Differential Systems with Strongly Indefinite Variational Structure

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We study the existence of non-trivial solutions for a class of problems containing in particular the following system of two coupled semilinear Poisson equations:

$$
\text{(I)} \begin{cases} -\Delta v = f(u) \quad \text{in } \Omega, & (1) \\ -\Delta u = g(v) \quad \text{in } \Omega, & (2) \\ u = v = 0 \qquad \text{on } \partial\Omega. & (3) \end{cases}
$$

Here Ω is a bounded domain in $\mathbf{R}^N$ with a smooth boundary, and Δ is the Laplace operator. Problem (I) allows for a variational formulation, i.e. solutions arise as critical points of the Lagrangian

$$
\mathcal{L}(\mathbf{u}) = \int_\Omega \nabla u \nabla v - \int_\Omega F(u) - \int_\Omega G(v), \quad \mathbf{u} = (u, v), \tag{4}
$$

where the functions F and G are the primitives of f and g. The objective of this paper is to establish a natural functional analytic frame work for the study of $\mathcal{L}$, and to obtain existence results for critical points of $\mathcal{L}$ by means of the topological min-max approach due to Benci and Rabinowitz [BR].

The quadratic part of $\mathcal{L}$,

$$
A(\mathbf{u}) = \int_\Omega \nabla u \nabla v = (u, v)_{H_0^1},
$$

is strongly indefinite. Indeed, if we replace (u, v) by $(u, -v)$, this results in a sign change of A, so that $\mathbf{u} = (u, v) = (0, 0)$ is a saddle point for A having an infinite Morse index: any decomposition of the $\mathbf{u}$-space into two subspaces H_1 and H_2 such that A restricted to H_1 has a minimum in $(0, 0)$, and A restricted to H_2 has a maximum in $(0, 0)$, will necessarily imply that both H_1 are infinite dimensional. Hence neither the mountain pass theorem [AR], where one usually has a local minimum in the origin, nor the saddle point theorem [R1], which requires one of the two subspaces H_1 and H_2 to be finite dimensional, are of use here.

Strongly indefinite functionals in the context of Hamiltonian systems and semilinear wave equations were studied in [A,R3]. A direct min-max method for these functionals was introduced by Benci and Rabinowitz in [BR], see also [R2,Ho]. Their result, which we shall refer to in this paper as the "Indefinite Functional Theorem", is an extension of both the mountain pass theorem and the saddle point theorem to functionals, defined on a Hilbert space H, of the form

$$
\mathcal{L}(\mathbf{u}) = \frac{1}{2}(L\mathbf{u}, \mathbf{u})_H - \mathcal{H}(\mathbf{u}). \tag{5}
$$

Here L is a bounded linear selfadjoint operator on H, and $\mathcal{H} : H \to H$ is nonlinear, satisfying a number of assumptions. This theorem allows a decomposition of H into two infinite dimensional subspaces H_1 and H_2 such that L is positive on H_1 and negative on H_2. Benci and Rabinowitz

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applied this theorem to Hamiltonian systems, semilinear wave equations and to Lagrangians of the form

$$\frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega H(u,v). \tag{6}$$

We shall apply it to the Lagrangian (4) of Problem (I), where there is an interesting new interplay between the choice of function spaces and the growth conditions imposed on $F(u)$ and $G(v)$.

To get some feeling for the subtleties involved we first consider the case that $f \equiv g$ and $u \equiv v$, when Problem (I) reduces to the well known Dirichlet problem for the semilinear Poisson equation,

$$\text{(II)}\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and solutions are critical points of the Lagrangian

$$\tilde{\mathcal{L}}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F(u), \tag{7}$$

If we take

$$f(u) = u^p = |u|^{p-1}u, \tag{8}$$

then, for

$$1 < p < \frac{N+2}{N-2}, \tag{9}$$

a (positive) solution is easily obtained from a standard mountain pass argument [AR] applied to the functional $\tilde{\mathcal{L}}(u)$ in (7). The variable function u is taken in the Sobolev space $H_0^1(\Omega)$, which is the closure of the compactly supported (in Ω) smooth functions with respect to the (inner product) norm

$$\|u\|^2_{H_0^1} = \int_\Omega |\nabla u|^2. \tag{10}$$

This space embeds compactly into $L^{p+1}(\Omega)$ if p satisfies (9), i.e. if p is subcritical. On the other hand, if

$$p \geq \frac{N+2}{N-2}, \tag{11}$$

the Pohozaev identity [P] excludes the existence of (classical) solutions of (II) when Ω is starshaped. Moreover, if the inequality in (11) is strict, the functional $\tilde{\mathcal{L}}(u)$ is no longer well defined on $H_0^1(\Omega)$.

Now consider Problem (I). Looking at the pure power case first, we let f again be defined by (8), and set

$$g(v) = v^q = |v|^{q-1}v, \tag{12}$$

where also $q > 1$. At first glance, it seems natural to look for a critical point of $\mathcal{L}(\mathbf{u})$, varying both u and v over $H_0^1(\Omega)$. This space was also used by Benci and Rabinowitz when they dealt with (7). For $\mathcal{L}(\mathbf{u})$ to be well defined one is then however forced to assume that

$$p \leq \frac{N+2}{N-2} \quad \text{and} \quad q \leq \frac{N+2}{N-2}, \tag{13}$$

and most likely, for reasons of compactness, that these inequalities are strict.

The main point of this paper is that for (4) this is to restrictive and therefore *not* the right approach, because the natural assumption on p and q is

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}. \tag{14}$$

Fig. 1. The subcritical values of p and q, $N > 4$.

This is in agreement with a generalization of the Pohozaev identity due to Mitidieri [Mi] and to the second author of this paper [vdV1], which shows that positive classical solutions on starshaped domains can only exist if (14) holds. For solutions of Problem (I) this identity reads

$$\int_\Omega \Big(NF(u) - \alpha u f(u)\Big) + \int_\Omega \Big(NG(v) - (N-2-\alpha)vg(v)\Big) = \oint_{\partial\Omega} \frac{\partial u}{\partial \nu}\frac{\partial v}{\partial \nu}(x,\nu), \tag{15}$$

where ν is the outward unit normal on $\partial\Omega$. We note that this identity also follows from a variant of Emmy Noether's first theorem [N,vdV1,HV].

Substituting (8) and (12) into (15), and choosing e.g.

$$\alpha = \frac{N}{p+1}, \tag{16}$$

it is easily seen that the existence of a positive solution pair, combined with the boundary point lemma [PW], implies that (14) must hold.

A growth condition of the type $f(u) = O(|u|^p)$ and $g(v) = O(|v|^q)$, with p and q satisfying (14), together with a number of other assumptions, was shown to be sufficient for the existence of a positive solution pair if Ω is a ball in [PV], and for general convex domains in [CFM]. In fact the celebrated Gidas-Ni-Nirenberg Theorem has been generalized by Troy [Tr] to positive solution pairs of (I) under the condition that the right hand sides are nonnegative. In both [PV] and [CFM] this is used to derive a priori estimates which allow the construction of a solution by fixed point arguments.

Returning to the variational approach, we shall use suitable inter- and extrapolation spaces to replace $H_0^1(\Omega) \times H_0^1(\Omega)$, and extend the variational formulation of (I) in a Hilbert space setting to the parameter range defined by (14). The key idea is to destroy the symmetry between u and v, by demanding more regularity of u then of v if p is large and q is small, and vice versa. Considering $-\Delta$ as an unbounded positive operator in $L^2(\Omega)$, with zero Dirichlet boundary conditions, we let $\Theta^r(\Omega)$ be the domain of $(-\Delta)^{r/2}$, i.e.

$$\Theta^r(\Omega) = D((-\Delta)^{r/2}), \tag{17}$$

and set

$$E^r(\Omega) = \Theta^r(\Omega) \times \Theta^{2-r}(\Omega). \tag{18}$$

Here $r \in (0,2)$ depending on p and q is chosen in such a way that $\mathcal{L}(\mathbf{u})$ has a continuously differentiable extension to $E^r(\Omega)$ if p and q satisfy (14). We then apply the "Indefinite Functional Theorem" to Problem (I), and in doing so we extend the existence results of [CFM] and [PV]. Our main result is

Theorem 1. *Suppose the functions f and g satisfy the hypotheses*

$$\begin{aligned}
(h1):&\quad f,g \in C(\mathbf{R},\mathbf{R}), f(0)=g(0)=0 \ \text{ and } \ F,G \geq 0 \ \text{ on } \ \mathbf{R},\\
(h2):&\quad f(u) = O(|u|^p),\ \ g(v) = O(|v|^q) \ \text{ as } \ |u|,|v| \longrightarrow \infty\\
&\qquad\qquad \text{with} \quad \frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N},\ \ p,q>1,\ \ N \geq 1,\\
(h3):&\quad f(u) = o(|u|),\ \ g(v) = o(|v|) \ \text{ as } \ |u|,|v| \longrightarrow 0,\\
(h4):&\quad \text{there exist constants } \ \gamma > 2 \ \text{ and } \ \eta > 0 \ \text{ such that}\\
&\qquad\qquad 0 < \gamma F(u) \leq uf(u),\ \ 0 < \gamma G(v) \leq vg(v) \quad \text{for} \ \ |u|,|v| \geq \eta.
\end{aligned}$$

Then Problem (I) has at least one non-trivial solution $(u,v) \in (C^2(\Omega) \cap C^1(\overline{\Omega}))^2$. Moreover, if both f and g are nonnegative on $[0,\infty)$, then (I) has a solution with positive components.

By an adaptation of the methods in the proof of Theorem 1 we obtain a similar result for

$$\text{(III)}\begin{cases} -\Delta v = \lambda u + f(u) & \text{in } \Omega, \\ -\Delta u = \mu v + g(v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

Here λ and μ are fixed real numbers.

Theorem 2. *Suppose the functions f and g satisfy the hypotheses (h1-4). Then for every real λ and μ there exists at least one non-trivial solution $(u, v) \in (C^2(\Omega) \cap C^1(\overline{\Omega}))^2$ of Problem (III). Moreover, if f and g are nonnegative on $[0, \infty)$, λ and μ are nonnegative, and if $0 \leq \lambda\mu < \lambda_1^2$, where λ_1 is the first eigenvalue of $-\Delta$ with zero boundary conditions, then (III) has a solution with positive components.*

The conditions in Theorem 2 are natural in view of the results in [vdV1] for

$$\begin{cases} f(u) = \lambda u + u^p, \\ g(v) = \mu v + v^q, \end{cases}$$

when $\lambda\mu \geq \lambda_1^2$, λ and μ nonnegative, excludes the existence of positive solutions. We suspect however that for general λ and μ our result for positive solutions is not yet optimal.

Problem (III) is a special case of

$$\text{(IV)}\begin{cases} -\Delta v = \lambda u + H_u(u, v) & \text{in } \Omega, \\ -\Delta u = \mu v + H_v(u, v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

Solutions of Problem (IV) are critical points of the functional

$$\mathcal{L}(\mathbf{u}) = \int_\Omega \nabla u \nabla v - \int_\Omega H(u, v), \tag{19}$$

to which our methods also apply. A typical example is if we consider the Hamiltonian

$$H(u, v) = \frac{1}{p+1} u^{p+1} + \frac{1}{q+1} v^{q+1} + \sum_{i=1}^{m} u^{\alpha_i} v^{\beta_i},$$

with $\alpha_i, \beta_i > 1$, and

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N} \quad \text{and} \quad \frac{\alpha_i}{p+1} + \frac{\beta_i}{q+1} = 1,$$

which agree with related non-existence results [vdV1,Mi].

Finally we mention a generalization of Theorem 2 to the case of higher order operators. For $K \in \mathbf{N}$ consider

$$
\text{(V)} \begin{cases} (-\Delta)^K v = \lambda u + f(u) & \text{in } \Omega, \\ (-\Delta)^K u = \mu v + g(v) & \text{in } \Omega, \\ u = \dots = (-\Delta)^{K-1} u = 0 & \text{on } \partial\Omega, \\ v = \dots = (-\Delta)^{K-1} v = 0 & \text{on } \partial\Omega, \end{cases}
$$

Using the function space $H = E^r_K(\Omega) = \Theta^r(\Omega) \times \Theta^{2K-r}(\Omega)$, $r \in [0, 2K]$, one proves along the same lines as Theorem 2:

Theorem 3. *Suppose the functions f and g satisfy the hypotheses*

$$
\begin{aligned}
(p1): &\quad f, g \in C(\mathbf{R}, \mathbf{R}),\ f(0) = g(0) = 0 \ \ \textit{and} \ \ F, G \ge 0, \\
(p2): &\quad f(u) = O(|u|^p), \ \ g(v) = O(|v|^q) \ \ \textit{as} \ \ |u|, |v| \longrightarrow \infty \\
&\qquad \textit{with} \ \ \frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2K}{N}, \ \ p, q > 1, \ \ N \ge 1 \\
(p3): &\quad f(u) = o(|u|), \ \ g(v) = o(|v|) \ \ \textit{as} \ \ |u|, |v| \to 0, \\
(p4): &\quad \textit{there exist constants} \ \ \gamma > 2 \ \ \textit{and} \ \ \eta > 0 \ \ \textit{such that} \\
&\qquad 0 < \gamma F(u) \le u f(u), \ \ 0 < \gamma G(v) \le v g(v) \quad \textit{for} \ \ |u|, |v| \ge \eta,
\end{aligned}
$$

Then for every real λ and μ there exists at least one non-trivial solution $(u, v) \in (C^{2K}(\Omega) \cap C^{2K-1}(\overline{\Omega}))^2$ of Problem (V). Moreover, if f and g are nonnegative on $[0, \infty)$, λ and μ are nonnegative, and if $0 \le \lambda\mu < \lambda_1^{2K}$, where λ_1 is the first eigenvalue of $-\Delta$ with zero boundary conditions, then (V) has a solution with positive components.

The plan of this paper is as follows. In Section 1 we introduce the space $E^r(\Omega)$ in considerable detail, and extend the quadratic form $A(\mathbf{u})$ to $E^r(\Omega)$. Then, in Section 2, we obtain critical points of $\mathcal{L}$, and prove Theorem 1 in a weak sense. In Section 3 we obtain the regularity of these solutions, thus completing the proof of Theorem 1. In Section 4, we describe how to adjust the proof of Theorem 1 to the case of Theorem 2. The proof of Theorem 3 is left to the reader.

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1. The functional analytic frame work.

In this section we discuss the spaces $\Theta^r(\Omega)$, the product spaces $E^r(\Omega) = \Theta^r(\Omega) \times \Theta^{2-r}(\Omega)$, and the precise definition of $\mathcal{L}(\mathbf{u})$ for $\mathbf{u} = (u, v)$ in $E^r(\Omega)$ in detail. The choice of the parameter r will depend on the nonlinearities f and g.

We begin with the spaces $\Theta^r(\Omega)$ which, following [LM], are defined in terms of the domains of powers of the Laplacian in $L^2(\Omega)$ with zero Dirichlet boundary conditions, i.e. the strictly positive selfadjoint operator $-\Delta$ with domain $H^2(\Omega) \cap H^1_0(\Omega) \subset L^2(\Omega)$. For simplicity we use the Fourier coefficients of u and v with respect to a fixed orthonormal basis of $L^2(\Omega)$, consisting of eigenfunctions $\phi_1, \phi_2, \phi_3, \dots$ of $-\Delta$, $\phi_1 > 0$, corresponding to positive eigenvalues $\lambda_1 < \lambda_2 \le \lambda_3 \le \dots \uparrow \infty$, counted with multiplicity. Throughout this paper these eigenfunctions are normalized in $L^2(\Omega)$:

$$
\int_\Omega \phi_i \phi_j = \delta_{ij}. \tag{1.1}
$$

Thus

$$
L^2(\Omega) = \{u = \sum_{k=1}^\infty \xi_k \phi_k : \sum_{k=1}^\infty \xi_k^2 < \infty\}, \tag{1.2}
$$

and

$$(u,v)_{L^2} = \int_\Omega uv = \sum_{k=1}^\infty \xi_k \eta_k, \quad u = \sum_{k=1}^\infty \xi_k \phi_k, \quad v = \sum_{k=1}^\infty \eta_k \phi_k. \tag{1.3}$$

The operators $(-\Delta)^{r/2}$ are defined by

$$(-\Delta)^{r/2} u = \sum_{k=1}^\infty \lambda_k^{r/2} \xi_k \phi_k, \tag{1.4}$$

with domain

$$D((-\Delta)^{r/2}) = \Theta^r(\Omega) = \{\sum_{k=1}^\infty \xi_k \phi_k \in L^2(\Omega) : \sum_{k=1}^\infty \lambda_k^r \xi_k^2 < \infty\}, \tag{1.5}$$

if $r \geq 0$. We note that $\Theta^r(\Omega) = H_0^r(\Omega) = H^r(\Omega)$ for $0 < r < 1/2$, $\Theta^{1/2}(\Omega) = H_{00}^{1/2}(\Omega)$ (see [LM]), $\Theta^r(\Omega) = H_0^r(\Omega)$ for $1/2 < r \leq 1$, and that $\Theta^r(\Omega) = H^r(\Omega) \cap H_0^1(\Omega)$ for $1 < r \leq 2$.

If $r < 0$ the domain is equal to the whole of $L^2(\Omega)$, and extends in a natural way to a larger space which we shall define for $-2 < r < 0$ in a moment.

For $r \geq 0$ the space $\Theta^r(\Omega)$ is a Hilbert space with inner product

$$((u,v))_{\Theta^r} = (u,v)_{L^2} + ((-\Delta)^{r/2} u, (-\Delta)^{r/2} v)_{L^2}. \tag{1.6}$$

The corresponding norm is the graph norm of the operator $(-\Delta)^{r/2}$. We can identify $\Theta^r(\Omega)$ with the space

$$\varpi^r = \{\xi = (\xi_1, \xi_2, \ldots) : \sum_{k=1}^\infty \lambda_k^r \xi_k^2 < \infty\}, \tag{1.7}$$

which is a Hilbert space with inner product

$$(\xi, \eta)_r = \sum_{k=1}^\infty \lambda_k^r \xi_k \eta_k. \tag{1.8}$$

If $r = 0$ this is just the standard separable Hilbert space l^2. Since u and v in Θ^r are represented by ξ and η in ϖ^r, (1.8) is equal to the second term in the right hand side of (1.6):

$$((u,v))_{\Theta^r} = (\xi, \eta)_0 + (\xi, \eta)_r. \tag{1.9}$$

We shall write

$$(u,v)_{\Theta^r} = ((-\Delta)^{r/2} u, (-\Delta)^{r/2} v)_{L^2(\Omega)} = (\xi, \eta)_r, \tag{1.10}$$

and clearly this defines an equivalent inner product on Θ^r. In fact we have, denoting the norms corresponding to (1.10) by $\|u\|_{\Theta^r} = |\xi|_r$, that

$$\|u\|_{L^2(\Omega)} = \|\xi\|_{l^2} = |\xi|_0 \leq \lambda_1^{-r/2} |\xi|_r = \lambda_1^{-r/2} \|u\|_{\Theta^r}, \tag{1.11}$$

which can be viewed as a generalized Poincaré inequality.

Next we consider $\Theta^{-r}(\Omega)$ with $r > 0$. Note that $\sum_{k=1}^\infty \xi_k \phi_k$ can not lie in $L^2(\Omega)$ if ξ is not in l^2. The standard way to overcome this difficulty is to define $\Theta^{-r}(\Omega)$ as a representation of the dual space $\Theta^r(\Omega)'$. Since $\Theta^r(\Omega)$ can be identified with ϖ^r, $\Theta^r(\Omega)'$ can be identified with the dual of ϖ^r, which is easily seen to be represented as ϖ^{-r} setting

$$< \xi, \eta > = \sum_{k=1}^\infty \xi_k \eta_k, \quad \xi \in \varpi^r, \quad \eta \in \varpi^{-r}. \tag{1.12}$$

In this way the space $\Theta^r(\Omega)'$ is isomorphic to $(\varpi^r)'$ and ϖ^{-r} , and, for $0 < r < 2$, to

$$\Theta^{-r}(\Omega) = \{\sum_{k=1}^{\infty} \xi_k \phi_k \in (H^2(\Omega) \cap H_0^1(\Omega))' : \sum_{k=1}^{\infty} \lambda_k^{-r} \xi_k^2 < \infty\}. \tag{1.13}$$

From ϖ^{-r} the space $\Theta^{-r}(\Omega)$ inherits its inner product. Because $H^2(\Omega) \cap H_0^1(\Omega)$ is dense in $\Theta^r(\Omega)$, it follows that $\Theta^{-r}(\Omega)$ is the subspace of all elements in $(H^2(\Omega) \cap H_0^1(\Omega))'$ which are bounded with respect to the $\Theta^r(\Omega)$-norm. The identification of $\Theta^{-r}(\Omega)$ and $\Theta^r(\Omega)'$ is made explicit by writing

$$< u, v >=< \sum_{k=1}^{\infty} \xi_k \phi_k, \sum_{k=1}^{\infty} \eta_k \phi_k >=< \xi, \eta >= \sum_{k=1}^{\infty} \xi_k \eta_k, \tag{1.14}$$

for $u \in \Theta^{-r}(\Omega)$ and $v \in \Theta^r(\Omega)$. We observe that $(-\Delta)^s : \Theta^r(\Omega) \to \Theta^{r-2s}(\Omega)$ is an isomorphism.

The motivation to introduce these spaces was to extend $A(\mathbf{u}) = \int \nabla u \nabla v$ to functions u and v with different regularity properties. Since

$$A(\mathbf{u}) = \sum_{k=1}^{\infty} \lambda_k \xi_k \eta_k = \sum_{k=1}^{\infty} \lambda_k^{r/2} \xi_k \lambda_k^{1-r/2} \eta_k,$$

we have

$$|A(\mathbf{u})| \leq \{\sum_{k=1}^{\infty} \lambda_k^r \xi_k^2\}^{\frac{1}{2}} \{\sum_{k=1}^{\infty} \lambda_k^{2-r} \eta_k^2\}^{\frac{1}{2}} = \|u\|_{\Theta^r} \|v\|_{\Theta^{2-r}}.$$

Thus defining the product Hilbert spaces

$$E^r(\Omega) = \Theta^r(\Omega) \times \Theta^{2-r}(\Omega), \quad 0 < r < 2, \tag{1.15}$$

the quadratic form $A(\mathbf{u})$ uniquely extends to $E^r(\Omega)$.

We observe that $-\Delta$ is an isometric map from the first (second) component of $E^r(\Omega)$ onto the dual of the second (first) component. Hence

$$-\Delta_S = \begin{pmatrix} 0 & -\Delta \\ -\Delta & 0 \end{pmatrix} : \; E^r(\Omega) \to \Theta^{-r}(\Omega) \times \Theta^{r-2}(\Omega) = (E^r(\Omega))', \tag{1.16}$$

defined by

$$-\Delta_S \mathbf{u} = (-\Delta v, -\Delta u), \quad \mathbf{u} = (u, v) \in E^r(\Omega), \tag{1.17}$$

is an isometry, so that there exists a unique selfadjoint isometry $L : E^r(\Omega) \to E^r(\Omega)$, such that

$$\frac{1}{2} < -\Delta_S \mathbf{u}, \mathbf{u} >= \frac{1}{2}(L\mathbf{u}, \mathbf{u})_{E^r} \quad \mathbf{u} = (u, v) \in E^r(\Omega). \tag{1.18}$$

For $u, v \in H^2(\Omega) \cap H_0^1(\Omega)$ we have

$$\frac{1}{2} < -\Delta_S \mathbf{u}, \mathbf{u} >= \frac{1}{2} \int_\Omega (-\Delta u) v + \frac{1}{2} \int_\Omega u(-\Delta v) = \int_\Omega \nabla u \nabla v = A(\mathbf{u}),$$

so

$$\begin{aligned} \frac{1}{2} < -\Delta_S \mathbf{u}, \mathbf{u} > &= ((-\Delta)^{r/2} u, (-\Delta)^{1-r/2} v)_{L^2} \\ &= \frac{1}{2}((-\Delta)^{r/2} u, (-\Delta)^{r/2} (-\Delta)^{1-r} v)_{L^2} + \\ &\quad \frac{1}{2}((-\Delta)^{1-r/2} (-\Delta)^{r-1} u, (-\Delta)^{1-r/2} v)_{L^2} \\ &= \frac{1}{2}((-\Delta)^{r-1} u, v)_{\Theta^r} + \frac{1}{2}(u, (-\Delta)^{1-r} v)_{\Theta^{2-r}} \\ &= \frac{1}{2}\left(\begin{pmatrix} 0 & (-\Delta)^{1-r} \\ (-\Delta)^{r-1} & 0 \end{pmatrix} \mathbf{u}, \mathbf{u} \right)_{E^r}, \end{aligned} \tag{1.19}$$

which explicitly gives the unique extension of $A(\mathbf{u})$ to $E^r(\Omega)$, and yields

$$L = \begin{pmatrix} 0 & (-\Delta)^{1-r} \\ (-\Delta)^{r-1} & 0 \end{pmatrix}. \tag{1.20}$$

Strictly speaking we should replace $\mathbf{u}$ by its transpose $\mathbf{u}^t$ in these formulas, but for notational convenience we shall not bother to do so.

In the functional analytic set-up of Problem (I) the duality map $-\Delta_S$ plays the same role as the familiar duality map $-\Delta:\ H_0^1(\Omega) \to H^{-1}(\Omega)$ for Problem (II).

Introducing the "diagonals"

$$E^{\pm} = \{(u, \pm(-\Delta)^{r-1}u):\ u \in \Theta^r\}, \tag{1.21}$$

we have

$$E^r(\Omega) = E^+ \oplus E^- = \{\mathbf{u} = \mathbf{u}^+ + \mathbf{u}^-,\ \mathbf{u}^{\pm} \in E^{\pm}\}. \tag{1.22}$$

Note that

$$L\mathbf{u}^{\pm} = L(u, \pm(-\Delta)^{r-1}u) = (\pm u, (-\Delta)^{r-1}u) = \pm\mathbf{u}^{\pm}, \tag{1.23}$$

so that E^+ and E^- are the mutually orthogonal eigenspaces of the eigenvalues 1 and -1 of L. Orthonormal bases consisting of eigenvectors of $E^{\pm}$ are given by

$$\{\frac{1}{\sqrt{2}}(\lambda_k^{-\frac{r}{2}}\phi_k, \pm\lambda_k^{\frac{r}{2}-1}\phi_k) : k = 1, 2, \dots\}. \tag{1.24}$$

For $\mathbf{u} = \mathbf{u}^+ + \mathbf{u}^-$ we have

$$A(\mathbf{u}) = \frac{1}{2} < -\Delta_S\mathbf{u}, \mathbf{u} >= \frac{1}{2}(L\mathbf{u}, \mathbf{u})_{E^r(\Omega)} = A(\mathbf{u}^+) + A(\mathbf{u}^-), \tag{1.25}$$

whereas

$$A(\mathbf{u}^+) - A(\mathbf{u}^-) = \frac{1}{2}\|\mathbf{u}\|^2_{E^r(\Omega)}, \tag{1.26}$$

The derivative of $A(\mathbf{u})$ defines a bilinear form

$$B(\mathbf{u}, \mathbf{\Phi}) = A'(\mathbf{u})\mathbf{\Phi} = (L\mathbf{u}, \mathbf{\Phi})_{E^r(\Omega)} =< -\Delta_S\mathbf{u}, \mathbf{\Phi} >, \quad \mathbf{u}, \mathbf{\Phi} \in E^r(\Omega), \tag{1.27}$$

with

$$A(\mathbf{u}) = \frac{1}{2}B(\mathbf{u}, \mathbf{u}) \text{ and } B(\mathbf{u}^+, \mathbf{u}^-) = 0. \tag{1.28}$$

Next we examine for which values of r, and for which nonlinearities, the other terms appearing in the Lagrangian are well defined on $E^r(\Omega)$. Observe (see [LM]) that the injection $\Theta^r(\Omega) \to H^r(\Omega)$ is continuous.

By the Sobolev embeddings,

$$\Theta^r(\Omega) \to H^r(\Omega) \to L^p(\Omega), \text{ if } 1 \le p \le \frac{2N}{N-2r} < \infty \tag{1.29}$$

is bounded for $0 < r < 2N$. The second injection is compact if $1 \le p < \frac{2N}{N-2r}$. If $2r \ge N$, these statements hold for any $1 \le p < \infty$. Also $\Theta^{r_1}(\Omega) \to \Theta^{r_2}(\Omega)$ is compact if $r_1 > r_2$ (for the corresponding spaces ϖ^r this can be proved directly). An immediate consequence of (1.29) and (1.15) is that for $N > 2r$ and $N > 4 - 2r$,

$$E^r(\Omega) \to L^{p+1}(\Omega) \times L^{q+1}(\Omega), \tag{1.30}$$

if

$$1 \le p + 1 \le \frac{2N}{N-2r},\ 1 \le q + 1 \le \frac{2N}{N+2r-4}. \tag{1.31}$$

This embedding is compact if both inequalities bounding p and q from above are strict. If $r \geq 2N$, there is no restriction on p, and if $N \leq 4 - 2r$, there is no restriction on q.

For the functional $\mathcal{L}$ to be well defined, we find that p and q in assumption (h2) have to satisfy (1.31), where we are still free in our choice of $0 < r < 2$. We restrict to this range of r, because we shall need the compactness of the embedding $E^r(\Omega) \to L^2(\Omega) \times L^2(\Omega)$. If we put equality signs in (1.31) we obtain a parametrization of a section of the "critical" hyperbola

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N} \tag{1.32}$$

in the first quadrant of the (p, q)-plane, which vanishes for $N \leq 2$. Clearly, if p and q satisfy (h2), then $\mathcal{L}$ is well defined if we choose $0 < r < 2$ such that (1.31) holds.

2. Existence of critical points.

In this section we shall prove that $\mathcal{L}$ has a critical point $\mathbf{u} = (u, v)$. Using the extension of the the quadratic form $A(\mathbf{u}) = \int \nabla u \nabla v$ to $E^r(\Omega)$, we thus consider the Lagrangian

$$\mathcal{L}(\mathbf{u}) = \frac{1}{2}(L\mathbf{u}, \mathbf{u})_{E^r(\Omega)} - \mathcal{H}(\mathbf{u}), \tag{2.1}$$

where L is defined by (1.20), and

$$\mathcal{H}(\mathbf{u}) = \int_\Omega F(u)dx + \int_\Omega G(v)dx \tag{2.2}$$

is the "Hamiltonian".

Theorem 2.1. *Let the assumptions (h1-4) be satisfied, and let $0 < r < 2$ be such that (1.31) holds with strict inequalities. Then $\mathcal{L}$ has a non-trivial critical point $\mathbf{u}$ in $E^r(\Omega)$.*

The proof is based on an application of the following theorem of Benci and Rabinowitz [BR].

Theorem 2.2. *(Indefinite Functional Theorem.)*

Let H be a real Hilbert space with $H = H_1 \oplus H_2$. Suppose $\mathcal{L} \in C^1(H, \mathbf{R})$, satisfies the Palais-Smale condition, and

(I_1) $\mathcal{L}(\mathbf{u}) = \frac{1}{2}(L\mathbf{u}, \mathbf{u})_H - \mathcal{H}(\mathbf{u})$, where $L : H \to H$ is bounded and self-adjoint, and L leaves H_1 and H_2 invariant;

(I_2) $\mathcal{H}'$ is compact;

(I_3) there exists a subspace $\tilde{H} \subset H$ and sets $S \subset H$, $Q \subset \tilde{H}$ and constants $\alpha > \omega$ such that

(i) $S \subset H_1$ and $\mathcal{L}|_S \geq \alpha$,

(ii) Q is bounded and $\mathcal{L} \leq \omega$ on the boundary ∂Q of Q in $\tilde{H}$.

(iii) S and ∂Q link.

Then $\mathcal{L}$ possesses a critical value $c \geq \alpha$.

Before we apply this theorem to $\mathcal{L}$ we discuss the notion of linking, and the min-max characterisation of the critical value. Starting with two sets Q and S such that Q and S have nonempty intersection, while the intersection of ∂Q and S is empty, we consider deformations of Q. The idea is to consider a class of deformations $\psi(t, Q)$, $t \in [0, 1]$, having the property that as long as $\psi(t, \partial Q)$ and S do not intersect, the intersection of $\psi(t, Q)$ and S must be nonempty. We then know that the supremum of the values taken by $\mathcal{L}$ on $\psi(t, Q)$ is larger then α. If at the same time we are able to keep the values of $\mathcal{L}$ below $\omega < \alpha$ on $\psi(t, \partial Q)$, the critical value is then obtained by minimizing this supremum over a suitable class of such deformations.

In our application of Theorem 2.2 we shall choose for the set Q the product of a closed ball in H_2 and a closed line segment in H_1, and for S a sphere contained in H_1, with S and Q intersecting in precisely one (interior) point of Q. The difficulty here is that if H_2 has infinite dimension, one can make a hole in Q by a continuous deformation of Q leaving the boundary of Q invariant. Thus the intersection of Q and S may simply disappear. We note however that the difference of such a deformation and the identity can never be compact. In [BR] this was the reason to introduce the following class of deformations:

$$\Psi = \big\{\psi \in C([0,1] \times H, H) \mid \psi(0,\mathbf{u}) = \mathbf{u},$$
$$P_2\psi(t,\mathbf{u}) = P_2\mathbf{u} - K(t,\mathbf{u}),\ K : [0,1] \times H \to H_2 \ \text{ continuous and compact}\big\}.$$

The sets S and ∂Q are then said to link if, for all $\psi \in \Psi$ with $\psi([0,1], \partial Q) \cap S = \emptyset$,

$$\psi(t,Q) \cap S \neq \emptyset \ \ \forall t \in [0,1],$$

and this is indeed the case for the choice of S and Q above. It is important to observe that this concept of linking depends on the choice of Ψ and in particular on the choice of H_2.

In the proof of Theorem 2.2 [BR] the critical value c is actually obtained as the min-max over a different class of deformations of Q. The choice of this class is related to the pseudo-gradient flow of $\mathcal{L}$ and the application of the deformation lemma in the proof of Theorem 2.2. Let Γ be the subset of $C([0,1] \times H, H)$ consisting of all functions h satisfying

$$\begin{aligned} &h(0,\mathbf{u}) = \mathbf{u}, \\ &h(t,\mathbf{u}) = \mathbf{u} \ \text{ for } \ \mathbf{u} \in \partial Q, \\ &h(t,\mathbf{u}) = e^{\theta(t,\mathbf{u})L}\mathbf{u} + K(t,\mathbf{u}), \ \text{ where } \ \theta \in C([0,1] \times H, \mathbf{R}), \ \ K \text{ compact.} \end{aligned}$$

Then

$$c = \inf_{h \in \Gamma} \sup_{Q} \mathcal{L}(h(1,\mathbf{u})) \geq \alpha.$$

Proof of Theorem 2.1. We apply Theorem 2.2 with $H = E^r(\Omega)$, $H_1 = E^+$, and $H_2 = E^-$. To show that $\mathcal{L}$ is continuously differentiable and satisfies (I_1) and (I_2), it suffices to observe that $\mathcal{H}(\mathbf{u})$, defined by (2.2), is continosly differentiable with compact derivative $\mathcal{H}'(\mathbf{u})$. Apart from the choice of r, this is standard, and follows from conditions (h1-2), and (1.31). See e.g. [BR,R2,F].

Step 1. The geometric conditions $(I_3(i,ii,iii))$ are satisfied. Let ρ, $s_1 > \rho$ and s_2 be positive numbers to be specified later on, and let $\mathbf{e}^\pm$ be the first vector in the basis (1.24) of $E^\pm$. We set $[0, s_1\mathbf{e}^+] = \{s\mathbf{e}^+;\ \ 0 \leq s \leq s_1\}$, and

$$Q = [0, s_1\mathbf{e}^+] \oplus \left(\overline{B}_{s_2} \cap E^-\right), \quad \tilde{H} = \text{span}\ [\mathbf{e}^+] \oplus E^-, \quad S = \partial B_\rho \cap E^+, \tag{2.3}$$

where B_R denotes an open ball with radius R centered at the origin.

On E^+ the quadratic part $A(\mathbf{u})$ of $\mathcal{L}$ reduces to $(1/2)\|\mathbf{u}\|^2_{E^r}$, so that A has a strict local minimum on E^+ at $\mathbf{u}^+ = 0$. Condition (h3) implies that the same is true for $\mathcal{L}$. Indeed we have the familiar estimates

$$F(u) \leq \epsilon u^2 + C(\epsilon)|u|^{p+1}, \ \ G(v) \leq \epsilon v^2 + C(\epsilon)|v|^{q+1}, \ \ \forall u, v \in \mathbf{R}, \tag{2.4}$$

with $\epsilon > 0$ arbitrarily small, and combining (2.4) with the Poincaré inequality (1.11), (1.31), and the Sobolev embedding (1.30), it is straightforward to show that

$$\mathcal{L}(\mathbf{u}^+) \geq (\frac{1}{2} - \epsilon)\|\mathbf{u}^+\|^2_{E^r} - C(\epsilon)\|\mathbf{u}\|^{p+1}_{E^r} - C(\epsilon)\|\mathbf{u}\|^{q+1}_{E^r}, \tag{2.5}$$

for all $\mathbf{u}^+ \in E^+$. Thus we can fix $\rho > 0$ and $\alpha > 0$ such that $\mathcal{L}(\mathbf{u}) \geq \alpha$ on S, and $(I_3(i))$ is satisfied.

Fig. 2. The sets Q and S in $E^r(\Omega) = E^+ \oplus E^-$.

Next we show that for suitable choices of s_1 and s_2 the function $\mathcal{L}(\mathbf{u})$ is nonpositive on ∂Q. Thus we prove that $(I_3(ii))$ holds with $\omega = 0$. Note that the boundary ∂Q of the cylinder Q is taken in the space $\tilde{H}$, and consists of three parts, namely the bottom $Q \cap \{s = 0\}$, the lid $Q \cap \{s = s_1\}$, and the 'lateral' boundary $[0, s_1\mathbf{e}^+] \oplus (\partial B_{s_2} \cap E^-)$. Clearly $\mathcal{L}(\mathbf{u}) \le 0$ on the bottom because $A(\mathbf{u}) \le 0$ in E^-, and by (h1) the functional $\mathcal{H}(\mathbf{u})$ is nonnegative. For the remaining two parts of the boundary we first observe that, for $\mathbf{u} = \mathbf{u}^- + s\mathbf{e}^+ \in \tilde{H}$,

$$\mathcal{L}(\mathbf{u}^- + s\mathbf{e}^+) = \frac{1}{2}s^2 - \frac{1}{2}\|\mathbf{u}^-\|^r_E(\Omega)^2 - \mathcal{H}(\mathbf{u}^- + s\mathbf{e}^+). \tag{2.6}$$

By (h4)

$$\mathcal{H}(\mathbf{u}^- + s\mathbf{e}^+) \ge B_1 \int_\Omega |\mathbf{u}^- + s\mathbf{e}^+|^\gamma - B_2|\Omega|, \tag{2.7}$$

where B_1, B_2 are constants. Thus, writing $\mathbf{u}^- = t\mathbf{e}^- + \mathbf{u}_2^-$, where t is a real number, and $\mathbf{u}_2^- \in E^-$ is perpendicular to $\mathbf{e}^-$ in $E^r(\Omega)$, $\mathbf{u}_2^-$ is also perpendicular to $\mathbf{e}^-$ and $\mathbf{e}^+$ in $L^2 \times L^2$, and we proceed from (2.7) to conclude

$$\begin{aligned}
\mathcal{H}(\mathbf{u}^- + s\mathbf{e}^+) &\ge B_1 \int_\Omega |\mathbf{u}_2^- + t\mathbf{e}^- + s\mathbf{e}^+|^\gamma - B_2|\Omega| \\
&\ge B_3 \Big(\int_\Omega |\mathbf{u}_2^- + t\mathbf{e}^- + s\mathbf{e}^+|^2 \Big)^{\gamma/2} - B_4 \\
&= B_3 \Big(\int_\Omega |\mathbf{u}_2^-|^2 + \int_\Omega |t\mathbf{e}^- + s\mathbf{e}^+|^2 \Big)^{\gamma/2} - B_4 \\
&\ge B_3 \Big(\int_\Omega |t\mathbf{e}^- + s\mathbf{e}^+|^2 \Big)^{\gamma/2} - B_4 \\
&\ge B_3 \Big(s^2 \sin^2\chi \int_\Omega |\mathbf{e}^+|^2 \Big)^{\gamma/2} - B_4 = B_5 s^\gamma - B_4.
\end{aligned} \tag{2.8}$$

Here χ is the (positive) angle between $\mathbf{e}^+$ and $\mathbf{e}^-$ with respect to the inner product in $L^2(\Omega) \times L^2(\Omega)$.

For $\mathcal{L}$ this yields

$$\mathcal{L}(\mathbf{u}^- + s\mathbf{e}^+) \le \frac{1}{2}s^2 - B_5 s^\gamma + B_4 - \frac{1}{2}\|\mathbf{u}^-\|^r_E(\Omega)^2. \tag{2.9}$$

Now choose s_1, s_2 such that

$$\psi(s) = \frac{1}{2}s^2 - B_5 s^\gamma + B_4 \le 0 \quad \forall s \ge s_1, \quad s_2^2 > 2 \max_{s \ge 0} \psi(s), \tag{2.10}$$

to make $\mathcal{L}$ negative on the lid and on the lateral boundary respectively. Then $(I_3(ii))$ is satisfied with $\omega = 0$.

For the proof of $(I_3(iii))$ we refer to [BR].

Step 2. The Palais-Smale condition. We have to show that any sequence $\{\mathbf{u}_n\}$ in $E^r(\Omega)$ satisfying

$$\mathcal{L}(\mathbf{u}_n) \ \text{ is bounded in } E^r(\Omega), \quad \mathcal{L}'(\mathbf{u}_n) \to 0, \ \text{ as } \ n \to \infty, \tag{2.11}$$

has a convergent subsequence. The key point here is to prove that such a sequence is necessarily bounded in $E^r(\Omega)$. For then the compactness of $\mathcal{H}'$ implies, since $\mathcal{L}'(\mathbf{u}_n)$ converges in $(E^r(\Omega))'$, that a subsequence of $A'(\mathbf{u}_n)$ also converges. In view of (1.27), we then also have $L\mathbf{u}_n$ and $\mathbf{u}_n$ converging in $E^r(\Omega)$, because L is invertible.

To prove that $\mathbf{u}_n$ is bounded we proceed as follows. By (2.1), (1.27), (1.28), (2.11) and (h4), for some $M > 0$, and arbitrarily small $\epsilon_n > 0$, omitting the subscripts,

$$\begin{aligned} M + \epsilon\|\mathbf{u}\|_{E^r(\Omega)} &\geq \mathcal{L}(\mathbf{u}) - \frac{1}{2} < \mathcal{L}'(\mathbf{u}), \mathbf{u} > \\ &= \int_\Omega \big(\frac{1}{2}uf(u) - F(u) + \frac{1}{2}vg(v) - G(v)\big)dx \\ &\geq (\frac{1}{2} - \frac{1}{\gamma})\{\int_\Omega |u|\,|f(u)|dx + \int_\Omega |v|\,|g(v)|dx\} - C, \end{aligned} \tag{2.12}$$

where C is a constant taking care of the u, v-values between $-\eta$ and η. Hence, for some new constants $C, \epsilon > 0$,

$$\int_\Omega |u|\,|f(u)|dx + \int_\Omega |v|\,|g(v)|dx \leq C + \epsilon\|\mathbf{u}\|_{E^r}. \tag{2.13}$$

Still omitting subsripts, and writing $\mathbf{u}^\pm = (u^\pm, v^\pm)$, we also have, by (1.23) and (2.11),

$$\begin{aligned} \|\mathbf{u}^\pm\|^2_{E^r} - \epsilon\|\mathbf{u}^\pm\|_{E^r} &\leq \big|(L\mathbf{u}, \mathbf{u}^\pm)_{E^r} - < \mathcal{L}'(\mathbf{u}), \mathbf{u}^\pm > \big| \\ &= \big| < \mathcal{H}'(\mathbf{u}), \mathbf{u}^\pm > \big| = \big| \int_\Omega f(u)u^\pm dx + \int_\Omega g(v)v^\pm dx\big| \\ &\leq \{\int_\Omega |f(u)|^{\frac{p+1}{p}}\}^{\frac{p}{p+1}}\|u^\pm\|_{L^{p+1}} + \{\int_\Omega |g(v)|^{\frac{q+1}{q}}\}^{\frac{q}{q+1}}\|v^\pm\|_{L^{q+1}} \\ &\leq C\{\int_\Omega |f(u)||f(u)|^{\frac{1}{p}}\}^{\frac{p}{p+1}}\|\mathbf{u}^\pm\|_{E^r} + C\{\int_\Omega |g(v)||g(v)|^{\frac{1}{q}}\}^{\frac{q}{q+1}}\|\mathbf{u}^\pm\|_{E^r} \\ &\leq C\Big\{1 + \{\int_\Omega |f(u)|\,|u|\}^{\frac{p}{p+1}} + \{\int_\Omega |g(v)|\,|v|\}^{\frac{q}{q+1}}\Big\}\|\mathbf{u}^\pm\|_{E^r}. \end{aligned}$$

Dividing the first and the last expression by $\|\mathbf{u}^\pm\|_{E^r}$ we obtain

$$\|\mathbf{u}^\pm\|_{E^r} - \epsilon \leq C\Big\{1 + \{\int_\Omega |f(u)|\,|u|\}^{\frac{p}{p+1}} + \{\int_\Omega |g(v)|\,|v|\}^{\frac{q}{q+1}}\Big\}. \tag{2.14}$$

Combining for $\mathbf{u} = \mathbf{u}^+ + \mathbf{u}^-$, together with (2.13), it follows that, possibly for some new constant,

$$\|\mathbf{u}\|_{E^r} \leq C\Big\{1 + \{C + \epsilon\|\mathbf{u}\|_{E^r}\}^{\frac{p}{p+1}} + \{C + \epsilon\|\mathbf{u}\|_{E^r}\}^{\frac{q}{q+1}}\Big\}, \tag{2.15}$$

which keeps $\|\mathbf{u}\|_{E^r}$ away from infinity. This implies that the Palais-Smale condition is satisfied, and thereby concludes the proof of the theorem. □

Proposition 2.3. *Suppose the functions f and g satisfy the hypotheses (h1-4), and let $0 < r < 2$ be such that (1.31) holds. Then a critical point $\mathbf{u} = (u, v)$ of $\mathcal{L}$ is a solution of the Euler-Lagrange equations*

$$\begin{cases} -\Delta v = f(u) & \text{in } \Theta^{-r}(\Omega), \\ -\Delta u = g(v) & \text{in } \Theta^{r-2}(\Omega), \end{cases}$$

which can be seen as the appropiate weak formulation of Problem (I).

Proof of Proposition 2.3. Let $\mathbf{u}$ be such a critical point. Then, for all $\mathbf{\Phi} = (\phi, \psi) \in E^r(\Omega)$,

$$B(\mathbf{u}, \mathbf{\Phi}) = (L\mathbf{u}, \mathbf{\Phi})_{E^r} = \int_\Omega f(u)\phi dx + \int_\Omega g(v)\psi dx, \tag{2.16}$$

so that, substituting $\psi = 0$, for all $\phi \in \Theta^r(\Omega)$,

$$\int_\Omega f(u)\phi dx = ((-\Delta)^{1-r}v, \phi)_{\Theta^r} =$$

$$((-\Delta)^{r/2}(-\Delta)^{1-r}v, (-\Delta)^{r/2}\phi)_{L^2} =< -\Delta v, \phi >, \tag{2.17}$$

where the brackets in the last term are the same as in (1.14). Thus $-\Delta v = f(u)$ in $\Theta^{-r}(\Omega)$, and likewise, $-\Delta u = g(v)$ in $\Theta^{r-2}(\Omega)$. □

Theorem 2.4. *Let the assumptions (h1-4) be satisfied, let $0 < r < 2$ be such that (1.31) holds with strict inequalities, and in addition assume that f and g are nonnegative on $[0, \infty)$. Then $\mathcal{L}(\mathbf{u})$ has a non-trivial critical point in $E^r(\Omega)$, with both components nonnegative.*

Proof of Theorem 2.4. We redefine f and g to be zero on $(-\infty, 0]$. Applying Corollary 2.3 combined with the positivity properties of $(-\Delta)^{-1}$, the components u and v of any critical point have to be nonnegative. Thus it suffices to adapt the proof of Theorem 2.1, and in particular the construction of the set Q in Step 2.

In (2.7) we take $s_1 = s_2$. This immediately implies that the quadratic part of $\mathcal{L}(\mathbf{u})$ is nonpositive not only on the bottom, but also on the lateral boundary of $Q = Q_s$. Hence the same holds for $\mathcal{L}(\mathbf{u})$ because $\mathcal{H}(\mathbf{u})$ is nonnegative.

It remains to establish the nonpositivity of $\mathcal{L}$ on the lid of Q_s, which is the disk $D_s = \{(s\mathbf{e}^+, \mathbf{u}^-) : \|\mathbf{u}^-\|_{E^r} \le s\}$, and as in the proof of Theorem 2.1, this will be done by estimating $\mathcal{H}$ from below on D_s. From (h4) we have

$$F(u) \ge C(u_+^\gamma - 1), \quad G(v) \ge C(v_+^\gamma - 1), \tag{2.18}$$

where the subscript refers to taking the positive part, and is not be confused with the superscripts refering to elements in $E^\pm$. Fixing $s = 1$ we claim that there exists a constant $\delta > 0$, such that

$$\int_\Omega (u_+^\gamma + v_+^\gamma) > \delta \qquad \forall \mathbf{u} = (u, v) \in D_1. \tag{2.19}$$

Assuming (2.19) for the moment, we conclude, using homogenuity, that $\mathcal{H} \ge \delta s^\gamma - C$ on D_s. Combining with (2.9), $\mathcal{L}$ must be negative on D_s if s is large because $\gamma > 2$. Thus the proof of Theorem 2.4 will be complete if we prove (2.19).

Every $\mathbf{u}$ in D_1 is of the form $\mathbf{u} = \mathbf{e}^+ + t\mathbf{e}^- + \mathbf{u}_2^-$, with t real and $\mathbf{u}_2^- \in E^-$ perpendicular in $E^r(\Omega)$ to $\mathbf{e}^-$, as well as in $L^2 \times L^2$. Note that $\mathbf{e}^-$ has one positive and one negative component, and that the components of $\mathbf{u}_2^-$ both change sign (unless they are identical to zero), because they are perpendicular in $L^2(\Omega)$ to ϕ_1. This implies that the integral in (2.19) is always positive. Moreover, γ is subcritical, so the integral is easily seen to be continuous with respect to the weak topology in $E^r(\Omega)$, and D_1 is weakly compact. Hence the existence of a $\delta > 0$ such that (2.19) holds follows. This completes the proof. □

3. Regularity of solutions of Problem (I).

In this section we prove that critical points of $\mathcal{L}$ are classical solutions of Problem (I). Our method applies to subcritical and critical growth rates, and is similar to the one for fourth-order equations [vdV2].

Theorem 3.1. *Suppose the functions f and g satisfy the hypotheses (h1-4), and let $0 < r < 2$ be such that (1.29) holds. Then every critical point of $\mathcal{L}$ has $u, v \in L^\alpha(\Omega)$ for all $1 \le \alpha < \infty$.*

Proof of Theorem 3.1. We assume that $N \ge 2$. Because of (h3) we can rewrite the conclusion of Proposition 2.3 as

$$\begin{cases} -\Delta v = a(x)u \quad \text{in} \quad \Theta^{-r}(\Omega), & (3.1) \\ -\Delta u = b(x)v \quad \text{in} \quad \Theta^{r-2}(\Omega), & (3.2) \end{cases}$$

where

$$a(x) = \frac{f(u(x))}{u(x)}, \quad b(x) = \frac{g(v(x))}{v(x)}. \tag{3.3}$$

By (h2,h3)

$$a(x) \in L^{\frac{p+1}{p-1}}, \quad b(x) \in L^{\frac{q+1}{q-1}}(\Omega). \tag{3.4}$$

Now let $\epsilon > 0$. Then (see [vdV2]) there exists functions $q_\epsilon \in L^{\frac{p+1}{p-1}}(\Omega)$, and $f_\epsilon \in L^\infty(\Omega)$, such that

$$a(x)u(x) = q_\epsilon(x)u(x) + f_\epsilon(x), \quad \text{a.e. in } \ \Omega, \quad \|q_\epsilon\|_{L^{\frac{p+1}{p-1}}} < \epsilon. \tag{3.5}$$

Inserting (3.5) into (3.1-3.2) we obtain

$$\begin{cases} -\Delta v = q_\epsilon(x)u + f_\epsilon(x) & \text{in } \ \Theta^{-r}(\Omega), \qquad (3.6) \\ -\Delta u = b(x)v & \text{in } \ \Theta^{r-2}(\Omega), \qquad (3.2) \end{cases}$$

We denote the multiplication operators $u(x) \to b(x)u(x)$ and $u(x) \to q_\epsilon(x)u(x)$ by $\mathcal{B}$ and $\mathcal{Q}^\epsilon$. Then, inverting the Laplacians in (3.6-3.2), and eliminating v, we obtain,

$$u = (-\Delta)^{-1}\mathcal{B}(-\Delta)^{-1}\mathcal{Q}^\epsilon u + (-\Delta)^{-1}\mathcal{B}(-\Delta)^{-1} f_\epsilon,$$

or

$$(I - K^\epsilon)u = h_\epsilon, \tag{3.7}$$

where I is the identity operator, and

$$K^\epsilon = (-\Delta)^{-1}\mathcal{B}(-\Delta)^{-1}\mathcal{Q}^\epsilon, \quad h_\epsilon = (-\Delta)^{-1}\mathcal{B}(-\Delta)^{-1} f_\epsilon. \tag{3.8}$$

Thus the proof will be complete if we show for all large $\alpha < \infty$ that h_ϵ is in L^α, and that the operator $I - K^\epsilon : L^\alpha \to L^\alpha$ is invertible. The latter will be achieved by a bound of order ϵ on the operator norm of K^ϵ.

We observe that, as a consequence of the Hardy-Littlewood-Sobolev inequality, the operator $(-\Delta)^{-1}$ is bounded from L^{α_1} to L^{α_2} with

$$\frac{1}{\alpha_2} = \frac{1}{\alpha_1} - \frac{2}{N}, \quad 1 < \alpha_1 < \alpha_2 < \infty. \tag{3.9}$$

Also, because of Hölder's inequality, the multiplication operator $\mathcal{C}$ corresponding to a function $c(x) \in L^s$ is bounded from L^{β_1} to L^{β_2} with

$$\frac{1}{\beta_2} = \frac{1}{\beta_1} + \frac{1}{s}, \quad 1 \le s, \beta_1, \beta_2 \le \infty, \tag{3.10}$$

and its operator norm is equal to $\|c\|_{L^s}$. Note that s and β_1 have to be sufficiently large in order to keep β_2 larger then one, which is needed, because in (3.8) each multiplication operator is succeeded by $(-\Delta)^{-1}$.

Combining these two results we find that K^ϵ is bounded from L^α to L^β with

$$\frac{1}{\beta} = \frac{1}{\alpha} + \frac{p-1}{p+1} - \frac{2}{N} + \frac{q-1}{q+1} - \frac{2}{N} = \frac{1}{\alpha} + 2\Big(\frac{N-2}{N} - \frac{1}{p+1} - \frac{1}{q+1}\Big), \tag{3.11}$$

and that, denoting the two constants appearing in the two Hardy-Littlewood-Sobolev inequalities we apply by C_1, C_2,

$$\|K^\epsilon u\|_{L^\beta} \le C_1 \|b\|_{L^{\frac{q+1}{q-1}}} C_2 \|q_\epsilon(x)\|_{L^{\frac{p+1}{p-1}}} \|u\|_{L^\alpha} \le C_1 \|b\|_{L^{\frac{q+1}{q-1}}} C_2 \epsilon \|u\|_{L^\alpha}, \tag{3.12}$$

because of (3.5). Here we have to be careful, because we use (3.10) two times, first with $\beta_1 = \alpha$ and $s = \frac{p+1}{p-1}$ for the operator $\mathcal{Q}^\epsilon$, and then for the operator $\mathcal{B}$ with

$$\frac{1}{\beta_1} = \frac{1}{\alpha} + \frac{p-1}{p+1} - \frac{2}{N} \quad \text{and} \quad s = \frac{q+1}{q-1}.$$

Thus we have to choose α so large that

$$\frac{1}{\alpha} + \frac{p-1}{p+1} < 1 \quad \text{and} \quad \frac{1}{\alpha} + \frac{p-1}{p+1} - \frac{2}{N} + \frac{q-1}{q+1} = \frac{1}{\alpha} + 2\Big(\frac{N-1}{N} - \frac{1}{p+1} - \frac{1}{q+1}\Big) < 1.$$

This is possible because p and q are (sub)critical in the sense of (14). It follows that $\beta > \alpha$, and that $\beta = \alpha$ if $N > 2$ and p and q are critical. In the latter case the arguments above are valid for all $p + 1 \le \alpha < \infty$. Moreover, because of (3.12), the norm of K^ϵ can be made arbitrarily small by choosing ϵ small, so that $I - K^\epsilon : L^\alpha \to L^\alpha$ is invertible for all large $\alpha < \infty$. Along the same lines one has h_ϵ in every L^α with $1 < \alpha < \infty$. Thus we conclude from (3.7) that $u \in L^\alpha$ for all $1 \le \alpha < \infty$. For v the argument is similar. $\square$

Remark. Note that it is only at this stage that we know that both u and v satisfy the boundary conditions in the sense that $u, v \in H_0^1(\Omega)$.

Corollary 3.2. *Critical points of $\mathcal{L}$ are classical solutions of Problem (I), and in Theorem 2.4 "nonnegative" can be replaced by "strictly positive".*

Proof of Corollary 3.2. This is now standard. One has the L^p-estimates for the second derivatives due to Agmon, Douglis & Nirenberg [ADN], so that Sobolev embeddings and Schauder estimates [GT] finish the proof. $\square$

Remark. Combining Theorem 2.1 and Corollary 3.2 completes the proof of Theorem 1.

4. Solutions of Problem (III).

In this section we generalize Theorems 2.1 and 2.4 to the case of Problem (III). Thus we consider the Lagrangian

$$\mathcal{L}_*(\mathbf{u}) = \int_\Omega \nabla u \nabla v - \frac{\lambda}{2}\int_\Omega u^2 - \frac{\mu}{2}\int_\Omega v^2 - \int_\Omega F(u) - \int_\Omega G(v), \quad \mathbf{u} = (u, v). \tag{4.1}$$

The quadratic part can again be written as

$$A_*(\mathbf{u}) = \frac{1}{2}(L_*\mathbf{u}, \mathbf{u})_{E^r(\Omega)} = \int_\Omega \nabla u \nabla v - \frac{\lambda}{2}\int_\Omega u^2 - \frac{\mu}{2}\int_\Omega v^2, \tag{4.2}$$

where now

$$L_* = \begin{pmatrix} -\lambda(-\Delta)^{-r} & (-\Delta)^{1-r} \\ (-\Delta)^{r-1} & -\mu(-\Delta)^{r-2} \end{pmatrix} \tag{4.3}$$

is bounded and selfadjoint. Unlike L, L_* is not an isometry.

In order to determine the spectrum of L_*, we note that $E^r(\Omega)$ is the direct Hilbertspace sum of the spaces E_k, $k = 1, 2, \ldots$, where E_k is the two-dimensional subspace of $E^r(\Omega)$, spanned by $(\phi_k, 0)$ and $(0, \phi_k)$. An orthonormal basis of E_k is given by

$$\{\frac{1}{\sqrt{2}}(\lambda_k^{-\frac{r}{2}}\phi_k, 0), \frac{1}{\sqrt{2}}(0, \lambda_k^{\frac{r}{2}-1}\phi_k)\}. \tag{4.4}$$

Every E_k is invariant under L_*, and in E_k the restriction of L_* is given by the symmetric matrix

$$L^k = \begin{pmatrix} -\lambda\lambda_k^{-r} & 1 \\ 1 & -\mu\lambda_k^{r-2} \end{pmatrix}.$$

The eigenvalues of L^k are given by

$$\mu_k^{\pm} = -\frac{\lambda\lambda_k^{-r} + \mu\lambda_k^{r-2}}{2} \pm \sqrt{(\frac{\lambda\lambda_k^{-r} + \mu\lambda_k^{r-2}}{2})^2 + 1 - \frac{\lambda\mu}{\lambda_k^2}}, \tag{4.5}$$

with corresponding eigenvectors

$$\left(1, \frac{\lambda\lambda_k^{-r} - \mu\lambda_k^{r-2}}{2} \pm \sqrt{1 + (\frac{\lambda\lambda_k^{-r} - \mu\lambda_k^{r-2}}{2})^2}\right). \tag{4.6}$$

We have $\mu_k^- < 0 < \mu_k^+$ if $\lambda\mu < \lambda_k^2$. If $\lambda\mu > \lambda_k^2$ the signs of μ_k^+ and μ_k^- are the same: positive (negative) if λ and μ are negative (positive). If $\lambda\mu = \lambda_k^2$, then $\mu_k^+ = 0$ ($\mu_k^- = 0$) if λ and μ are positive (negative). Also note that

$$\mu_k^{\pm} \to \pm 1 \quad \text{as} \quad k \to \infty. \tag{4.7}$$

Let E^+ (E^-) be the subspace spanned by eigenvectors with positive (negative) eigenvalues, and E^0 the nullspace of L_*. Then

$$E^r(\Omega) = E^+ \oplus E^0 \oplus E^- = \{\mathbf{u} = \mathbf{u}^+ + \mathbf{u}^0 + \mathbf{u}^-,\ \mathbf{u}^\pm \in E^\pm,\ \mathbf{u}^0 \in E^0\}. \tag{4.8}$$

It follows that both E^+ and E^- are infinite dimensional, and that E^0 has finite dimension: $\lambda\mu \neq \lambda_k^2$ implies $\dim E^0 = 0$ while for $\lambda\mu = \lambda_k^2$ the dimension of E^0 is equal to the multiplicity of λ_k.

We introduce a new equivalent (inner product) norm $\|\cdot\|_*$ on $E^r(\Omega)$ by

$$(L_*\mathbf{u}^+, \mathbf{u}^+) - (L_*\mathbf{u}^-, \mathbf{u}^-) + \|\mathbf{u}^0\|^2_{L^2\times L^2} = \frac{1}{2}\|\mathbf{u}\|_*^2, \tag{4.9}$$

in analogy to (1.26). Note that the equivalence of (4.9) and $\|\cdot\|_{E^r}$ follows from (4.7) and the fact that E^0 is finite dimensional.

Theorem 4.1. *Let the assumptions (h1-4) be satisfied, let λ and μ be fixed reals, and let $0 < r < 2$ be such that (1.31) holds with strict inequalities. Then $\mathcal{L}_*$ has a non-trivial critical point $\mathbf{u}$ in $E^r(\Omega)$.*

Proof of Theorem 4.1. We adjust the proof of Theorem 2.1. If $\lambda\mu \neq \lambda_k^2$ for all k we have $\dim E^0 = 0$, and (4.9) reduces to

$$(L_*\mathbf{u}^+, \mathbf{u}^+) - (L_*\mathbf{u}^-, \mathbf{u}^-) = \frac{1}{2}\|\mathbf{u}\|_*^2. \tag{4.10}$$

We take for $\mathbf{e}^+$ an eigenvector in E^+, such that $\mathbf{e}^+$ belongs to some E_k with the other eigenvector $\mathbf{e}^-$ in E_k belonging to E^- ($\mathbf{e}^+$ and $\mathbf{e}^-$ normalized with respect to $\|\cdot\|_*$). Note that $\mathbf{e}^-$ is the only eigenvector of L_* not perpendicular to $\mathbf{e}^+$ in $L^2(\Omega)\times L^2(\Omega)$. Using $\|\cdot\|_*$ instead of $\|\cdot\|_{E^r}$, the proof is then identical to the proof of Theorem 2.1.

If $\lambda\mu = \lambda_k^2$ for some k, the proof is slightly more complicated. We replace again $\|\cdot\|_{E^r}$ by $\|\cdot\|_*$, choose $\mathbf{e}^\pm$ as above, and set $H_1 = E^+$, $H_2 = E^0 \oplus E^-$, and

$$Q = [0, s_1\mathbf{e}^+] \oplus \left(\overline{B}_{s_2} \cap H_2\right), \quad \tilde{H} = \operatorname{span}\,[\mathbf{e}^+] \oplus H_2, \quad S = \partial B_\rho \cap H_1. \tag{4.11}$$

Elements of $H_1 = E^+$ are denoted by $\mathbf{u}^+$, and elements of $H_2 = E^0 \oplus E^-$ by $\mathbf{u}^0 + \mathbf{u}^-$.

To verify the geometric conditions in Step 1, we have to estimate

$$\mathcal{L}_*(\mathbf{u}^0 + \mathbf{u}^- + s\mathbf{e}^+) = \frac{1}{2}s^2 - \frac{1}{2}\|\mathbf{u}^-\|_*^2 - \mathcal{H}(\mathbf{u}^0 + \mathbf{u}^- + s\mathbf{e}^+) \tag{4.12}$$

on the boundary of the cylinder Q. The lateral boundary however is no longer given by $\|\mathbf{u}^-\|_* = s_2$ but by $\|\mathbf{u}^0 + \mathbf{u}^-\|_* = s_2$. Thus if s_2 is large, the norm $\|\mathbf{u}^-\|_*$ can still be small on the lateral boundary, provided $\|\mathbf{u}^0\|_*$ is large. Therefore we now estimate $\mathcal{H}(\mathbf{u}^0 + \mathbf{u}^- + s\mathbf{e}^+)$ from below by changing (2.8) as follows

$$\begin{aligned}
\mathcal{H}(\mathbf{u}^0 + \mathbf{u}^- + s\mathbf{e}^+) &\geq B_3\Big(\int_\Omega |\mathbf{u}^- + \mathbf{u}^0 + s\mathbf{e}^+|^2\Big)^{\gamma/2} - B_4 \\
&\geq B_3\Big(\sin^2\chi \int_\Omega |\mathbf{u}^0 + s\mathbf{e}^+|^2\Big)^{\gamma/2} - B_4 \\
&= B_3 \sin^\gamma \chi\big(\|\mathbf{u}^0\|_*^2 + s^2\big)^{\frac{\gamma}{2}} - B_4 \\
&\geq B_5\|\mathbf{u}^0\|_*^2 + B_5 s^\gamma - B_6.
\end{aligned} \tag{4.13}$$

Here χ is the (positive) angle between E^- and $E^0 \oplus [\mathbf{e}^+]$ with respect to the inner product in $L^2(\Omega)\times L^2(\Omega)$. The analogon of (2.9) for $\mathcal{L}_*$ is

$$\begin{aligned}
\mathcal{L}_*(\mathbf{u}^0 + \mathbf{u}^- + s\mathbf{e}^+) &\leq \frac{1}{2}s^2 - B_5 s^\gamma + B_6 - B_7\big(\|\mathbf{u}^-\|_*^2 + \|\mathbf{u}^0\|_*^2\big) \\
&= \frac{1}{2}s^2 - B_5 s^\gamma + B_6 - B_7\|\mathbf{u}^- + \mathbf{u}^0\|_*^2.
\end{aligned} \tag{4.14}$$

The proof now proceeds along the same lines as before.

It remains to show that $\mathcal{L}_*$ satisfies the Palais-Smale condition. So let $\{\mathbf{u}_n\}$ in $E^r(\Omega)$ be a sequence with $\mathcal{L}_*(\mathbf{u}_n)$ bounded in $E^r(\Omega)$, and $\mathcal{L}'_*(\mathbf{u}_n) \to 0$. We have to do some extra work to show that such a sequence is bounded. Using the decomposition (4.8) estimates (2.13-14) for $\mathbf{u}^\pm$ remain the same. Thus we have, in stead of (2.15),

$$\|\mathbf{u}^\pm\|_* \leq C\Big\{1 + \{C + \epsilon\|\mathbf{u}\|_*\}^{\frac{p}{p+1}} + \{C + \epsilon\|\mathbf{u}\|_*\}^{\frac{q}{q+1}}\Big\}. \tag{4.15}$$

To controle the component $\mathbf{u}^0$ we modify (2.12) and derive

$$\begin{aligned} M + \epsilon\|\mathbf{u}\|_{E^r(\Omega)} &\geq \int_\Omega \big(\frac{1}{2}uf(u) - F(u) + \frac{1}{2}vg(v) - G(v)\big)dx \\ &\geq \big(\frac{\gamma}{2} - 1\big)\mathcal{H}(\mathbf{u}) \\ &\geq B_1 \int_\Omega |\mathbf{u}|^\gamma - B_2|\Omega| \\ &\geq B_3\Big(\int_\Omega |\mathbf{u}^- + \mathbf{u}^0 + \mathbf{u}^+|^2\Big)^{\gamma/2} - B_4 \\ &\geq B_3\Big(\sin^2\chi \int_\Omega |\mathbf{u}^0|^2\Big)^{\gamma/2} - B_4 \\ &= B_5\|\mathbf{u}^0\|_*^\gamma - B_4, \end{aligned} \tag{4.15}$$

where χ is the angle between E^0 and $E^+ \oplus E^-$ in $L^2(\Omega) \times L^2(\Omega)$. Combining (4.15) and (4.16) we obtain

$$\|\mathbf{u}\|_* \leq C\Big\{1 + \{C + \epsilon\|\mathbf{u}\|_*\}^{\frac{p}{p+1}} + \{C + \epsilon\|\mathbf{u}\|_*\}^{\frac{q}{q+1}} + \{C + \epsilon\|\mathbf{u}\|_*\}^{\frac{1}{\gamma}}\Big\}, \tag{4.17}$$

and as before this implies that the sequence $\mathbf{u}_n$ is bounded. This completes the proof of Theorem 4.1. ☐

Theorem 4.2. *Let the assumptions (h1-4) be satisfied, let λ and μ be fixed nonnegative reals with $\lambda\mu < \lambda_1^2$, let $0 < r < 2$ be such that (1.31) holds with strict inequalities, and assume that f and g are nonnegative on $[0, \infty)$. Then $\mathcal{L}(\mathbf{u})$ has a non-trivial critical point in $E^r(\Omega)$, with both components nonnegative.*

Proof of Theorem 4.2. We have to adjust the proof of Theorem 2.4, which was based on the maximum principle, and on the existence of an eigenvector $\mathbf{e}^+$ with positive components. For Problem (III) the maximum principle holds provided λ and μ are nonnegative with $\lambda\mu < \lambda_1^2$ [FMi]. With respect to $\mathbf{e}^+$ we note that in the proofs of Theorems 2.1 and 4.1 the choice of $\mathbf{e}^+$ in E^+ was rather arbitrary. In the proof of Theorems 2.4 however the positivity properties of $\mathbf{e}^\pm$ were crucial, so adjusting this proof to the case of $\mathcal{L}_*$ we need an eigenvector with positive components. Such an eigenvector can only exist in E_1, so we look at the eigenvalues and eigenvectors of L_*^1. Thus we must take $k = 1$ and a plus sign in (4.5-6). To make μ_1^+ positive it is necessary and sufficient to assume that $\lambda\mu < \lambda_1^2$. The corresponding eigenvector has positive components, so we can choose $\mathbf{e}^+$ as desired. Since λ and μ are nonnegative we can use (4.10) again instead of (4.9). The proof is then a trivial variant of the proof of Theorem 2.4. The details are left to the reader. ☐

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