula is an eigenfunction for λ_2. We shall show that there must be a minimizer φ_2 such that Proposition 6.5 holds. So let us assume that ψ is a minimizer for λ_2. Since $\int \psi\varphi_1 = 0, \psi$ must change sign. Let G_1 be a component of $\{x \in \Omega : \psi(x) > 0\}$, G_2 a component of $\{x \in \Omega : \psi(x) < 0\}$, and let

$$\psi_1 = \psi\chi_{G_1} \quad \text{and} \quad \psi_2 = \psi\chi_{G_2},$$

where χ_A deotes the characteristic function of A. Then $\psi_1 \geq 0$ and $\psi_2 \leq 0$. Now set

$$\varphi = a\psi_1 + b\psi_2.$$

Then

$$\begin{aligned}
\int_\Omega \{|\nabla\varphi|^2 - p|u|^{p-1}\varphi^2\} &= \int_\Omega \{a^2|\nabla\psi_1|^2 + 2ab\nabla\psi_1\nabla\psi_2 + b^2|\nabla\psi_2|^2 \\
&\quad - p|u|^{p-1}(a^2\psi_1^2 + 2ab\psi_1\psi_2 + b^2\psi_2^2)\} \\
&= a^2\int_\Omega \{|\nabla\psi_1|^2 - p|u|^{p-1}\psi_1^2\} + b^2\int_\Omega \{|\nabla\psi_2|^2 - p|u|^{p-1}\psi_2^2\} \\
&= a^2\int_\Omega \{\nabla\psi\nabla\psi_1 - p|u|^{p-1}\psi\psi_1\} + b^2\int \{\nabla\psi\nabla\psi_2 - p|u|^{p-1}\psi\psi_2\} \\
&= \lambda_2 a^2\int_\Omega \psi\psi_1 + \lambda_2 b^2\int_\Omega \psi\psi_2 \\
&= \lambda_2\{a^2\int_\Omega \psi_1^2 + b^2\int_\Omega \psi_2^2\} = \lambda_2\int_\Omega (a\psi_1 + b\psi_2)^2 = \lambda_2\int_\Omega \varphi^2,
\end{aligned}$$

so choosing $a, b \in \mathbf{R}$ such that

$$\int_\Omega (a\psi_1 + b\psi_2)\varphi_1 = 0,$$

we have another minimizer, which we call φ_2, for which the proposition holds.

N.B. Since eigenfunctions cannot vanish in an open subset, it follows that every eigenfunction of λ_2 satisfies Proposition 6.5.

Completion of the proof of Theorem 6.4 We have to be precise in deciding between Case I and Case II. We decompose $\partial\Omega$ as

$$\partial\Omega = S^+ \cup S^- \cup S^0,$$

where

$$\frac{\partial\varphi_2}{\partial\nu} \quad \begin{cases} < 0 & \text{on} \quad S^+ \\ = 0 & \text{on} \quad S^0 \\ > 0 & \text{on} \quad S^- \end{cases}$$

Clearly, S^+ and S^- are open in $\partial\Omega$. Observe that $S^0 \neq \partial\Omega$, for otherwise we could construct a compactly supported solution of

$$\Delta\varphi + p|u|^{p-1}\varphi = 0 \quad \text{on} \quad \mathbf{R}^2$$

by extending u and φ_2 to the whole of $\mathbf{R}^2$ setting $u \equiv \varphi_2 \equiv 0$ on Ω^c. In view of Theorem 6.6, which is a variant of [Carleman, C.R. Vol 197, 1933, pp. 471-474], it is not difficult to show that this implies $\varphi_2 \equiv 0$ on Ω, absurd. Thus S^+ and S^- are not both empty.

Case I. S^+ or S^- is empty. Without loss of generality, we assume $\partial\Omega = S^+ \cup S^0$. Clearly the argument in the sketch of the proof applies.

Case II. S^+ and S^- are both nonempty. We now observe that S^+ and S^- are connected. Indeed, if x_0 and y_0 are in S^+, then because Ω^+ is connected, there must be a smooth curve $\gamma \subset \Omega^+$ joining x_0 and y_0. Since Ω^- is also connected, on one side of γ we must have $\varphi_2 > 0$ by the strong maximum principle, and $\frac{\partial\varphi_2}{\partial\nu} < 0$ on $\partial\Omega$. Hence S^+ and also S^- are connected. Because S^+ and S^- are also open it follows that S^0 is the union of two disjoint closed intervals (probably singletons) in $\partial\Omega$. Now choose one point of each, say P_1 and P_2.

If the tangents L_1 and L_2 of $\partial\Omega$ at P_1 and P_2 intersect, then the argument in the sketch of the proof applies. So it remains to consider the situation in which they are parallel.

Observe that in Theorem 6.3 (ii) we may replace x by $x - x_0$ where x_0 is an arbitrary point. In fact this corresponds to choosing another origin. Subtracting the new formula from the old formula, we obtain

$$\int_{\partial\Omega} \frac{\partial u}{\partial\nu}\frac{\partial\varphi_2}{\partial\nu}(x_0, \nu) = 0 \quad \forall x_0 \in \mathbf{R}^2.$$

Without loss of generality, we may assume that in the figure above, L_2 coincides with the x_1-axis. Then, choosing $x_0 = (1,0)$, we have

$$0 = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\frac{\partial \varphi_2}{\partial \nu}\nu_1 = \int_{S^+} \frac{\partial u}{\partial \nu}\frac{\partial \varphi_2}{\partial \nu}\nu_1 + \int_{S^-} \frac{\partial u}{\partial \nu}\frac{\partial \varphi_2}{\partial \nu}\nu_1,$$

a contradiction, because both integrals on the right are nonzero and of the same sign. This completes the proof of the Theorem 6.4.

Remark. Carleman proves that solutions of a system of two first order equations in $\mathbf{R}^2$ cannot vanish identically in an open set without vanishing everywhere. In fact this is true for a large class of (hypo-)elliptic operators [H]. Below we adapt Carleman's proof to the result we used to conclude that S^+ and S^- in the proof above could not both be empty.

Theorem 6.6. *Suppose $u \in C^2(\Omega)$ is a solution of*

$$\Delta u + k(x)u = 0,$$

with $k \in C(\Omega)$. Here Ω is any domain in $\mathbf{R}^n$ containing 0. Then, if $u(x)|x|^{-m}$ is locally bounded for all $m \in \mathbf{N}$, it follows that $u \equiv 0$ in Ω.

Proof. We give the proof for $n = 2$, and use the complex structure of $\mathbf{R}^2$. Put, writing $z = x + iy$,

$$p = u_y \quad \text{and} \quad q = u_x.$$

Now consider $f(z) = p(x,y) + iq(x,y)$. Then

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f = p_x + iq_x + ip_y - q_y = u_{yx} + iu_{xx} + iu_{yy} - u_{xy} = -ik(x,y)u,$$

and, writing $f_n(z) = f(z)z^{-n}$,

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f_n(z) = -\frac{iku}{z^n}.$$

Note that f_n is as smooth as f because of the assumptions on u, and that

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(g(z)f(z)) = g(z)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(f(z))$$

for all $f \in C^1(\mathbf{R}^2;\mathbf{C})$ and all $g : \mathbf{C} \to \mathbf{C}$ analytic. We leave it to the reader check the following variant of Cauchy's formula:

$$f(\zeta) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-\zeta}dz - \frac{1}{2\pi}\int\int_D \frac{1}{z-\zeta}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(f(z))d(x,y).$$

Here D is a simply connected open set in Ω with $\gamma = \delta D \subset \Omega$ a smooth contour going around $\zeta \in \Omega$, and the function $f : \Omega \to \mathbf{C}$ must be real differentiable with continuous partial derivatives.

Applying this formula to f_n with $D = B_R$, we have for $\zeta \in B_R$,

$$f_n(\zeta) = \frac{1}{2\pi i} \oint_{\partial B_R} \frac{f_n(z)}{z-\zeta} dz + \frac{1}{2\pi} \int\int_{B_R} \frac{1}{z-\zeta} \frac{ku}{z^n} d(x,y),$$

so that, writing $\zeta = \xi + i\eta$,

$$\int\int_{B_R} |f_n(\zeta)| d(\xi,\eta) = \int\int_{B_R} \Big| \frac{1}{2\pi} \oint_{\partial B_R} \frac{f_n(z)}{z-\zeta} dz \Big| d(\xi,\eta)$$

$$+ \int\int_{B_R} \Big| \frac{1}{2\pi} \int\int_{B_R} \frac{1}{z-\zeta} \frac{ku}{z^n} d(x,y) \Big| d(\xi,\eta) \leq$$

$$\oint_{\partial B_R} |f_n(z)| \Big\{ \frac{1}{2\pi} \int\int_{B_R} \frac{d(\xi,\eta)}{|z-\zeta|} \Big\} |dz| + \int\int_{B_R} \Big| \frac{ku}{z^n} \Big| \Big\{ \frac{1}{2\pi} \int\int_{B_R} \frac{d(\xi,\eta)}{|z-\zeta|} \Big\} d(x,y).$$

Hence, since

$$\frac{1}{2\pi} \int\int_{B_R} \frac{d(x,y)}{|z-\zeta|} \leq 2R,$$

we find

$$\int\int_{B_R} \Big| \frac{f(z)}{z^n} \Big| d(x,y) \leq 2R^{1-n} \oint_{\partial B_R} |f(z)||dz| + 2R \max_{B_R} |k| \int\int_{B_R} \Big| \frac{u}{z^n} \Big| d(x,y).$$

We can absorb the second term on the right hand in the left side, because in B_R,

$$u(x,y) = \int_{(0,0)}^{(x,y)} (u_x dx + u_y dy) = \int_{(0,0)}^{(x,y)} q dx + p dy,$$

the integral being taken over a straight line joining $(0,0)$ and (x,y), so that in polar coordinates,

$$\Big| u(r\cos\theta, r\sin\theta) \Big| \leq \int_0^r \Big| f(\rho\cos\theta, \rho\sin\theta) \Big| d\rho,$$

whence

$$\int\int_{B_R} \Big| \frac{u}{z^n} \Big| d(x,y) \leq \int_0^{2\pi} \int_0^R \frac{1}{r^n} \int_0^r \Big| f(\rho\cos\theta, \rho\sin\theta) \Big| d\rho \; r dr d\theta \leq$$

$$\int_0^{2\pi} \int_0^R \int_0^R \frac{f(\rho\cos\theta, \rho\sin\theta)}{\rho^n} \rho d\rho dr d\varphi = R \int\int_{B_R} \Big| \frac{f(z)}{z^n} \Big| d(x,y).$$

Thus, for R small,

$$\int\int_{B_R} \Big| \frac{f(z)}{z^n} \Big| d(x,y) \leq \frac{2R}{1 - 2\|k\|_{L^\infty(B_R)} R^2} \frac{1}{R^n} \oint_{\partial B_R} |f(z)| dz.$$

Now assume that $f(\rho e^{i\theta}) \neq 0$ for some $0 < \rho < R$, $\theta \in (0, 2\pi)$. Then, as $n \to \infty$, clearly the left hand side is bounded from below by a multiple of ρ^{-n}, while the right hand side is bounded from above by a multiple of R^{-n}. Since $0 < \rho < R$, this is a contradiction.

7. A priori estimates.

In order to continue solutions of

$$(P) \qquad \begin{cases} \Delta u + u^p = 0 \quad \text{in} \quad \Omega, \\ u > 0 \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{in} \quad \partial\Omega, \end{cases}$$

as p (or Ω) is varied, we need estimates to ensure that solutions branches do not run out off the Hilbert space $H_0^1(\Omega)$. We shall first show that solutions of (P) are uniformly bounded by a constant, assuming the domain Ω is convex, which allows us to keep the proofs simple. See [BrT,FLN,GS].

Theorem 7.1. *Let $\Omega \subset \mathbf{R}^n$ be a bounded convex domain with smooth boundary $\partial\Omega$, and let $1 < p \leq \overline{p} < \frac{n}{n-2}$. Then if $u \in H_0^1(\Omega)$ is a weak solution of (P), we have*

$$\|u\|_{L^\infty(\Omega)} \leq M^{\frac{2}{p-1}},$$

where $M = M(\overline{p}, N)$ is a constant depending only on $\overline{p}$, N, and Ω.

This estimate can be established under much weaker assumptions. In particular $\frac{n}{n-2}$ can be replaced by $\frac{n+2}{n-2}$. The stronger assumption $p < \frac{n}{n-2}$ however allows for a much simpler proof.

In view of the regularity result for weak solutions (Section 10) we shall assume from here on that all solutions under consideration are smooth up to the boundary. We first establish a nonexistence result for positive solutions on $\mathbf{R}^n$, which is at the hart of the proof of Theorem 7.1.

Theorem 7.2. *Suppose $u : \mathbf{R}^n \to \mathbf{R}^+$ is a solution of $\Delta u + u^p = 0$ with $1 \leq p < \frac{n}{n-2}$. Then $u \equiv 0$.*

Proof. First we take $n = 2$. In polar coordinates we would have

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\varphi\varphi} + u^p = 0, \tag{7.1}$$

where $u = u(r, \varphi)$, $r \geq 0$, $\varphi \in (0, 2\pi)$. Integrating (7.1) with respect to φ, we find, if we define,

$$\omega(r) = \frac{1}{2\pi}\int_0^{2\pi} u(r, \varphi)d\varphi, \tag{7.2}$$

that

$$0 = \frac{1}{2\pi}\{\int_0^{2\pi} u_{rr}(r,\varphi)d\varphi + \int_0^{2\pi} \frac{1}{r}u_r(r,\varphi)d\varphi + \int_0^{2\pi} \frac{1}{r^2}u_{\varphi\varphi}(r,\varphi)d\varphi + \int_0^{2\pi} u(r,\varphi)^p d\varphi\}$$

$$= w''(r) + \frac{w'(r)}{r} + \frac{1}{2\pi}\int_0^{2\pi} u(r,\varphi)^p d\varphi \geq \omega''(r) + \frac{w'(r)}{r} + \left(\frac{1}{2\pi}\int_0^{2\pi} u(r,\varphi)d\varphi\right)^p$$

$$= \omega''(r) + \frac{w'(r)}{r} + \omega(r)^p, \tag{7.3}$$

so that ω is a radially symmetric function satisfying

$$\Delta\omega + \omega^p \leq 0. \tag{7.4}$$

By the mean value theorem for superharmonic functions we have

$$\omega(0) = u(0) \geq \frac{1}{2\pi}\int_0^{2\pi} u(r,\varphi)d\varphi = \omega(r). \tag{7.5}$$

But then, writing $r = e^{-t}$, we find

$$\frac{d^2\omega}{dt^2} + e^{-2t}\omega^p \leq 0, \quad t \in \mathbf{R}, \tag{7.6}$$

so that ω is a concave function on $\mathbf{R}$, which is nonnegative and bounded. This implies $\omega = 0$, so that also $u \equiv 0$.

Next we consider $n \geq 3$. Again we set

$$\omega(r) = \frac{1}{n\omega_n r^{n-1}} \oint_{\partial B_R} u, \tag{7.7}$$

where ω_n is the measure of the unit ball, and arrive at (7.4) with $\omega(0) \geq \omega(r)$. Thus

$$\omega'' + \frac{n-1}{r}\omega' + \omega^p \leq 0 \tag{7.8}$$

so that, writing $r = t^{-\frac{1}{n-2}}$,

$$(n-2)^2\frac{d^2\omega}{dt^2} + t^{-\frac{2(n-1)}{n-2}}\omega^p \leq 0, \quad t \in \mathbf{R}^+. \tag{7.9}$$

Hence ω is now a bounded concave nonnegative function of t on $\mathbf{R}^+$. Clearly we have, writing $\omega(t)$ instead of $\omega(r)$,

$$\omega'(t) = \int_t^\infty -\omega''(s)ds \geq \frac{1}{(n-2)^2}\int_t^\infty s^{-\frac{2(n-1)}{n-2}}\omega(s)^p ds$$

$$\geq \frac{\omega(t)^p}{(n-2)^2}\int_t^\infty s^{-\frac{2(n-1)}{n-2}}\,ds = \frac{\omega(t)^p}{n(n-2)}t^{-\frac{n}{n-2}},$$

so that

$$(\omega(t)^{1-p})' \leq \frac{1-p}{n(n-2)}t^{-\frac{n}{n-2}},$$

whence

$$\omega(\infty)^{1-p} - \omega(t)^{1-p} \leq \frac{1-p}{2n}t^{-\frac{2}{n-2}},$$

or

$$\omega(t) \leq \left(\omega(\infty)^{1-p} + \frac{p-1}{2n}t^{-\frac{2}{n-2}}\right)^{-\frac{1}{p-1}} \leq \left(\frac{p-1}{2n}\right)^{-\frac{1}{p-1}} t^{\frac{2}{(n-2)(p-1)}}. \tag{7.10}$$

Clearly (7.10) forces $\omega \equiv 0$ if the exponent in (7.10) is larger than one, i.e.

$$\frac{2}{(n-2)(p-1)} > 1,$$

because otherwise ω cannot be concave and nonnegative near $t = 0$. This completes the proof.

The next step is to narrow down the region where u can have its maximum. We use the moving plane method, see [GNN]. First we introduce some notation. We assume that Ω is convex. For $x_0 \in \partial\Omega$ and $\varepsilon > 0$ let

$$T_\varepsilon(x_0) = \{x \in \mathbf{R}^n : (ex_0 - x, \nu(x_0)) = \varepsilon\},$$

and

$$\Sigma_\varepsilon(x_0) = \{x \in \Omega : (ex_0 - x, \nu(x_0)) < \varepsilon\}.$$

Here $\nu(x_0)$ is the outward normal on $\partial\Omega$ in x_0. Now let

$$S_{\varepsilon,x_0} : \mathbf{R}^n \to \mathbf{R}^n$$

denote the orthogonal reflection in $T_\varepsilon(x_0)$, and let

$$\Sigma'_\varepsilon(x_0) = \{S_{\varepsilon,x_0}(x) : x \in \Sigma_\varepsilon(x_0)\}.$$

We say that $\varepsilon > 0$ is *admissible* for $x_0 \in \partial\Omega$ if

$$\Sigma'_\varepsilon(x_0) \subset \Omega$$

and

$$(\nu(x), \nu(x_0)) > 0 \quad \text{for all} \quad x \in T_\varepsilon(x_0) \cap \partial\Omega$$

Theorem 7.3. *Let Ω be a bounded convex domain with smooth boundary $\partial\Omega$, let $x_0 \in \partial\Omega$. Then the set of admissible $\varepsilon > 0$ is a nonempty bounded open interval of the form $(0, \varepsilon^*)$. Let u be a classical solution of P. Then*

$$\forall \varepsilon \in (0, \varepsilon^*] \quad \forall x \in \Sigma_\varepsilon(x_0) \qquad u(S_{\varepsilon,x_0}(x)) \geq u(x),$$

and, fixing $\nu = \nu(x_0)$,

$$\frac{\partial u}{\partial \nu} < 0 \quad \text{in} \quad \Sigma_\varepsilon(x_0).$$

Moreover, the first inequality is strict unless $\varepsilon = \varepsilon^*$ and both u and Ω are symmetric in $T_\varepsilon(x_0)$.

Proof. The existence of ε^* is obvious. Also the inequalities in Theorem 7.3 are obviously valid for small $\varepsilon > 0$ because [GT,PW] by the boundary point lemma $\frac{\partial u}{\partial \nu} < 0$ in x_0. Suppose there exists $\varepsilon_1 \in (0, \varepsilon^*)$ such that the inequalities are not valid for some $\varepsilon \geq \varepsilon_1$. We may assume that ε_1 is minimal in this respect, so that the inequalities do hold for all $\varepsilon \in (0, \varepsilon_1)$. We consider in $\Sigma_{\varepsilon_1}(x_0)$ the functions

$$\omega(x) = u(S_{\varepsilon_1, x_0}(x)) - u(x),$$

and

$$q(x) = \frac{\partial u}{\partial \nu}(x).$$

By continuity we have $w \geq 0$ and $q \leq 0$, because for all $0 < \varepsilon < \varepsilon_1$ the inequalities hold. Since $\varepsilon_1 < \varepsilon^*$, it follows that

$$\omega(x) = u(S_{\varepsilon_1, x_0}(x)) > 0 \quad \partial\Sigma_{\varepsilon_1, x_0}(x) \cap \partial\Omega.$$

Also in $\Sigma_{\varepsilon_1}(x_0)$

$$\Delta\omega(x) = -u(S_{\varepsilon_1, x_0}(x))^p + u(x)^p \leq 0,$$

so that [GT,PW] by the strong minimum principle for superharmonic functions,

$$\omega > 0 \quad \text{in} \quad \Sigma_{\varepsilon_1}(x_0).$$

Since $\omega = 0$ on $T_{\varepsilon_1}(x_0)$ the boundary point lemma again implies that

$$\frac{\partial \omega}{\partial \nu} > 0 \quad \text{on} \quad \Omega \cap T_{\varepsilon_1}(x_0).$$

Hence

$$\frac{\partial u}{\partial \nu} < 0 \quad \text{on} \quad \Omega \cap T_{\varepsilon_1}(x_0),$$

and, because ε_1 admissible, using the boundary point lemma once more,

$$\frac{\partial u}{\partial \nu} < 0 \quad \text{on} \quad \partial\Omega \cap \partial\Sigma_{\varepsilon_1}(x_0).$$

Thus

$$q = \frac{\partial u}{\partial \nu} < 0 \quad \text{on} \quad \partial\Sigma_{\varepsilon_1}(x_0),$$

and since $\Delta q = -pu^{p-1}q \geq 0$ in $\partial\Sigma_{\varepsilon_1}(x_0)$, we have

$$\frac{\partial u}{\partial \nu} < 0 \quad \text{on} \quad cl\big(\Sigma_{\varepsilon_1}(x_0)\big).$$

Together with $\omega > 0$ in $\Sigma_{\varepsilon_1}(x_0)$ this implies that the inequalities in Theorem 7.3 also hold for $\varepsilon > \varepsilon_1$ sufficiently close to ε_1. This is a contradiction, so ε_1 cannot exist.

To complete the proof we repeat the argument above with ε_1 replaced by ε^*. Again we find that inequalities hold for $\varepsilon = \varepsilon^*$, and that the inequalities are strict, unless $\omega = 0$ on $\partial\Sigma_{\varepsilon^*}(x_0)$, which implies

$$\Omega \backslash T_{\varepsilon^*}(x_0) = \Sigma_{\varepsilon^*}(x_0) \cup \Sigma'_{\varepsilon^*}(x_0),$$

so that Ω is symmetric in $T^*_\varepsilon(x_0)$. Interchanging the role of $u(x)$ and $u(S_{\varepsilon^*,x_0}(x))$ we conclude that also u must be symmetric in $T_{\varepsilon^*}(x_0)$. This completes the proof.

Corollary 7.4. *[GNN] If Ω is a ball, then every solution of (P) is radially symmetric.*

Corollary 7.5. *There exists a $\delta > 0$ such that if a solution u of (P) obtains its maximum in $\overline{x} \in \Omega$, then $B_\delta(\overline{x}) \subset\subset \Omega$. This δ depends only on the shape of the domain.*

We are now ready to prove Theorem 7.1

Proof of Theorem 7.1 Choose δ as in Corollary 7.5. Consider the class of all convex bounded domains Ω in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$, and also all $p \in (1, \overline{p})$ and all corresponding solutions u of (P). We may restrict our attention to solutions u having a global maximum in $0 \in \Omega$. We then define for each u,

$$\omega(x) = \frac{1}{u(0)} u(xu(0)^{\frac{1-p}{2}}), \tag{7.11}$$

which is certainly well defined for

$$|x| \leq \delta u(0)^{\frac{p-1}{2}}, \tag{7.12}$$

and clearly satisfies

$$0 < \omega \leq 1 = \omega(0), \tag{7.13}$$

and

$$\Delta\omega + \omega^p = 0. \tag{7.14}$$

We now claim that the right hand side of (7.12) is uniformily bounded by a constant $M = M(\overline{p}, N)$, for if it is not, then there exist arbitrarily large balls B_R and functions $\omega \in C^2(\overline{B_R})$ satisfying (7.13) and (7.14). A straight forward limiting argument then yields at least one $p \in [1, \overline{p}]$ for which (7.13) and (7.14) have a solution on the whole of $\mathbf{R}^n$. This is a contradiction in view of Theorem 7.2, and completes the proof of Theorem 7.1.

Corollary 7.6. *In the context of Theorem 7.1,*

$$\int_\Omega |\nabla u|^2 = \int_\Omega u^{p+1} \le |\Omega| M^{2\frac{p+1}{p-1}}.$$

We conclude this section by translating these a priori estimates to the problem

$$(\tilde{P}) \qquad \begin{cases} \Delta\varphi + \{\int_\Omega |\nabla\varphi|^2\}\varphi^p = 0 & \text{in} \quad \Omega; \\ \varphi > 0 \quad \text{in} \quad \Omega; \\ \varphi = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

The difference between (P) and $(\tilde{P})$ becomes clear if we take φ as a test function, to obtain

$$\left\{\int_\Omega |\nabla\varphi|^2\right\}\left\{\int_\Omega \varphi^{p+1} - 1\right\} = 0,$$

so that

$$\int_\Omega \varphi^{p+1} = 1.$$

In particular this implies that for $p = 1$ the solution of $(\tilde{P})$ is unique, namely the normalized positive eigenfunction of the first eigenvalue of the Laplacian on $H_0^1(\Omega)$. For $p > 1$ solutions u of (P) and solutions φ of $(\tilde{P})$ are in a $1-1$ correspondence:

$$u = \|\varphi\|_{H_0^1}^{\frac{2}{p-1}}\varphi = \|\varphi\|_{H_0^1}^{\frac{p+1}{p-1}}\frac{\varphi}{\|\varphi\|_{H_0^1}}, \quad \varphi = \|u\|_{H_0^1}^{-\frac{2}{p+1}}u = \|u\|_{H_0^1}^{\frac{p-1}{p+1}}\frac{u}{\|u\|_{H_0^1}}.$$

Thus we have

Corollary 7.7. *In the context of Theorem 7.1 if φ is a solution of $(\tilde{P})$, then*

$$\int_\Omega |\nabla\varphi|^2 \le |\Omega|^{\frac{p-1}{p+1}} M^2.$$

8. Uniqueness for $p \downarrow 1$.

Throughout this section Ω is a bounded convex domain with smooth boundary $\partial\Omega$. We shall show that for $p > 1$ close to one, (P) has exactly one solution. Here it is more convenient to work with $(\tilde{P})$ which has a unique solution if $p = 1$, whereas (P) either has no solutions or infinitely many if $p = 1$. Also recall that the mountain pass solution u we constructed corresponds to a minimizer φ of

$$\min_{\varphi\in H_0^1 \ \int|\varphi|^{p+1}=1} \int_\Omega |\nabla\varphi|^2,$$

and that φ is in fact a solution of $(\tilde{P})$.

Proposition 8.1. *For $p = 1$, problem $(\tilde{P})$ has a unique solution $\varphi_1 > 0$, and, writing*

$$\lambda_1 = \int_\Omega |\nabla\varphi_1|^2,$$

we have

$$\begin{cases} \Delta\varphi_1 + \lambda_1\varphi_1 = 0 & \text{in} \quad \Omega; \\ \varphi_1 > 0 \quad \text{in} \quad \Omega; \\ \varphi_1 = 0 \quad \text{in} \quad \partial\Omega, \end{cases}$$

that is to say, λ_1 is the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$, and φ_1 is the corresponding eigenfunction, normalized by $\|\varphi_1\|_{L^2} = 1$.

Proof: Obvious, see [Br].

Next we show that in this set-up the implicit function theorem can be applied around φ_1. Thus let

$$\tilde{f} : [1, \frac{n+2}{n-2}) \times H_0^1(\Omega) \to H^{-1}(\Omega)$$

be defined by

$$\tilde{f}(p,u) = \Delta u + \{\int_\Omega |\nabla u|^2\} u^p. \tag{8.1}$$

Note that we write u again instead of φ. Since obviously the map

$$u \in H_0^1(\Omega) \to \|u\|^2_{H_0^1} = \int_\Omega |\nabla u|^2$$

is continuously differentiable, it follows that $\tilde{f}$ has the same differentiability properties as the function $f(u) = \Delta u + u^p$ encountered earlier. Thus

$$\tilde{f} \in C^1\Big([1, \frac{n+2}{n-2}) \times H_0^1(\Omega); H^{-1}(\Omega)\Big).$$

Computing $d_u\tilde{f}$ using the product rule, we find

$$\langle d_u\tilde{f}(p,u), \varphi\rangle = \Delta\varphi + (\int_\Omega |\nabla u|^2) p|u|^{p-1}\varphi + \big(2\int_\Omega \nabla u \nabla\varphi\big) u^p, \tag{8.2}$$

so that, for $p = 1$ and $u = \varphi_1$,

$$\langle d_u\tilde{f}(1,\varphi_1), \varphi\rangle = \Delta\varphi + \lambda_1\varphi + 2\varphi\int \nabla\varphi_1\nabla\varphi = \Delta\varphi + \lambda_1\varphi + 2\lambda_1\varphi\int_\Omega \varphi_1\varphi. \tag{8.3}$$

Now consider the set of all eigenvalues, counted with multiplicity, $\lambda_1 < \lambda_1 \le \lambda_3 \le \lambda_4 \le \ldots$, of $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$, with corresponding eigenfunctions $\varphi_1, \varphi_2, \varphi_3, \varphi_4 \ldots$. Then

$$\langle d_u\tilde{f}(1,\varphi_1), \varphi_1\rangle = 2\lambda_1\varphi_1,$$

and for $i \ne 1$,

$$\langle d_u\tilde{f}(1,\varphi_1), \varphi_i\rangle = (\lambda_1 - \lambda_i)\varphi_i.$$

Since λ_1 is a simple eigenvalue, it follows that

$$d_u\tilde{f}(1,\varphi_1) : H_0^1(\Omega) \to H^{-1}(\Omega)$$

is invertible. Thus, as a consequence of the implicit function theorem, we have

Proposition 8.2. *There exists $\epsilon > 0$ and a continuously differentiable map $U : [1, 1+\epsilon) \to H_0^1(\Omega)$, such that $u = U(p)$ is a solution of*

$$\Delta u + \{\int_\Omega |\nabla u|^2\} u^p = 0. \tag{8.4}$$

Moreover, every solution $u \in H_0^1(\Omega)$ of $(\tilde{P})$ with $1 \le p < 1+\epsilon$ and $\|u - \varphi_1\| < \epsilon$ is also of the form $u = U(p)$.

Proof. Apply the implicit function theorem.

As a consequence of this theorem we now can prove

Theorem 8.3. *There exists $\epsilon > 0$ such that for all $1 < p < 1+\epsilon$ both (P) and $(\tilde{P})$ have a unique solution. The solution of $(\tilde{P})$ is the unique positive minimizer $\varphi \in H_0^1(\Omega)$ of*

$$\min_{\substack{\varphi \in H_0^1 \\ \int |\varphi|^{p+1} = 1}} \int_\Omega |\nabla \varphi|^2,$$

and the solution u of (P) is given by $u = \|\varphi\|^{\frac{2}{p-1}} \varphi$.

Proof. In view of Proposition 8.2 it suffices to show that when $p \downarrow 1$ all solutions of $(\tilde{P})$ tend to φ_1. This will be so if every sequence of solutions $(\psi_n)_{n=1}^\infty$ of $(\tilde{P})$ with $p = p_n$ and $p_n \downarrow 1$ has a subsequence which converges to φ_1. Recall that ψ_n corresponds to a solution $u_n = \|\psi_n\|^{\frac{2}{p-1}} \psi_n$ of (P). In view of Corollary 7.7, the sequence ψ_n is uniformily bounded in $H_0^1(\Omega)$. Also, since

$$\|\psi_n\|_{H_0^1} \ge \min_{\substack{\varphi \in H_0^1 \\ \int |\varphi|^p = 1}} \|\varphi\|_{H_0^1} = m_p,$$

and $\frac{1}{m_p}$ is the norm of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$, it follows that m_p is bounded away from zero, and hence the same holds for $\|\psi_n\|$.

Next, because (ψ_n) is a bounded sequence in $H_0^1(\Omega)$, we may extract a subsequence, again denoted by (ψ_n), such that $\psi_n \rightharpoonup \overline{\psi}$ (weak convergence) in $H_0^1(\Omega)$, i.e.

$$\int_\Omega \nabla \psi_n \nabla \varphi \longrightarrow \int_\Omega \nabla \overline{\psi} \nabla \varphi \quad \forall \varphi \in H_0^1(\Omega).$$

Since the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ is compact for $p < \frac{n+2}{n-2}$, we may extract a further subsequence, such that $\psi_n \longrightarrow \overline{\psi} \ge 0$ strongly in $L^{p+1}(\Omega)$ for every $1 \le p < \frac{n+2}{n-2}$.

Note that $\|\psi_n\|_{L^{p_n}} = 1$. For convenience we assume now that the measure of Ω is one. This can be established by a simple scaling. Then we have

$$\|\psi_n\|_{L^2} \le \|\psi_n\|_{L^{1+p_n}} \le \|\psi_n\|_{L^{2+\epsilon}}$$

for $1 < p_n < 1+\epsilon$, so that, taking the limit,

$$\|\overline{\psi}\|_{L^2} \le 1 \le \|\overline{\psi}\|_{L^{2+\epsilon}},$$

and hence, letting $\epsilon \downarrow 0$, we find

$$\int_\Omega \overline{\psi}^2 = 1.$$

Now consider the weak formulation of ψ_n being a solution of $(\tilde{P})$ with $p = p_n$, i.e. for all $\varphi \in H_0^1(\Omega)$

$$\int_\Omega \nabla\psi_n \nabla\varphi = \int_\Omega |\nabla\psi_n|^2 \int_\Omega \psi_n^{p_n}\varphi. \tag{8.5}$$

The left hand side of (8.5) converges to $\int \nabla\overline{\psi}\nabla\varphi$, and, extracting a further subsequence, the first factor in the right hand side converges to a positive number μ. Thus, since $\varphi \in H_0^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$ and, as we shall see in a moment,

$$\psi_n^{p_n} \longrightarrow \overline{\psi} \quad \text{in} \quad L^{\frac{2n}{n+2}}(\Omega), \tag{8.6}$$

we obtain, taking he limit in (8.5),

$$\int_\Omega \nabla\overline{\psi}\nabla\varphi = \mu \int_\Omega \overline{\psi}\varphi \quad \forall \varphi \in H_0^1(\Omega).$$

Since $\overline{\psi} \geq 0$ and $\int_\Omega \overline{\psi}^2 = 1$, it follows that $\mu = \lambda_1$ and $\overline{\psi} = \varphi_1$. Moreover

$$\mu = \lambda_1 = \int_\Omega |\nabla\overline{\psi}|^2,$$

so that also $\|\psi_n\|_{H_0^1} \longrightarrow \|\overline{\psi}\|_{H_0^1}$. Thus

$$\psi_n \longrightarrow \overline{\psi} \quad \text{strongly in} \quad H_0^1(\Omega),$$

and hence by Proposition 8.2, for n large, $\psi_n = U(p_n)$. We conclude that there cannot be a sequence (ψ_n) of solutions for $p = p_n \downarrow 1$ with $\psi_n \neq U(p_n)$ for all n. This completes the proof, if we are able to establish (8.6).

We have pointwise

$$|\psi_n^{p_n} - \overline{\psi}| \leq |\psi_n^{p_n} - \overline{\psi}^{p_n}| + |\overline{\psi}^{p_n} - \overline{\psi}| \leq p_n(\psi_n + \overline{\psi})^{p_n-1}|\psi_n - \overline{\psi}| + \overline{\psi}|\overline{\psi}^{p_n-1} - 1|. \tag{8.7}$$

Since $\psi_n \longrightarrow \overline{\psi}$ in all L^{p+1} with p subcritical, we have for all $1 \leq r < \infty$

$$\limsup_{n\to\infty} ||(\psi_n + \overline{\psi})^{p_n-1}||_{L^r}^r = \limsup_{n\to\infty} \int_\Omega (\psi_n + \overline{\psi})^{r(p_n-1)} < \infty,$$

so that the first term on the right hand side of (8.7) tends to zero in all $L^q(\Omega)$ with $1 \leq q < \frac{2n}{n-2}$. The same holds for the second term in view of the dominated convergence theorem. Hence (8.6) is valid.

9. Uniqueness on convex planar domains.

From now on we assume that $\Omega \subset \mathbf{R}^2$ is a bounded convex domain with smooth boundary $\partial\Omega$. So $1 < p < \infty$.

Theorem 9.1 There is exactly one positive solution $u \in H_0^1(\Omega)$ of (P) with the property that the linearized problem , i.e.

$$\begin{cases} -\Delta\varphi - pu^{p-1}\varphi = \lambda\varphi & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

has precisely one negative eigenvalue. Equivalently, there is exactly one positive minimizer φ (global or local) for

$$m_p = \min_{\varphi \in H_0^1 \ \|\varphi\|_{L^{p+1}}=1} \|\varphi\|_{H_0^1}.$$

Proof. We have already seen that for $p > 1$ close to one the positive solution is unique. Now consider a fixed $p = \overline{p}$ and a corresponding positive solution $\overline{u}$, for which the linearization has only one negative eigenvalue. Then, by Proposotion 6.4, the second eigenvalue is necessarily positive, so we can apply the implicit function theorem to obtain a continuously differentiable function

$$U : [\overline{p} - \overline{\epsilon}, \overline{p} + \overline{\epsilon}] \longrightarrow H_0^1(\Omega),$$

such that $u = U(p)$ is a solution of $\Delta u + u^p = 0$. We need however a little more, namely, all $U(p)$ should be positive, and all $U(p)$ should have exactly one negative eigenvalue.

Concerning the positivity of $U(p)$ we first observe that regularity results yield the continuity of

$$U : [\overline{p} - \overline{\epsilon}, \overline{p} + \overline{\epsilon}] \longrightarrow C^1(\overline{\Omega}).$$

Since $\overline{u} > 0$ in Ω and $\frac{\partial \overline{u}}{\partial \nu} < 0$ on $\partial\Omega$, this implies, together with the strong maximum principle and the boundary point lemma, that the same is true for all $u = U(p)$ with $|p - \overline{p}| \leq \overline{\epsilon}$.

Next we consider the eigenvalues of the linearization. Observe that a byproduct of the application of the implicit function theorem is that along the curve

$$p \in [\overline{p} - \overline{\epsilon}, \overline{p} + \overline{\epsilon}] \longrightarrow U(p)$$

the linearized operator is invertible, i.e. zero cannot be an eigenvalue for any $U(p)$. We denote the first eigenvalue λ_1 of the linearization around $U(p)$ by $\lambda_1(p)$. Clearly $\lambda_1(p) < 0$ for all $p \in [\overline{p} - \overline{\epsilon}, \overline{p} + \overline{\epsilon}]$. The corresponding positive eigenfunction, normalized by setting its $L^2(\Omega)$-norm equal to one, is denoted by $\varphi_1(p)$. It follows from general results on simple eigenvalues (see [CR]) that

$$\lambda_1 : [\overline{p} - \overline{\epsilon}, \overline{p} + \overline{\epsilon}] \longrightarrow \mathbf{R}^-,$$

and

$$\varphi_1 : [\overline{p} - \overline{\epsilon}, \overline{p} + \overline{\epsilon}] \longrightarrow H_0^1(\Omega)$$

are continuously differentiable.

Note that

$$-\int_\Omega |\nabla\varphi_1|^2 + p\int_\Omega u^{p-1}\varphi_1 = -\lambda_1\int_\Omega \varphi_1^2,$$

and that $U(p)$ is bounded uniformly in both $L^\infty(\Omega)$ and $H_0^1(\Omega)$. Consequently the curve $\{U(p) : \overline{p}-\overline{\epsilon} \leq p \leq \overline{p}+\overline{\epsilon}\}$ is compact in $H_0^1(\Omega)$. Thus also $\lambda_1(p)$ and $\varphi_1(p)$ are uniformly bounded in $\mathbf{R}^-$ and $H_0^1(\Omega)$ respectively, so that also the curve $\{\varphi_1(p) : \overline{p}-\overline{\epsilon} \leq p \leq \overline{p}+\overline{\epsilon}\}$ is compact in $H_0^1(\Omega)$.

For the second eigenvalue, which we denote by $\lambda_2(p)$, we have

$$\lambda_2(p) = \min_{\int \varphi\varphi_1(p)=0} \frac{\int_\Omega |\nabla\varphi|^2 - p\int_\Omega U(p)^{p-1}\varphi^2}{\int_\Omega \varphi^2},$$

this minimum being achieved by $\varphi_2(p)$, normalized again in L^2. Now, taking $p \neq \overline{p}$, we project $\varphi_2(p)$ onto the subspace

$$L = \big\{\varphi \in L^2(\Omega) : \int \varphi\varphi_1(\overline{p}) = 0\big\}.$$

Simplifying notation by $\varphi_1 = \varphi_1(p)$, $\varphi_2 = \varphi_2(p)$, $\overline{\varphi}_1 = \varphi_1(\overline{p})$, $\overline{\varphi}_2 = \varphi_2(\overline{p})$, $\lambda_1 = \lambda_1(p)$, $\overline{\lambda}_1 = \lambda_1(\overline{p})$, etc., we denote the projection of φ_2 on L by $\tilde{\varphi}_2$. Thus

$$\tilde{\varphi}_2 = \varphi_2 - (\varphi_2, \overline{\varphi}_1)_{L^2}\overline{\varphi}_1.$$

Next we insert $\tilde{\varphi}_2$ into the quotient in the formula for $\overline{\lambda}_2 = \lambda_2(\overline{p})$. The denominator becomes, with Pythogoras,

$$\int_\Omega \tilde{\varphi}_2^2 = \int_\Omega \varphi_2^2 - (\varphi_2,\overline{\varphi}_1)_{L^2}^2 = \int_\Omega \varphi_2^2 + (\varphi_2,\varphi_1)_{L^2}^2 - (\varphi_2,\overline{\varphi}_1)_{L^2}^2 =$$

$$1 + (\varphi_2, \varphi_1+\overline{\varphi}_1)_{L^2}(\varphi_2,\varphi_1-\overline{\varphi}_1)_{L^2}, \tag{9.1}$$

and for the numerator we find

$$\int_\Omega |\nabla\tilde{\varphi}_2|^2 - \int_\Omega \overline{p}\overline{u}^{\overline{p}-1}\tilde{\varphi}_2^2 = \int_\Omega \Big|\nabla(\varphi_2 - (\varphi_2,\overline{\varphi}_1)_{L^2}\overline{\varphi}_1)\Big|^2 - \int_\Omega \overline{p}\overline{u}^{\overline{p}-1}(\varphi_2 - (\varphi_2,\overline{\varphi}_1)_{L^2}\overline{\varphi}_1)^2$$

$$= \Big\{\int_\Omega |\nabla\varphi_2|^2 - \int_\Omega \overline{p}\overline{u}^{\overline{p}-1}\varphi_2^2\Big\} - 2(\varphi_2,\overline{\varphi}_1)_{L^2}\Big\{\int_\Omega \nabla\varphi_2\nabla\overline{\varphi}_1 - \int_\Omega \overline{p}\overline{u}^{p-1}\varphi_2\overline{\varphi}_1\Big\}$$

$$+(\varphi_2,\overline{\varphi}_1)_{L^2}^2\Big\{\int_\Omega |\nabla\overline{\varphi}_1|^2 - \int_\Omega \overline{p}\overline{u}^{\overline{p}_1}\overline{\varphi}_1^2\Big\} =$$

$$\Big\{\int_\Omega |\nabla\varphi_2|^2 - \int_\Omega pu^{p-1}\varphi_2^2\Big\} + \int_\Omega (pu^{p-1} - \overline{p}\overline{u}^{\overline{p}-1})\varphi_2^2 - 2(\varphi_2,\overline{\varphi}_1)_{L^2}\overline{\lambda}_1(\varphi_2,\overline{\varphi}_1)_{L^2} + (\varphi_2,\overline{\varphi}_1)_{L^2}^2\overline{\lambda}_1$$

$$= \lambda_2 + \int_\Omega (pu^{p-1} - \overline{p}\overline{u}^{\overline{p}-1})\varphi_2^2 + \overline{\lambda}_1(\varphi_2,\varphi_1+\overline{\varphi}_1)_{L^2}(\varphi_2,\varphi_1-\overline{\varphi}_1)_{L^2}. \tag{9.2}$$

Because of the continuity results on $\lambda_1(p), U(p)$ and $\varphi_1(p)$ we conclude that for all $\epsilon > 0$ there exists $\delta > 0$ such that the last term in (9.1) and the two last terms in (9.2) are in absolute value smaller than ϵ if $|p - \overline{p}| < \delta$, and the same holds if we repeat the argument with $\overline{p}$ replaced by any other $p^* \in [\overline{p} - \overline{\epsilon}, \overline{p} + \overline{\epsilon}]$, the choice of δ depending on ϵ only, and not on p^*. Hence there exists for every $\epsilon > 0$ a $\delta > 0$ such that $\lambda_2(p_1) \leq \lambda_2(p_2) + \epsilon$ for all $p_1, p_2 \in [\overline{p} - \overline{\epsilon}, \overline{p} + \overline{\epsilon}]$ with $|p_1 - p_2| < \delta$. This implies that $\lambda_2 : [\overline{p} - \overline{\epsilon}, \overline{p} + \overline{\epsilon}] \to \mathbf{R}$ is continuous. Since $\lambda_2(p) \neq 0$ and $\lambda_2(p) \geq 0$ we find $\lambda_2(p) > 0$ for all $[\overline{p} - \overline{\epsilon}, \overline{p} + \overline{\epsilon}]$.

Next we show that the function $U(p)$ is in fact defined for all $p \in (1, \infty)$ and that $U : (1, \infty) \to H_0^1(\Omega)$ is continuously differentiable. Indeed let $(a, b) \ni p$ be the maximal interval on which such a function $U(p)$ is defined with $\lambda_2(p) > 0$ on (a, b). Then if $a > 1$, we cannot have a sequence $p_n \downarrow a$ with $\lambda_2(p_n) \downarrow 0$, for this would, taking a convergent subsequence $U(p)$ (a priori bounds!), imply the existence of a limit solution U^* for $p = a$ with second eigenvalue $\lambda_2^* = 0$. Hence $\lambda_2(p)$ is bounded away from zero as $p \downarrow 1$. This implies that

$$\frac{dU}{dp} = -(d_u f)^{-1} d_p f$$

is bounded in $H_0^1(\Omega)$ as $p \downarrow a$, so that $\lim_{p \downarrow a} U(p)$ exists in $H_0^1(\Omega)$. But then a cannot be minimal because $\lambda_2(a) > 0$. Conclusion: $a = 1$, and, similarily, $b = \infty$. This completes the proof in view of the already (Section 8) established uniqueness for p close to 1.

10. Regularity for Problem (P)

Let us recall that in Section 6 we used that the classical solution of

$$(P) \quad \begin{cases} -\Delta u = u^p, & u > 0 \quad \text{in} \quad \Omega; \\ u = 0 & \qquad\quad\ \ \text{on} \quad \partial\Omega, \end{cases}$$

is a function $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, satisfying (P) pointwise. The main result of this section establishes this regularity for weak solutions of (P). Our proofs are along the lines of [vdV2], where a similar regularity result is obtained for higher order equations.

Theorem 10.1. *Suppose $\partial\Omega$ is smooth, and $1 < p < \frac{n+2}{n-2}$. Let $u \in H_0^1(\Omega)$ be a weak solution of Problem (P). Then $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$.*

Before going into the proof of Theorem 10.1, we need some preliminary regularity results for the linear problem

$$(L) \quad \begin{cases} -\Delta u = f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $f \in L^2(\Omega)$.

The first step towards regularity is the proof that $-\Delta$ is an unbounded selfadjoint operator on $L^2(\Omega)$ with domain $D(-\Delta) = H^2 \cap H_0^1(\Omega)$. Here the crucial a priori estimate is

$$\|u\|_{H^2} \leq C\|f\|_{L^2}, \tag{10.1}$$

where $H^2(\Omega)$ is the space of all functions in $L^2(\Omega)$ with first and second order derivatives in $L^2(\Omega)$, see [A]. For a detailed proof of this regularity result we refer to [Br].

More complicated is the L^p-theory [ADN,CZ] of Problem (L), which establishes the estimate

$$\|u\|_{H^{2,p}} \leq C\|f\|_{L^p}. \tag{10.2}$$

See [A] again for the definition of the $H^{k,p}$-spaces. We observe that $\|\Delta u\|_{L^p}$ can serve as an equivalent norm on $H^{2,p} \cap H_0^{1,p}(\Omega)$.

Once (10.2) is obtained, the Sobolev embeddings [A] will give, provided p is sufficiently large, that u is uniformly Hölder continuous. Finally the Schauder estimates [GT] are needed to prove that u is a classical solution if f is uniformly Hölder continuous.

Proof of Theorem 10.1. The proof will consist of three steps. The first step is to establish that $u \in L^q$ for all $1 \leq q < \infty$. The second step consists of proving that $u \in H^{2,q}(\Omega) \ \ \forall\, 1 \leq q < \infty$. Finally using the Sobolev embeddings for $H^{2,q}(\Omega)$ and the Schauder estimates [GT] to finish the proof.

Step 1.

In the distributional sense we have

$$-\Delta u = a(x)u, \quad a(x) = |u|^{p-1}, \tag{10.3}$$

where $a \in L^{n/2}(\Omega)$, since $u \in H_0^1(\Omega)$. We decompose $a(x)u$ as

$$a(x)u(x) = q_\epsilon(x)u(x) + f_\epsilon(x), \tag{10.4}$$

with

$$\|q_\epsilon\|_{L^{n/2}} < \epsilon, \quad \|f_\epsilon\|_{L^\infty} \leq K_\epsilon. \tag{10.5}$$

To prove (10.4) and (10.5) we proceed as follows. Define

$$A_k = \{x \in \Omega; |a| < k\}, \quad B_l = \{x \in \Omega; |u| < l\},$$

where k, l are chosen such that

1. $\|a\|_{L^{n/2}(A_k^c)} < \frac{1}{4}\epsilon, \quad \|a\|_{L^{n/2}(B_l^c)} < \frac{1}{4}\epsilon,$
2. $A_k \cap B_l \neq \emptyset,$
3. $a(x) \not\equiv 0 \quad \text{on} \quad A_k \cap B_l.$

It is not hard to see that these conditions can indeed be met. By construction $|A_k| \to |\Omega|$ and $|B_l| \to |\Omega|$ as $k, l \to \infty$, so $|A_k^c| \to 0, |B_l^c| \to 0$ as $k, l \to \infty$, which ensures 1, whenever k, l are sufficiently large. Condition 2 is true if k, l are large enough. Suppose not, then $A_k \cap B_l = \emptyset$ for all k, l, and so $|A_k \cup B_l| = |A_k| + |B_l|$. Clearly if k, l are large $|A_k| + |B_l| > |\Omega|$ and so $|A_k \cup B_l| > |\Omega|$, which contradicts $|A_k \cup B_l| \leq |\Omega|$. Condition 3 follows because if $k, l \to \infty$, then $|A_k \cap B_l| \to |\Omega|$. Observe that if $a \equiv 0$ on $A_k \cap B_l \ \ \forall k, l$, then $\|a\|_{L^{n/2}((A_k \cap B_l)^c)} \neq 0$, while $|(A_k \cap B_l)^c| \to 0$, which is again a contradiction.

Next we set, with m to be chosen later on,

$$q_\epsilon(x) = \begin{cases} \dfrac{1}{m}a(x) & \text{on} \quad A_k \cap B_l \\ a(x) & \text{on} \quad A_k^c \cup B_l^c, \end{cases}$$

and

$$f_\epsilon(x) = \{a(x) - q_\epsilon(x)\}u(x).$$

Clearly

$$f_\epsilon(x) = 0 \quad \forall x \in (A_k \cap B_l)^c.$$

Let us now estimate the $L^{n/2}$-norm of q_ϵ.

$$\begin{aligned} \|q_\epsilon\|_{L^{n/2}}^{n/2} &= \int_\Omega |q_\epsilon|^{n/2} = \int_{A_k \cap B_l} |q_\epsilon|^{n/2} + \int_{(A_k \cap B_l)^c} |q_\epsilon|^{n/2} \\ &\leq \int_{A_k \cap B_l} |q_\epsilon|^{n/2} + \int_{A_k^c} |q_\epsilon|^{n/2} + \int_{B_l^c} |q_\epsilon|^{n/2} \\ &= (\frac{1}{m})^{n/2} \int_{A_k \cap B_l} |a|^{n/2} + \int_{A_k^c} |a|^{n/2} + \int_{B_l^c} |a|^{n/2} \\ &< (\frac{1}{m})^{n/2} \int_{A_k \cap B_l} |a|^{n/2} + 1/2\epsilon. \end{aligned}$$

This gives

$$\|q_\epsilon\|_{L^{n/2}} < \frac{1}{m}\|a\|_{L^{n/2}} + \frac{1}{2}\epsilon. \tag{10.6}$$

Hence, for $m > m_\epsilon = 2\|a\|_{L^{n/2}}/\epsilon$, we have

$$\|q_\epsilon\|_{L^{n/2}} < \epsilon.$$

Following the same procedure we find for f_ϵ,

$$\|f_\epsilon\|_{L^\infty} < kl < \infty.$$

This proves (10.4) and (10.5).

Let us go back now to equation (10.3). We define $A^\epsilon u = (-\Delta)^{-1}(q_\epsilon u)$, and thus solutions of

$$u - A^\epsilon u = h_\epsilon = (-\Delta)^{-1} f_\epsilon \tag{10.7}$$

are solutions of (10.3) and vice versa. We continue with (10.7), and we show that ϵ can be chosen such that $\|A^\epsilon\|_{L^p \to L^q} < \frac{1}{2}$ for all $1 \leq q < \infty$.

From the Hardy-Littlewood-Sobolev[1] inequality we have

$$\|(-\Delta)^{-1} w\|_{L^q} \leq C(q)\|w\|_{L^r}, \tag{10.8}$$

[1] *This version of the Hardy-Littlewood-Sobolev inequality for $L^r(\Omega)$ can be derived from the Sobolev embedding of $H^{2,r}(\Omega)$ into $L^q(\Omega)$, using the equivalent norm $\|\Delta u\|_{L^r}$.*

with

$$\frac{1}{r}=\frac{1}{q}+\frac{2}{n}. \tag{10.9}$$

Now let $v\in L^q$, then by (10.7), (10.8)

$$\|A^\epsilon v\|_{L^q}=\|(-\Delta)^{-1}(q_\epsilon v)\|_{L^q}\le C(q)\|q_\epsilon v\|_{L^r}.$$

By Hölder's inequality

$$\|q_\epsilon v\|_{L^r}\le\|q_\epsilon\|_{L^{n/2}}\|v\|_{L^q},$$

so

$$\|A^\epsilon v\|_{L^q}\le C(q)\|q_\epsilon\|_{L^{n/2}}\|v\|_{L^q}. \tag{10.10}$$

This gives that A^ϵ acts from L^q to L^q when $q\ge\frac{2n}{n-2}$, with norm less than 1/2 (choose $\epsilon=\epsilon^*=\frac{1}{2C(q)}$). Also

$$(I-A^{\epsilon^*})^{-1}:\ L^q\to L^q,\quad\text{with}\quad\|(I-A^{\epsilon^*})^{-1}\|_{L^q\to L^q}<2. \tag{10.11}$$

From (10.7) we have then $u=(I-A^{\epsilon^*})^{-1}h_{\epsilon^*}$, and

$$\begin{aligned}\|(I-A^{\epsilon^*})^{-1}h_{\epsilon^*}\|_{L^q}&<2\|h_{\epsilon^*}\|_{L^q}=2\|(-\Delta)^{-1}f_{\epsilon^*}\|_{L^q}\\&\le 2C(q)\|f_{\epsilon^*}\|_{L^q}\le\hat{C}(q)\|f_{\epsilon^*}\|_{L^\infty},\end{aligned}$$

which yields the estimate

$$\|u\|_{L^q}<\hat{C}(q)\|f_{\epsilon^*}\|_{L^\infty},\quad\forall q\ge\frac{2n}{n-2},$$

whence

$$u\in L^q(\Omega),\quad\forall 1\le q<\infty. \tag{10.12}$$

Step 2. Set $g(x)=u^p=a(x)u\in L^q$. From the L^q-estimates (10.2) we deduce

$$u\in H^{2,q}(\Omega),\ \text{and}\quad\|u\|_{H^{2,q}}\le C_q\|g\|_{L^q}=C_q\|u^p\|_{L^q}.$$

Step 3.

Let us recall the Sobolev-embedding of $H^{2,q}(\Omega)$ (see e.g. [A]):

$$H^{2,q}(\Omega)\hookrightarrow C^{1,\alpha}(\overline{\Omega}),\qquad q>n,\quad\alpha\le 1-n/q.$$

From Step 2 we know that $u\in H^{2,q}(\Omega)$ for all $q>n$. Thus also $u\in C^{1,\alpha}(\overline{\Omega})\subset C^1(\overline{\Omega})$. Because $u\in C^{1,\alpha}(\overline{\Omega})$ we have $u^p\in C^{0,\alpha}(\overline{\Omega})$. Using the Schander-estimates [GT,Br] we obtain $u\in C^{2,\alpha}(\Omega)\subset C^2(\Omega)$, so that certainly

$$u\in C^2(\Omega)\cap C^1(\overline{\Omega}).$$

Remark. If the solution u is strictly positive in Ω, u^p is infinitely many times differentiable. Repeating the above procedure then, we obtain $u \in C^\infty(\Omega)$.

Appendix: Applications of group theory.

In studying the equation $\Delta u + u^p = 0$ and its linearization $\Delta\varphi + p|u|^{p-1}\varphi = 0$, integral identities for solutions of the corresponding Dirichlet problem played an important role [Mi,N,O,P,PS,vdV1]. These identities can be derived in a local form from group theoretical considerations. In this appendix we shall outline this approach. Our starting point is a group of transformations

$$X_\epsilon : \mathbf{R}^n \to \mathbf{R}^n.$$

We write

$$\tilde{x} = X_\epsilon(x) = X(x,\epsilon).$$

For the moment we assume that $X(x,\epsilon)$ is a real analytic function of $x \in \mathbf{R}^n$ and $\epsilon \in E$, where $E \subset \mathbf{R}$ is an open interval equipped with a group operation, i.e. a function

$$\varphi : E \times E \to E$$

such that with

$$a \circ b = \varphi(a,b),$$

E is a group. We assume that φ is also real analytic. For the family $\{X_\epsilon\}_{\epsilon\in E}$ we now assume that

$$X(x,0) = x \quad \forall x \in \mathbf{R}^n.$$

In addition we assume that every X_ϵ is invertible, and also that the Jacobian matrices of X_ϵ are always invertible.

A first observation in this context is that, possibly after a reparametrisation of E, the curves

$$\epsilon \longrightarrow X(x,\epsilon)$$

are in fact *integral curves* of an *autonomous system* of first order differential equations in $\mathbf{R}^n$, i.e.

$$\frac{dx}{dt} = \xi(x),$$

where $\xi : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ is again analytic. This is *Lie's first theorem*, and can be seen as follows.

Let

$$\xi(x) = \frac{\partial X}{\partial \epsilon}(x,0).$$

Then, writing ϵ^{-1} for the inverse of ϵ,

$$\frac{\partial \tilde{x}}{\partial \epsilon} = \frac{d}{dh} X(x,\epsilon+h)\Big|_{h=0} = \frac{d}{dh} X(x,\epsilon\circ\epsilon^{-1}\circ\epsilon+h)\Big|_{h=0} =$$

$$\frac{d}{dh}X(X(x,\epsilon),\varphi(\epsilon^{-1},\epsilon+h))\Big|_{h=0}=$$

(writing φ_i for the partial derivatives of φ)

$$\frac{d}{dh}X(\tilde{x},\varphi(\epsilon^{-1},\epsilon+h))\Big|_{h=0}=\xi(\tilde{x})\varphi_2(\epsilon^{-1},\epsilon).$$

Thus if we define

$$t(\epsilon)=\int_0^\epsilon \varphi_2(s^{-1},s)ds,$$

we find

$$\frac{\partial\tilde{x}}{\partial t}=\frac{d\epsilon}{dt}\frac{\partial\tilde{x}}{\partial\epsilon}=\xi(\tilde{x}).$$

Also we have that

$$t(\epsilon_1\circ\epsilon_2)=\int_0^{\varphi(\epsilon_1,\epsilon_2)}\varphi_2(\epsilon^{-1},\epsilon)d\epsilon=\int_0^{\epsilon_1}\varphi_2(\epsilon^{-1},\epsilon)d\epsilon+\int_{\epsilon_1}^{\varphi(\epsilon_1,\epsilon_2)}\varphi_2(\epsilon^{-1},\epsilon)d\epsilon=$$

$$t(\epsilon_1)+\int_0^{\varphi(\epsilon_1^{-1},\varphi(\epsilon_1,\epsilon_2))}\varphi_2(\varphi(\epsilon_1,s)^{-1},\varphi(\epsilon_1,s))\varphi_2(\epsilon_1,s)ds=t(\epsilon_1)+t(\epsilon_2),$$

because

$$\varphi_2(s^{-1},s)=\frac{d}{dh}\varphi(s^{-1},s+h)\Big|_{h=0}=\frac{d}{dh}\varphi(\varphi(\epsilon_1,s)^{-1},\varphi(\epsilon_1,s+h))\Big|_{h=0}$$

$$=\varphi_2(\varphi(\epsilon_1,s)^{-1},\varphi(\epsilon_1,s))\varphi_2(\epsilon_1,s).$$

In view of these computations, it is no restiction to restrict ourselves to the case that $E=\mathbf{R}$ with the standard additive group structure. Thus we write t instead of ϵ, and

$$\tilde{x}=X(x,t),\quad \frac{d\tilde{x}}{dt}=\xi(\tilde{x}),$$

with ξ defined by

$$\xi(x)=\frac{\partial X(x,t)}{\partial t}|_{t=0}.$$

For any analytic function $F:\mathbf{R}^n\longrightarrow\mathbf{R}$ we now have

$$\frac{d}{dt}F(\tilde{x})|_{t=0}=\frac{d}{dt}F(X(x,t))\Big|_{t=0}=\sum_{i=1}^n\frac{\partial F}{\partial x_i}(x)\xi_i(x)=vF(x),$$

where v is called the *generator* of the transformation group, and is given by

$$v=\sum_{i=1}^n\xi_i(x)\frac{\partial}{\partial x_i},$$

i.e. v is a first order differential operator.

Repeated differentiation yields

$$\frac{d^k}{dt^k} F(\tilde{x})\Big|_{t=0} = v^k F(x) = \Big(\sum_{i=1}^{n} \xi_i(x) \frac{\partial}{\partial x_i}\Big)^k F(x),$$

so that, writing

$$e^{tv} = \sum_{k=0}^{\infty} \frac{t^k}{k!} v^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{i=1}^{n} \Big(\xi_i(x) \frac{\partial}{\partial x_i}\Big)^k,$$

we have that the power series of $F(\tilde{x}) = F(X(x,t))$ around $t = 0$ is given by

$$F(X(x,t)) = e^{tv} F(x),$$

and has a positive radius of convergence. Clearly this notation extends component wise to $F : \mathbf{R}^n \longrightarrow \mathbf{R}^n$, so that in particular, with $m = n$, $F(x) = x$,

$$X(x,t) = e^{tv} x.$$

So far we have assumed that all functions under consideration are used analytic. In fact we only shall be needing that

$$\frac{d}{dt} F(X(x,t))\Big|_{t=0} = vF(x),$$

so that only the chain role and the unique solvibility of the initial value problem for

$$\frac{dx}{dt} = \xi(x)$$

are needed.

Next we turn our attention to groups acting on expressions of the form

$$L(x, u(x), \nabla u(x)),$$

where we want to derive something like

$$\frac{d}{dt} L(\tilde{x}, \tilde{u}(\tilde{x}), \tilde{\nabla}\tilde{u}(\tilde{x}))\Big|_{t=0} = vL(x, u(x), \nabla u(x)).$$

The problem here is that we have to work with functions of independent variables and dependent variables, and in doing so we must understand how they transform under the group action.

We consider the case that $u(x)$ is a real valued function of x. In order to make a clear distinction between u as a variable and $u(x)$ as a function of x, we write $u = f(x)$, $p = \nabla f(x)$, and $L(x, u, p)$. We take transformations of the form

$$\tilde{x} = \varphi(x, u, t) = \varphi_t(x, u);$$

$$\tilde{u} = \psi(x, u.t) = \psi_t(x, u).$$

We assume that φ and ψ are sufficiently smooth, that $(\partial\varphi_i/\partial x_j)$ is an invertible matrix, and $\partial\psi/\partial u \neq 0$, regardless of the arguments. We may view (φ_t, ψ_t) as a transformation of $\mathbf{R}^{n+1}$, and we assume that as such, the (φ_t, ψ_t) form an additive group, with (φ_0, ψ_0) being the identity. For functions $F(x, u)$ we then have from what we saw earlier that

$$\frac{d}{dt}F(\tilde{x}, \tilde{u})\Big|_{t=0} = vF(x, u).$$

with

$$v = \sum_{i=1}^{n} h_i \frac{\partial}{\partial x_i} + a\frac{\partial}{\partial u}.$$

Here

$$h_i(x, u) = \frac{\partial\varphi_i(x, u, t)}{\partial t}\Big|_{t=0},$$

and

$$a(x, u) = \frac{\partial\psi(x, u, t)}{\partial t}\Big|_{t=0}.$$

Next we investigate how a relation of the form $u = f(x)$ transforms. Consider the graph of f and describe its image under (ψ_t, ψ_t) as $\tilde{u} = \tilde{f}(\tilde{x})$. We introduce the notation

$$Ix = x; \qquad \begin{pmatrix} I \\ f \end{pmatrix}(x) = \begin{pmatrix} x \\ f(x) \end{pmatrix}.$$

For x and u satisfying $u = f(x)$, where f is a fixed function, we have

$$\tilde{x} = \varphi_t(x, f(x)) = \varphi_t \circ \begin{pmatrix} I \\ f \end{pmatrix}(x);$$

$$\tilde{u} = \psi_t(x, f(x)) = \psi_t \circ \begin{pmatrix} I \\ f \end{pmatrix}(x) = \tilde{g}(x)$$

$$= \psi_t \circ \begin{pmatrix} I \\ f \end{pmatrix}(\varphi_t \circ \begin{pmatrix} I \\ f \end{pmatrix})^{-1}(\tilde{x}) = \tilde{f}(\tilde{x}).$$

Thus (φ_t, ψ_t) induces a transformation of functions

$$f \longrightarrow \psi_t \circ \begin{pmatrix} I \\ f \end{pmatrix} \circ (\varphi_t \circ \begin{pmatrix} I \\ f \end{pmatrix})^{-1}.$$

Not only f is transformed, but also ∇f, and this is important, because we consider $L(x, u(x), \nabla u(x))$. Thus we have to express $\partial\tilde{f}/\partial\tilde{x}_i$ in terms of $\partial f/\partial x_i$. Let

$$J = \left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_2}\cdots\frac{\partial}{\partial x_n}\right); \quad \tilde{J} = \left(\frac{\partial}{\partial\tilde{x}_1}\frac{\partial}{\partial\tilde{x}_2}\cdots\frac{\partial}{\partial\tilde{x}_n}\right).$$

Then Jf is defined for possibly vector valued functions $f(x)$, and $\tilde{J}$ acts on functions $\tilde{f}(\tilde{x})$. Note that

$$J\tilde{X} = J\varphi_t(x,u) = \left(\frac{\partial \varphi_i(x,u,t)}{\partial x_j}\right) = \left(\frac{\partial \tilde{x}_i}{\partial x_j}\right)$$

is the functional matrix of the transformation $x \to \tilde{x} = \varphi_t(x,u)$. Writing $\tilde{f}(\tilde{x}) = \tilde{g}(x)$, it follows that

$$\frac{\partial \tilde{f}}{\partial \tilde{x}_i} = \sum_{j=1}^{n} \frac{\partial \tilde{g}}{\partial x_j}\frac{\partial x_j}{\partial \tilde{x}_i},$$

or, in matrix notation,

$$\tilde{J}\tilde{f} = (J\tilde{g})(\tilde{J}x) = (J\tilde{g})(J\tilde{x})^{-1} = (J\psi_t \circ \begin{pmatrix} I \\ f \end{pmatrix})(J\varphi_t \circ \begin{pmatrix} I \\ f \end{pmatrix})^{-1},$$

so that

$$\tilde{J}\tilde{f}(\tilde{x}) = (J\psi_t(x, f(x)))(J\varphi_t(x, f(x)))^{-1}.$$

The right hand side can be written as a function $\xi_t(x, f(x), \nabla f(x))$. Putting $\tilde{p}_i = \partial \tilde{f}/\partial \tilde{x}_i$, we then have

$$\tilde{p}_i = \xi_t^i(x,u,p), \quad u = f(x), \ p = \nabla f(x).$$

We compute

$$\frac{d}{dt}\xi_t(x, f(x), \nabla f(x))\Big|_{t=0} = \frac{d}{dt}(J\psi_t(x, f(x)))(J\varphi_t(x, f(x)))^{-1}\Big|_{t=0} =$$

$$(J\frac{\partial \psi}{\partial t}(x, f(x), t))(J(\varphi_t(x, f(x)))^{-1}\Big|_{t=0} -$$

$$J\psi_t(x, f(x))(J\varphi_t(x, f(x)))^{-1}(J\frac{\partial \varphi}{\partial t}(x, f(x), t)(J\varphi_t(x, f(x)))^{-1}\Big|_{t=0}$$

$$= Ja(x, f(x)) - (Jf(x))(Jh(x, f(x))).$$

Written out in components,

$$\frac{d}{dt}\xi_t^i(x, f(x), \nabla f(x)\Big|_{t=0} = \frac{da(x, f(x))}{dx_i} - \sum_{j=1}^{n}\frac{df(x)}{dx_j}\frac{dh_j(x, f(x))}{dx_i} =$$

(writing $u = f(x)$ and $p = \nabla f(x)$)

$$\frac{\partial a}{\partial x_i}(x,u) + p_i\frac{\partial a}{\partial u}(x,u) - \sum_{j=1}^{n} p_j\frac{\partial h_j}{\partial x_i}(x,u) + \sum_{j=1}^{n} p_j p_i\frac{\partial h_j}{du}(x,u) =$$

(writing $D_i = \partial/\partial x_i + p_i\partial/\partial u$)

$$D_i a(x,u) - \sum_{j=1}^{n} p_j D_i h_j(x,u) = b_i(x,u,p).$$

Thus, instead of the operator,

$$v = \sum_{i=1}^{n} h_i(x,u)\frac{\partial}{\partial x_i} + a(x,u)\frac{\partial}{\partial u},$$

it seems natural to consider the so-called prolonged operator

$$\begin{aligned} v^{(1)} &= \sum_{i=1}^{n} h_i(x,u)\frac{\partial}{\partial x_i} + a(x,u)\frac{\partial}{\partial u} + \sum_{i=1}^{n} b_i(x,u,p)\frac{\partial}{\partial p_i} \\ &= v + \sum_{i=1}^{n} b_i(x,u,p)\frac{\partial}{\partial p_i}, \end{aligned}$$

for then we have, using the chain rule, if $u = f(x)$, $p = Jf(x)$, $\tilde{u} = \tilde{f}(\tilde{x})$, $\tilde{p} = \tilde{J}\tilde{f}(\tilde{x})$, that

$$\frac{d}{dt}L(\tilde{x},\tilde{u},\tilde{p})\Big|_{t=0} = \frac{d}{dt}L(\tilde{x},\tilde{f}(\tilde{x}),\tilde{J}\tilde{f}(\tilde{x}))\Big|_{t=0} = \frac{d}{dt}L(\varphi_t(x,u),\psi_t(x,u),\xi_t(x,u,p))\Big|_{t=0}$$

$$= \sum_{i=1}^{n}\frac{\partial L}{\partial x_i}(\tilde{x},\tilde{u},\tilde{p})\Big|_{t=0}\frac{\partial\varphi_t^i(x,u)}{\partial t}\Big|_{t=0} + \frac{\partial L}{\partial u}(\tilde{x},\tilde{u},\tilde{p})\Big|_{t=0}\frac{\partial\psi_t(x,u)}{\partial t}\Big|_{t=0}$$

$$+\sum_{i=1}^{n}\frac{\partial L}{\partial p_i}(\tilde{x},\tilde{u},\tilde{p})\Big|_{t=0}\frac{\partial\xi_t(x,f(x),Jf(x))}{\partial t}\Big|_{t=0}$$

$$= \sum_{i=1}^{n} h_i(x,u)\frac{\partial L}{\partial x_i}(x,u,p) + a(x,u)\frac{\partial L}{\partial u}(x,u,p) + \sum_{i=1}^{n} b_i(x,u,p)\frac{\partial L}{\partial p_i}(x,u,p)$$

$$= v^{(1)}L(u,x,p)\Big|_{u=f(x),\ p=Jf(x)}.$$

Next we rewrite $v^{(1)}$ and $v^{(1)}L$.

$$v^{(1)} = \sum_{i=1}^{n} h_i\frac{\partial}{\partial x_i} + a\frac{\partial}{\partial u} + \sum_{i=1}^{n}\Big(D_i a - \sum_{j=1}^{n} p_j D_i h_j\Big)\frac{\partial}{\partial p_i} =$$

$$\sum_{i=1}^{n} h_i D_i + (a - \sum_{i=1}^{n} p_i h_i)\frac{\partial}{\partial u} + \sum_{i=1}^{n}(D_i(a - \sum_{j=1}^{n} p_j h_j))\frac{\partial}{\partial p_i} =$$

$$\sum_{i=1}^{n} h_i D_i + Q\frac{\partial}{\partial u} + \sum_{i=1}^{n}(D_i Q)\frac{\partial}{\partial p_i},$$

where

$$Q = a - \sum_{i=1}^{n} p_i h_i.$$

Therefore,

$$v^{(1)}L = \sum_{i=1}^{n} h_i D_i L + Q\frac{\partial L}{\partial u} + \sum_{i=1}^{n}(D_i Q)\frac{\partial L}{\partial p_i} =$$

$$\sum_{i=1}^{n} D_i(h_i L) - \sum_{i=1}^{n} L D_i h_i + Q\frac{\partial L}{\partial u} + \sum_{i=1}^{n} D_i(Q\frac{\partial L}{\partial p_i}) - \sum_{i=1}^{n} Q D_i \frac{\partial L}{\partial p_i},$$

so that

$$v^{(1)}L + L\sum_{i=1}^{n} D_i h_i = Q(\frac{\partial L}{\partial u} - \sum_{i=1}^{n} D_i\frac{\partial L}{\partial p_i}) + \sum_{i=1}^{n} D_i(h_i L + Q\frac{\partial L}{\partial p_i}). \tag{1}$$

Untill now, all derivatives are pure partial derivatives, but now we insert again that $u = f(x)$ and $p = Jf(x)$. For any function $G(x, u, p)$ we then have for the total derivative with respect to x_i (denoted by d/dx_i)

$$\frac{d}{dx_i}G(x, f(x), Jf(x)) =$$

$$\frac{\partial G}{\partial x_i}(x, f(x), Jf(x)) + \frac{\partial G}{\partial u}(x, f(x), Jf(x))\frac{\partial f(x)}{\partial x_i} + \sum_{j=1}^{n}\frac{\partial G}{\partial p_j}(x, f(x), Jf(x))\frac{\partial p_j(x)}{\partial x_i}$$

$$= D_i G(x, f(x), Jf(x)) + \sum_{j=1}^{n}\frac{\partial G}{\partial p_j}(x, f(x), Jf(x))\frac{\partial^2 f}{\partial x_i \partial x_j}.$$

Thus if we replace D_i in (1) by d/dx_i we add extra terms containing partial derivatives with respect to p_j. In doing so we can neglect h_i, because h_i does not depend explicitly on p_j. Therefore we only have to look at the right hand side of (1). The extra terms from $-Q\sum_{i=1}^{n} D_i \partial L/\partial p_i$ cancel with the extra terms from $\sum_{i=1}^{n} D_i(Q\partial L/\partial p_i)$. This leaves us with just the extra terms from $\sum_{i=1}^{n}(\partial L/\partial p_i)D_i Q$ and $\sum_{i=1}^{n} h_i D_i L$ to consider. These add up to

$$-\sum_{i=1}^{n}\frac{\partial L}{\partial p_i}\sum_{j=1}^{n} h_j \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^{n} h_i \sum_{j=1}^{n}\frac{\partial L}{\partial p_j}\frac{\partial^2 f}{\partial x_i \partial x_j} = 0,$$

and all the extra terms cancel, so that from (1) we obtain

$$(v^{(1)}L)(x, f(x), Jf(x)) + L(x, f(x), Jf(x))\sum_{i=1}^{n}\frac{dh(x, f(x))}{dx_i} =$$

$$Q(x, f(x), Jf(x))\Big\{\frac{\partial L}{\partial u}(x, f(x), Jf(x)) - \sum_{i=1}^{n}\frac{d}{dx_i}\frac{\partial L}{\partial p_i}(x, f(x), Jf(x))\Big\} +$$

$$\sum_{i=1}^{n}\frac{d}{dx_i}\Big(h_i(x, f(x))L(x, f(x), Jf(x)) + Q(x, f(x), Jf(x))\frac{\partial L}{\partial p_i}(x, f(x), Jf(x))\Big)$$

$$= Q \cdot EL + \mathrm{div}\mathbf{P}.$$

Here

$$EL = \frac{\partial L}{\partial u} - \sum_{i=1}^{n} \frac{d}{dx_i} \frac{\partial L}{\partial p_i}$$

is the Euler-Lagrangian, i.e. the left hand side of the Euler-Lagrange equations, and $\mathbf{P}$ is the vectorfield

$$\mathbf{P} = hL + Q\frac{\partial L}{\partial p}.$$

In abbreviated form we write

$$v^{(1)}L + L\mathrm{div}h = Q \cdot EL + \mathrm{div}\mathbf{P}.$$

Next we observe that, writing $\det(A)$ for the determinant od a matrix A,

$$\frac{d}{dt}\det\left(\frac{d\varphi_t^i(x, f(x))}{dx_j}\right)\Big|_{t=0} = \mathrm{div}h(x, f(x)),$$

so that

$$\frac{d}{dt}\big(L(\tilde{x}, \tilde{f}(\tilde{x}), \tilde{J}\tilde{f}(\tilde{x})\big)\det\big(\frac{d\varphi_t^i(x, f(x))}{dx_j}\big)\Big|_{t=0} = v^{(1)}L - L\ \mathrm{div}h(x, f(x)).$$

Conclusion.

Writing $\tilde{L} = L(\tilde{x}, \tilde{f}(\tilde{x}), \tilde{J}\tilde{f}(\tilde{x}))$, $d\tilde{x}/dx = \det\big(d\tilde{x}_i/dx_j\big)$, our final extended version of Noether's formula reads

$$\frac{d}{dt}\tilde{L}\frac{d\tilde{x}}{dx}\Big|_{t=0} = Q \cdot EL + \mathrm{div}\mathbf{P}.$$

If the group of transformation that we started off with, satisfies

$$\frac{d}{dt}\tilde{L}\frac{d\tilde{x}}{dx}\Big|_{t=0} = 0,$$

and if $u = f(x)$ is a solution of the Euler-Lagrange equations, then

$$\mathrm{div}\mathbf{P} = 0.$$

We generalize this result to Lagrangians of the form $L(x, u(x), v(x), Ju(x), Jv(x))$, or $L(x, u(x), Ju(x))$, where $u(x)$ is a vector valued function. In the derivatives above we then have

$$f(x) = (f_k(x))_{k=1,..,m}; \quad Jf(x) = (Jf_k(x))_{k=1,..,m}.$$

We write

$$u^k = f_k(x); \quad p_i^k = \frac{\partial f_k}{\partial x_i}(x), \quad \text{etc.}$$

Noether's formula as we derived it above remains valid if we replace Q by

$$Q^k = a^k - \sum_{i=1}^{n} p_i^k h_i,$$

where

$$a^k = \frac{\partial \psi_t^k(x,u,p)}{\partial t}\Big|_{t=0},$$

$Q \cdot EL$ by

$$\sum_{k=1}^{m} Q^k (EL)^k,$$

where

$$(EL)^k = \frac{\partial L}{\partial u^k} - \sum_{i=1}^{n} \frac{d}{dx_i} \frac{\partial L}{\partial p_i^k},$$

and finally $\mathbf{P}$ by

$$\mathbf{P} = hL + \sum_{k=1}^{m} Q^k \frac{\partial L}{\partial p^k}.$$

Example. Let, writing $(u,v) = (u^1, u^2)$,

$$L(x,u,v,\nabla u,\nabla v) = \nabla u \nabla v - u^p v.$$

Then

$$-EL^{(1)} = \sum_{i=1}^{n} \frac{d}{dx_i} \frac{\partial v}{\partial x_i} + p|u|^{p-1} v = \Delta v + p|u|^{p-1} v,$$

and

$$-EL^{(2)} = \sum_{i=1}^{n} \frac{d}{dx_i} \frac{\partial u}{\partial x_i} + u^p = \Delta u + u^p.$$

For $\mathbf{P}$ we find

$$\mathbf{P} = h(\nabla u \nabla v + u^p v) + (a - h \cdot \nabla u)\nabla v + (b - h \cdot \nabla v)\nabla u,$$

where $a = a^1$ and $b = a^2$. We apply this to the transformation group

$$\begin{cases} \tilde{x} = e^{\tau} x \\ \tilde{u} = e^{-\frac{2}{p-1}\tau} \\ \tilde{v} = e^{\frac{N-(N-2)p}{p-1}\tau} v. \end{cases}$$

Then

$$h(x,u,v) = x; \quad a(x,u,v) = -\frac{2}{p-1} u; \quad b(x,u,v) = \frac{N-(N-2)p}{p-1} v.$$

Hence

$$\mathbf{P} = x(\nabla u \nabla v + u^p v) - (\frac{2}{p-1} u\nabla v + x\nabla u)\nabla v + \left(\frac{N-(N-2)p}{p-1} v - x\nabla v\right)\nabla u,$$

so that if ν is the normal on the boundary $\partial\Omega$ of a domain Ω, and if $u = v = 0$ on $\partial\Omega$,

$$\mathbf{P}\cdot\nu = -(x\cdot\nu)\frac{\partial u}{\partial\nu}\frac{\partial v}{\partial\nu}.$$

Finally

$$\tilde{L}\frac{d\tilde{x}}{dx} = \nabla u\nabla v - u^p v,$$

so that in particular

$$\frac{d}{dt}\left(\tilde{L}\frac{d\tilde{x}}{dx}\right)_{t=0} = 0.$$

We conclude that smooth solutions of

$$\begin{cases} \Delta u + u^p = 0 \qquad \Omega \\ \Delta v + p|u|^{p-1}v = 0 \qquad \Omega \\ u = v = 0 \qquad \partial\Omega \end{cases}$$

satisfy

$$\int_{\partial\Omega}(x\cdot\nu)\frac{\partial u}{\partial\nu}\frac{\partial v}{\partial\nu} = 0,$$

which is exactly the result we needed for the second eigenfunction.

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