SELFSIMILAR SOLUTIONS OF THE SECOND KIND FOR THE MODIFIED POROUS MEDIUM EQUATION

by

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Abstract

We construct compactly supported self-similar solutions of the Modified Porous Medium Equation (MPME)

$$u_t = \begin{cases} \Delta u^m & \text{for} \quad \Delta u^m > 0 \\ (1+\varepsilon)\Delta u^m & \text{for} \quad \Delta u^m < 0. \end{cases}$$

They have the form

$$u(x,t) = t^{-\alpha} U(x\, t^{-\beta}),$$

where the similarity exponents α and β depend on ε, m and the dimension N. This corresponds to what is known in the literature as anomalous exponents or self-similarity of the second kind, a not completely understood phenomenon. This paper performs a detailed study of the properties of the anomalous exponents of the MPME.

Keywords. Porous medium equation, self-similar solutions, anomalous exponents,

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1. Introduction.

This paper is concerned with the existence and properties of a family of special solutions of self-similar form for the Modified Porous Medium Equation (MPME)

$$u_t = \begin{cases} \Delta u^m & \text{for} \quad \Delta u^m > 0 \\ (1+\varepsilon)\Delta u^m & \text{for} \quad \Delta u^m < 0, \end{cases} \tag{1.1}$$

where $x \in \Re^N$ ($N \geq 1$) and $t > 0$ are the independent variables, and $m > 1$ and $\varepsilon \in (-1, \infty)$ are fixed parameters. This equation was proposed by G. I. Barenblatt and collaborators [B] as a model for the horizontal spreading of groundwater. The novelty is the idea that in the derivation of the model equations from Darcy's law and the conservation of mass principle, it is taken into account that the drying process is not just the inverse of the wetting process. After some simplifications this leads to an equation of the form (1.1). In fact they have $m = N = 2$, and $\varepsilon > 0$. See [B] and also [CGO] and [MvD] for references.

Equation (1.1) can be written as

$$F(u_t) = \Delta u^m, \tag{1.2.a}$$

where F is the nonlinear monotone function,

$$F(s) = s + (p-1)s^+, \qquad \text{with} \quad p = 1 + \varepsilon. \tag{1.2.b}$$

It is fully nonlinear, not in divergence form, and moreover, it degenerates at points where $u = 0$ because there the diffusivity, which is proportional to u^{m-1}, vanishes.

If we put $\varepsilon = 0$ then (1.1) is just the standard Porous Medium Equation (PME), whose main properties are by now well-known, [A]. In particular there exists a unique weak solution to the Cauchy problem, and this solution is continuous and bounded.

An important place in the study of the PME corresponds to a family of explicit solutions of self-similar form, the so-called Barenblatt solutions. They are

$$U(x,t) = t^{-\alpha} f(\xi), \quad \xi = x\, t^{-\beta}, \tag{1.3}$$

where the self-similarity exponents are $\alpha = N/(N(m-1)+2)$ and $\beta = \alpha/N$. The function f is given by

$$f(\xi; c) = (c^2 - d|\xi|^2)_+^{1/(m-1)}, \tag{1.4}$$

with $d = (m-1)\beta/(2m)$ and $c > 0$ a free constant, which can easily be put in correspondence with the mass of the solution, $c = c(M)$ with $M = \int u(x,t)\, dx$. The value of the similarity exponents is derived from two basic facts: (i) U is a solution of the PME, which implies that

$$\alpha(m-1) + 2\beta = 1, \tag{1.5}$$

and (ii) the mass $\int U(x,t)\,dx$ is a conserved quantity, which means that

$$\alpha = N\,\beta. \tag{1.6}$$

The importance of the Barenblatt solutions comes from the fact that they represent the asymptotic behaviour of all solutions to the Cauchy Problem with nonnegative and integrable initial data. In fact, any such solution converges as $t\to\infty$ towards a Barenblatt solution with the same mass. More precisely, we have (cf. [FK], [KV])

$$t^{\alpha}|u(x,t)-U(x,t;c)|\to 0 \quad \text{as} \quad t\to\infty\,, \tag{1.7}$$

where $c=c(M)$ and $M=\int u_0(x)\,dx$.

Our goal here is to extend this analysis to equation (1.1) with $\varepsilon\neq 0$. By means of the quadratic phase plane methods of [H] we construct in Section 2 a family of self-similar solutions $U(x,t;c,m,\varepsilon)$ which generalize the Barenblatt solutions. These solutions are again of the form (1.3), but now the similarity exponents cannot be obtained in such a simple way. Indeed, though relation (1.5) continues to hold, the conservation of mass fails. As a consequence we obtain so-called *anomalous exponents*, which are smooth functions of p, m and N. Notice that once a selfsimilar solution $U(x,t)$ of (1.1) of the form (1.3) is obtained, a whole family can be derived by means of the scaling

$$U(x,t;c)=c^{\frac{2}{m-1}}U(x/c,\,t)\,. \tag{1.8}$$

Consequently, we may forget the parameter c.

As a close precedent to the present work we should mention the studies on equation (1.1) with exponent $m=1$, the so-called Barenblatt elasto-plastic equation, cf. [BER]. Thus, in a recent paper, [KPV], the existence, uniqueness and properties of the self-similar solutions with anomalous exponents for such equation are given, as well as a proof of the asymptotic convergence of the solutions of the Cauchy Problem towards these self-similar solutions. The present paper carries out the first part of that programme for the case $m>1$. The second part will be dealt with in [HV].

Subsequently, in [AV] a technique was presented to study the dependence of the anomalous exponent on the parameters of the equation through the use of dynamical systems ideas and the Implicit Function Theorem. This was illustrated by two applications, one to the modified heat equation ($m=1$), and one to the so-called focusing problem for the porous medium equation in more dimensions. We apply this technique to the present case, thus obtaining the explicit value of the derivative of α with respect to ε. An investigation of these anomalous exponents using the formal techniques of Renormalization has recently been done in [CGO], but the authors point out that in order to do their calculations, they had to assume the existence of the selfsimilar compactly supported profiles, which now follows from this paper. Moreover, we recover the values of the derivative through a completely different and mathematically rigorous computation, namely

$$\frac{\partial\alpha}{\partial\varepsilon}(0)=\Big(\frac{N(m-1)}{(2+(m-1)N)}\Big)^{\frac{N}{2}}\,\Big(\frac{2}{(2+(m-1)N)}\Big)^{\frac{m}{m-1}}\;\frac{1}{B(\frac{1}{m-1},\frac{N}{2})}\,, \tag{1.9}$$

where B is the Beta-function. As a byproduct it follows that for all $\varepsilon > -1$

$$\frac{\partial \alpha}{\partial \varepsilon} > 0\,. \tag{1.10}$$

All this can be found in Section 3.

In principle it is also possible to compute the higher order derivatives, but the computations become more and more involved. In Section 4 we shall, with a little help of Maple5, compute the second derivative of the anomalous exponent when $m = N = 2$.

Let us also remark that there are a number of more or less related equations where anomalous exponents appear. A first result in that direction was proved in [BHV] for the dual porous medium equation, $u_t = |u_{xx}|^{m-1}u_{xx}$, and later also for the turbulent bursts equation [KV]. What all these equations have in common, is that the relevant similarity profile is an eigenfunction of a second order nonlinear degenerate ordinary differential equation, corresponding to an eigenvalue which is not a priori known from dimensional considerations.

Finally we note that the techniques in this paper work equally well for the cases $m = 1$ and $0 < m < 1$. And the ends of Sections 2 and 3 we shall state the corresponding results.

2. Compactly supported similarity solutions.

In this section we examine the existence of compactly supported radially symmetric nonnegative similarity solutions of equation (1.11), which we now write in the form

$$F(u_t) = \frac{1}{m}\Delta u^m\,, \tag{2.1}$$

where

$$F(s) = s + (p-1)s^+, \tag{2.2}$$

with $p > 0$ and $m > 1$. Recall that with the notation of (1.1) $p = 1 + \varepsilon$. For convenience we will write the similarity formula in the form

$$u(x,t) = t^{-\alpha} f(\beta^{1/2}|x|t^{-\beta})\,. \tag{2.3}$$

In order for (2.3) to be a solution of (2.1) we must have

$$2\beta + (m-1)\alpha = 1\,. \tag{2.4}$$

We introduce as parameter the ratio $k = \alpha/\beta$, so that

$$\alpha = \frac{k}{2+(m-1)k} \text{ and } \beta = \frac{1}{2+(m-1)k}\,. \tag{2.5}$$

The profile f has to satisfy the nonlinear ODE

$$\frac{1}{m} r^{1-N}(r^{N-1}(f^m)')' = F(-rf' - kf)\,. \tag{2.6}$$

where $r = \beta^{1/2}|x|t^{-\beta}$. We then have

Theorem 2.1. *For every $p > 0$ there exists a unique value $K(p)$ such that (2.1) has a compactly supported one-parameter family of radially symmetric and nonnegative similarity solutions of the form (2.3), parametrized by $f(0) > 0$, if and only if $k = K(p)$. Moreover, $K(p)$ is monotone increasing and smooth in p, with $K(0) = \max((N-2)/m, 0)$, $K(1) = N$ and $K(+\infty) = +\infty$.*

Proof. We want a solution f of (2.6) which has $f(0) > 0$, $f'(0) = 0$, and which decreases to zero as r runs from zero to $r = R$, with $f(R) = (f^m)'(R) = 0$. Because of scaling arguments, this only depends on the parameters, and not on the value of $f(0)$. Following [H] we set

$$X = \frac{rf'}{f}\,; \quad Y = r^2 f^{1-m}\,; \quad t = \log r\,, \tag{2.7}$$

which transforms (2.6) into

$$\begin{cases} dX/dt = X(2-N-mX) + F(-X-k)Y \\ dY/dt = Y(2+(1-m)X). \end{cases} \tag{2.8}$$

Note that (2.8) is a piecewise quadratic system. If $p \neq 1$, the equation for dX/dt depends on the sign of $X + k$. Nevertheless, it is locally Lipschitz continous in X, Y and k.

System (2.8) was studied extensively in [H] in the case that $p = 1$, when it comes from the standard porous medium equation. Then the Barenblatt solution ($k = N$) corresponds to the straight line $X + Y = 0$, which is an orbit of (2.8) coming out of the origin. The statement in the following lemma is easy to verify, see also [H].

Lemma 2.2. *For any k and p, every radial solution of (2.6) with $f(0) > 0$ is mapped by (2.7) into the unique orbit γ_r coming out of the critical point $(0, 0)$ along the eigenvector $(-k, N)$ corresponding to the eigenvalue 2 of the linearization of (2.8).*

Next we want to investigate whether the radial solutions of (2.6) have compact support. For $p = 1$ this was done in [H] by means of suitable polar coordinates which compactify the plane: every solution f which decreases to 0 as $r \uparrow R$ with $(f^m)'(R) = 0$ is contained in a unique orbit γ_i escaping to infinity in finite t-time parallel to the line $X + Y = 0$. For general $p > 0$ this statement is easily adapted as

Lemma 2.3. *For any $k > 0$ and $p > 0$, every radial solution of (2.6) which decreases to 0 as $r \uparrow R$ with $(f^m)'(R) = 0$ is mapped by (2.7) into the unique orbit γ_i going to infinity parallel to the line $X + pY = 0$.*

Thus the problem of finding the values of $k > 0$ and $p > 0$ for which (2.1) has a compactly supported radially symmetric nonnegative similarity solution reduces to finding the values of $k > 0$ and $p > 0$ for which $\gamma_r = \gamma_i$. For $p = 1$ this happens for $k = N$, which yields the Barenblatt solution.

We note here that in forward t-time γ_r has the choice out of three possibilities. If it is not as in Lemma 2.3 then either it goes to infinity with $X/Y \to -\infty$, or it goes to infinity along $X = -k$ (see [H] again for details). In particular γ_r always crosses the line $X = -k$.

It is convenient to consider besides (2.8) the standard quadratic system corresponding to $p = 1$, which simply reads

$$(2.9) \qquad \begin{cases} dX/dt = X(2 - N - mX) - (X + k)Y \\ dY/dt = Y(2 + (1 - m)X). \end{cases}$$

We do not rename the orbits γ_r and γ_i obtained from Lemma 2.2 and Lemma 2.3 by setting $p = 1$. Now any orbit of (2.8) lying in the region where $X + k < 0$ yields an orbit of (2.9) by setting $Y^* = pY$. The following lemma is an immediate consequence of this simple observation.

Lemma 2.4. *Let $k > 0$ be fixed. Suppose the orbits γ_r and γ_i of (2.9) intersect the line $X + k = 0$ in $Y = y_r$ and $Y = y_i$ respectively. Then for $p = y_i/y_r$ the orbits γ_r and γ_i of (2.8) coincide.*

Thus, we find p from the intersection heights $Y = y_r$ and $Y = y_i$. All our proofs will now be based on a careful study of y_r and y_i, and in particular of their dependence on k.

Lemma 2.5. *Let $k > 0$. The orbits γ_r and γ_i of (2.9) both intersect the line $X + k = 0$ if and only if $k > (N - 2)/m$.*

Proof. Immediate from inspection of the phase plane of (2.9). Note that for $N > 2$ and $k = (N-2)/m$ the line $X = -k = (2-N)/m$ is an orbit, while for $N > 2$ and $k < (N-2)/m$ the orbit γ_r is to the right of $X = -k$. $\square$

Lemma 2.6. *Let $k > 0$ and $p > 0$. There exists a strictly increasing function*

$$P : (\max(0, \frac{N-2}{m}), \infty) \to (0, \infty)$$

such that the orbits γ_r and γ_i of (2.9) coincide if and only if $p = P(k)$.

Proof. The existence of the function $P(k)$ follows from Lemma 2.4 and Lemma 2.5. To see that it is strictly increasing we observe that in both (2.8) and (2.9), $-dX/dt$ is increasing in k, while dY/dt is independent of k. Since the orbits γ_r and γ_i in Lemma 2.2 and Lemma 2.3 are unique, this implies that with increasing k, γ_r moves downwards and γ_i upwards. As a consequence we have that $P(k)$ is strictly increasing. This is not completely obvious, because the line $X + k$ also moves. (This flaw in the original proof was pointed out by Lucas du Croo de Jongh, a student at the University of Leiden.) In Section 3 we shall give the details of this monotonicity argument. $\square$

Lemma 2.7. *$P(k) \uparrow \infty$ if $k \uparrow \infty$.*

Proof. Consider the orbit γ_r of (2.9). As k increases it moves downwards. From [H] we know that for $k > N$ it goes to infinity with $X/Y \to -\infty$. But then using the monotonicity again, it follows, in the notation of Lemma 2.4, that $y_r/k \to 0$ as $k \to \infty$. On the other hand, γ_i moves upwards, so it lies above the line $X + Y = 0$. Hence y_i as defined in Lemma 2.4 satisfies $y_i > k$. Combining we have $p = y_i/y_r \to \infty$ as $k \to \infty$. $\square$

Lemma 2.8. *Let $N \leq 2$. Then $P(k) \downarrow 0$ if $k \downarrow 0$.*

Proof. Consider the orbit γ_i of (2.9) for $0 < k < N$. By similar arguments as before we have that it is below the line $X + Y = 0$ and that it moves downwards as k decreases. Then, regardless of the value of $N = 1, 2$, the orbit γ_i must come out of the critical point $(0, 0)$ along the negative X-axis. Using monotonicity again this implies that $y_i/k \to 0$ as $k \downarrow 0$. On the other hand γ_r is above the line $X + Y = 0$ for $0 < k < N$, so that y_r/k is bounded away from zero. Hence $y_i/y_r \to 0$ as $k \downarrow 0$. $\square$

Lemma 2.9. *Let $N > 2$. Then $P(k) \downarrow 0$ if $k \downarrow (N-2)/m$.*

Proof. Here the situation is a little different because of the critical point $((2-N)/m, 0)$, which lies on the negative X-axis. The origin is now a saddle, so that for $(N-2)/m < k < N$, the orbit γ_i cannot have come out of it. Since it is below $X + Y = 0$, it can only have come out of $((2-N)/m, 0)$. But then we find that $y_i \to 0$ as $k \downarrow (N-2)/m$. On the other hand γ_r is still above the line $X + Y = 0$, so that $y_r > k$. Hence $y_i/y_r \to 0$ as $k \downarrow (N-2)/m$. $\square$

The proof of Theorem 2.1 is now completed by letting K be the inverse function of the function P. $\square$

Remark 2.10. *Similarly one proves that for $N = 1$ there exists a family of similarity solutions of dipole type if and only if $k = D(p)$, where $D : (0, \infty) \to (0, \infty)$ is a smooth monotone increasing surjective function with $D(1) = 2$.*

We conclude this section with the cases $m = 1$ and $0 < m < 1$. For $m = 1$ the words "compactly supported" in Theorem 2.1 should be replaced by the words "exponentially decaying". This was proved by different methods in [KPV]. For $0 < m < 1$ the situation is a little more complicated. The special profiles are now the ones with so-called fast decay, which means a powerlike decay, the power being $-2/(1-m)$, as opposed to slow decay, when the power is $-k$, see also [BHV]. Moreover the corresponding Barenblatt exists only if $m > 1 - 2/N$.

For this case the proofs above are easily adapted. Essentially the only difference is that the role of γ_i is taken over by an orbit γ_f, which goes into the saddle point

$$(X, Y) = \left(-\frac{2}{1-m}, \frac{2}{1-m}\frac{2 - N(1-m)}{2 - k(1-m)}\right),$$

from below.

Theorem 2.11. *Let $0 < m < 1$ and $m > 1 - 2/N$. For every $p > 0$ there exists a unique value $K(p)$ such that (2.1) has a one-parameter family of radially symmetric and nonnegative similarity solutions with fast decay of the form (2.3), parametrized by $f(0) > 0$, if and only if $k = K(p)$. Moreover, $K(p)$ is monotone increasing and smooth in p, with $K(0) = \max((N-2)/m, 0)$, $K(1) = N$ and $K(+\infty) = 2/(1-m)$.*

3. Application of the Implicit Function Theorem: Proof of (1.9).

In this section we compute the derivatives of the anomalous exponents with respect to p at the value $p = 1$ when $m > 1$. As a byproduct we obtain a proof of the (strict) monotonicity of the function $P(k)$ in Lemma 2.6.

We consider again

$$(3.1) \qquad \begin{cases} dX/dt = X(2 - N - mX) - (X + k)Y \\ dY/dt = Y(2 + (1 - m)X), \end{cases}$$

and the intersections of the orbits γ_r and γ_i and the line $X + k = 0$ in $Y = y_r$ and $Y = y_i$ respectively. Note that both y_r and y_i are functions of the parameters m, N, and in particular of k. From here one we shall drop the m- and N-dependence from our notation.

Let us first observe that in suitable coordinats, namely

$$(3.2) \qquad X = \frac{\rho}{1-\rho}\cos\phi, \quad Y = \frac{\rho}{1-\rho}\sin\phi,$$

both γ_r and γ_i are orbits coming out of or going into a saddle point, and as such, they depend smoothly (in fact analytically) on the parameters. In fact [H], (3.2) transforms orbits of (3.1) into orbits of the system

$$(3.2) \qquad \begin{cases} \dot\rho = \rho(1-\rho)^2\{2 - \cos\phi(N\cos\phi + k\sin\phi)\} \\ \qquad\qquad - \rho^2(1-\rho)\cos\phi\{m - 1 + \cos\phi(\cos\phi + \sin\phi)\} \\ \dot\phi = \rho\sin\phi\cos\phi(\cos\phi + \sin\phi) + (1-\rho)\sin\phi(N\cos\phi + k\sin\phi). \end{cases}$$

The saddle points of (3.2) corresponding to γ_r and γ_i are the critical points $(\rho, \phi) = (0, \operatorname{arccot}(-k/N))$ and $(\rho, \phi) = (1, 3\pi/4)$.

We also know, by inspection of the phase plane of (3.1), that both γ_r and γ_i have monotone coordinats $X(t)$ and $Y(t)$ in the regions $\{X > -k\}$ and $\{X < -k\}$ respectively. This allows us to consider, in stead of (3.1), the first order equation

$$(3.3) \qquad \frac{dx}{dy} = H(x, y; k) = \frac{x(2 - N - mx) - (x + k)y}{y(2 + (1 - m)x)},$$

and to describe γ_r and γ_i as solutions

$$(3.4) \qquad x = X_r(y; k) \text{ and } x = X_i(y; k)$$

of (3.3). The functions X_r and X_i inherit the regularity properties of γ_r and γ_i. Therefore we can obtain differential equations for

$$(3.5) \qquad z_r(y; k) = \frac{\partial X_r}{\partial k}(y; k) \text{ and } z_i(y; k) = \frac{\partial X_i}{\partial k}(y; k),$$

by differentiating (3.3) with respect to k. For both z_r and z_i this yields the equation

$$(3.6) \qquad \frac{dz}{dy} - \frac{\partial H}{\partial x} z = \frac{\partial H}{\partial k}.$$

Here we have omitted the subscripts r and i. In both derivatives of H, (3.4) is tacitly assumed to be substituted.

For $k = N$, we have that γ_r and γ_i coincide, and are given by $x + y = 0$. We find

$$\frac{\partial H}{\partial x}(-y, y; N) = \frac{2-N}{2}\frac{1}{y} + (\frac{1}{m-1} + \frac{N}{2})\frac{m-1}{2+(m-1)y}\,, \tag{3.7}$$

and

$$\frac{\partial H}{\partial k}(-y, y; N) = -\frac{1}{2+(m-1)y}\,. \tag{3.8}$$

Hence we can write (3.6) as

$$\left(y^{\frac{N-2}{2}}\,(2+(m-1)y)^{-\frac{1}{m-1}-\frac{N}{2}}\,z(y)\right)' + y^{\frac{N-2}{2}}\,(2+(m-1)y)^{-\frac{m}{m-1}-\frac{N}{2}} = 0\,, \tag{3.9}$$

so that

$$z(y) = y^{\frac{2-N}{2}}\,(2+(m-1)y)^{\frac{1}{m-1}+\frac{N}{2}}\,\Big(C - \int_0^y \eta^{\frac{N-2}{2}}\,(2+(m-1)\eta)^{-\frac{m}{m-1}-\frac{N}{2}}d\eta\Big)\,. \tag{3.10}$$

For z_r and z_i only the constant C is different.

We first compute z_r. It follows from $X_r(0; k) = 0$ that $z_r(0; k) = 0$. If $N \geq 2$ this implies immediately that $C = 0$. For $N = 1$ we observe that the smoothness properties of $X_r(y; k)$ with respect to k, force $C = 0$. Thus

$$z_r(y; N) = -y^{\frac{2-N}{2}}\,(2+(m-1)y)^{\frac{1}{m-1}+\frac{N}{2}}\int_0^y \eta^{\frac{N-2}{2}}\,(2+(m-1)\eta)^{-\frac{m}{m-1}-\frac{N}{2}}d\eta\,. \tag{3.11}$$

For z_i we rewrite (3.10) as

$$z(y) = y^{\frac{2-N}{2}}\,(2+(m-1)y)^{\frac{1}{m-1}+\frac{N}{2}}\,\Big(C + \int_y^\infty \eta^{\frac{N-2}{2}}\,(2+(m-1)\eta)^{-\frac{m}{m-1}-\frac{N}{2}}d\eta\Big)\,. \tag{3.12}$$

We claim that $z_i(y; k)$ is sublinear as $y \to \infty$, which we shall show when we have finished these computations. As a consequence of this claim we have

$$z_i(y; N) = y^{\frac{2-N}{2}}\,(2+(m-1)y)^{\frac{1}{m-1}+\frac{N}{2}}\int_y^\infty \eta^{\frac{N-2}{2}}\,(2+(m-1)\eta)^{-\frac{m}{m-1}-\frac{N}{2}}d\eta\,. \tag{3.13}$$

Formulas (3.11) and (3.13) allow us to write down the derivatives of $y_r(k)$ and $y_i(k)$ in Lemma 2.4 with respect to k in $k = N$. We have, omitting the subscripts r and i again,

$$X(y(k); k) + k = 0\,, \tag{3.14}$$

whence

$$H(-k, y(k); k)y'(k) + z(y(k); k) + 1 = 0\,. \tag{3.15}$$

For $k=N$ and $x+y=0$ we have $H(x,y;k)=-1$ and $y(N)=-N$, so

$$y'(N)=z(N;N)+1\,. \tag{3.16}$$

Since $P(k)=y_i(k)/y_r(k)$ we obtain, also using $y_r(N)=y_i(N)=N$, that

$$P'(N)=\frac{y_i'(N)}{y_r(N)}-\frac{y_r'(N)y_i(N)}{y_r(N)^2}=$$

$$N^{-\frac{N}{2}}\,(2+(m-1)N)^{\frac{1}{m-1}+\frac{N}{2}}\,\int_0^\infty\,\eta^{\frac{N-2}{2}}\,(2+(m-1)\eta)^{-\frac{m}{m-1}-\frac{N}{2}}d\eta$$

$$=N^{-\frac{N}{2}}\,(2+(m-1)N)^{\frac{1}{m-1}+\frac{N}{2}}\,2^{-\frac{m}{m-1}}\,(m-1)^{-\frac{N}{2}}\,B(\frac{m}{m-1},\frac{N}{2})\,, \tag{3.17}$$

where $B(x,y)$ is the Beta-function:

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt=\int_0^\infty\frac{t^{x-1}}{(1+t)^{x+y}}dt\,. \tag{3.18}$$

Consequently

$$\frac{\partial k}{\partial\epsilon}(0)=\frac{\partial k}{\partial p}(1)=\frac{N^{\frac{N}{2}}\,2^{\frac{m}{m-1}}\,(m-1)^{\frac{N}{2}}}{(2+(m-1)N)^{\frac{1}{m-1}+\frac{N}{2}}\,B(\frac{m}{m-1},\frac{N}{2})}\,, \tag{3.19}$$

and

$$\frac{\partial\alpha}{\partial\epsilon}(0)=\frac{\partial\alpha}{\partial p}(1)=\frac{N^{\frac{N}{2}}\,2^{\frac{m}{m-1}+1}\,(m-1)^{\frac{N}{2}}}{(2+(m-1)N)^{\frac{m}{m-1}+1+\frac{N}{2}}\,B(\frac{m}{m-1},\frac{N}{2})}$$

$$=\Big(\frac{N(m-1)}{(2+(m-1)N)}\Big)^{\frac{N}{2}}\,\Big(\frac{2}{(2+(m-1)N)}\Big)^{\frac{m}{m-1}}\,\frac{1}{B(\frac{1}{m-1},\frac{N}{2})}\,. \tag{3.20}$$

For $m=N=2$, when the equation models the spreading of groundwater, we recover the value 1/8 in [CGO,3.29], while for the thermal waves propagation model, $N=3$ and $m=6$, we obtain the value

$$\frac{42}{4913}\frac{2^{1/5}17^{3/10}5^{1/2}3^{1/2}\Gamma(7/10)}{\pi^{1/2}\Gamma(1/5)}=0.01419432003\ldots\,.$$

It remains to show that the function $z_i(y;k)$ is sublinear as $y\to\infty$. Indeed we have that the orbit γ_i is decribed in terms of the polar coordinats (3.2) as a smooth function

$$\phi=\Phi(\rho;k)\,. \tag{3.21}$$

Thus we also have

$$x = \frac{\rho}{1-\rho} \cos \Phi(\rho; k) , \quad y = \frac{\rho}{1-\rho} \sin \Phi(\rho; k) . \tag{3.22}$$

Omitting the subscript i it follows that

$$x = X(y; k) = X(\frac{\rho}{1-\rho} \sin \Phi(\rho; k), k) . \tag{3.23}$$

Differentiating with respect to k we obtain

$$x_k = X_y y_k + X_k , \tag{3.24}$$

so that

$$-\frac{\rho}{1-\rho} \sin \Phi \Phi_k = X_y \frac{\rho}{1-\rho} \cos \Phi \Phi_k + X_k , \tag{3.25}$$

whence

$$\frac{z_i(y; k)}{y} = \frac{X_k}{y} = -(X_y \mathrm{cotan}\Phi + 1)\Phi_k . \tag{3.26}$$

Since $\Phi_k(1, k) = 0$ the sublinearity of $z_i(y; k)$ follows.

Let us make some comments about the arguments above. First of all, the computations we did gave an explicit formula because we started from an explicit solution. The implicit function theorem argument itself however, applies equally well when the solution is not explicit, which is the case for all $\varepsilon \neq 0$. In particular we still have the differential equation (3.6) for z_r and z_i, which we rewrite as

$$\frac{dz}{dy} + a(y; k)z = b(y; k) , \tag{3.27}$$

where

$$a(y; k) = -\frac{\partial H}{\partial x}(X(y; k), y; k) \tag{3.28}$$

and

$$b(y; k) = \frac{\partial H}{\partial k}(X(y; k), y; k) = -\frac{1}{2 + (1-m)X(y; k)} < 0 . \tag{3.29}$$

Here we have omitted again the subscripts r and i. Letting

$$A(y; k) = \int a(y; k)dy , \tag{3.30}$$

we find

$$z_r(y;k) = -e^{-A(y;k)} \int_0^y e^{A(\eta;k)} b(\eta;k)d\eta < 0\,, \tag{3.31}$$

and

$$z_i(y;k) = e^{-A(y;k)} \int_y^\infty e^{A(\eta;k)} b(\eta;k)d\eta > 0\,. \tag{3.32}$$

Since the coefficient of $y'(k)$ in (3.15) equals

$$\frac{-(2-N+mk)}{y(k)(2+(m-1)k)}\,,$$

we can evaluate (3.15) as

$$y'(k) = \frac{2+(m-1)k}{2-N+mk}(z(y(k);k)+1)y(k)\,, \tag{3.33}$$

so that the derivative of the function $P(k)$ in Lemma 2.6 is given by
(3.34)

$$P'(k) = \frac{y_i'(k)y_r(k) - y_r'(k)y_i(k)}{y_r(k)^2} = \frac{2+(m-1)k}{2-N+mk}(z_i(y_i(k);k) - z_r(y_r(k);k))P(k) > 0\,.$$

This implies that always

$$\frac{\partial\alpha}{\partial\varepsilon} > 0\,. \tag{3.35}$$

A second observation concerns the case $0 < m < 1$. The computation of z_r remains exactly the same. The other orbit, which we renamed γ_f at the end of Section 2, is now described as $x = X_f(y;k)$. The formula for $z_f(y;k) = (\partial X_f/\partial k)(y;k)$ at $k = N$ is the same as formula (3.13) above for $z_i(y, N)$, except that the upper bound $\eta = \infty$ has to be replaced by $\eta = \frac{2}{1-m}$. This results in the following formula for $P'(N)$:

$$\begin{aligned} P'(N) &= N^{-\frac{N}{2}}\,(2-(1-m)N)^{\frac{1}{m-1}+\frac{N}{2}} \int_0^{\frac{2}{1-m}} \eta^{\frac{N-2}{2}}\,(2-(1-m)\eta)^{-\frac{m}{m-1}-\frac{N}{2}}d\eta \\ &= N^{-\frac{N}{2}}\,(2-(1-m)N)^{\frac{1}{m-1}+\frac{N}{2}}\,2^{-\frac{m}{m-1}}\,(1-m)^{-\frac{N}{2}}\,B(\frac{1}{1-m}-\frac{N}{2},\frac{N}{2})\,. \end{aligned} \tag{3.36}$$

Finally we mention that for $m = 1$ the computation yields

$$P'(N) = \frac{1}{2}(\frac{2e}{N})^{\frac{N}{2}} \int_0^\infty t^{-1+\frac{N}{2}} e^{-t} dt = \frac{1}{2}(\frac{2e}{N})^{\frac{N}{2}}\Gamma(\frac{N}{2})\,,\ \alpha'(0) = \frac{(\frac{N}{2e})^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}\,, \tag{3.37}$$

where Γ is the Gamma-function. This agrees with the result in [AV].

4. Second order derivatives.

In this section we compute the second derivative of the anamolous exponent with respect to ε at $\varepsilon = 0$ in some special cases. To do so let

$$w_r(y;k) = \frac{\partial z_r}{\partial k}(y;k) \text{ and } w_i(y;k) = \frac{\partial z_i}{\partial k}(y;k)\,. \tag{4.1}$$

By differentiating (3.6) with respect to k we obtain for both w_r and w_i the equation

$$\frac{dw}{dy} - \frac{\partial H}{\partial x} w = \frac{\partial^2 H}{\partial x^2} z^2 + 2\frac{\partial^2 H}{\partial x \partial k} z + \frac{\partial^2 H}{\partial k^2}\,. \tag{4.2}$$

Here we have omitted once again the subscripts r and i.

Evaluating the partial derivatives of H at $x = -y$ and $k = N$ we find for the second order derivatives

$$\frac{\partial^2 H}{\partial x^2} = -2\frac{2+(m-1)N}{y(2+(m-1)y)^2}\,,\ \frac{\partial^2 H}{\partial k \partial x} = -\frac{m-1}{y(2+(m-1)y)^2}\,,\ \frac{\partial^2 H}{\partial k^2} = 0\,. \tag{4.3}$$

Thus it depends on the explicit form of $z = z_r$ and $z = z_i$ whether or not we can solve for w explicitly. The best possible case is when all the exponents in (3.12) and (3.13) are integers.

Before actually carrying out the computation, let us first show how we can find the values of $P''(N)$ and $\alpha''(0)$ from $w_r(N;N)$ and $w_i(N;N)$. First we differentiate (3.15) once more with respect to k. This yields

$$-Hy'' = \frac{\partial H}{\partial x} y'(z-1) + 2\frac{\partial H}{\partial k} y' + \frac{\partial H}{\partial y} y'^2 + w\,. \tag{4.4}$$

From (4.4) we obtain both $y_r''(N)$ and $y_i''(N)$, so, because $y_r(N) = y_i(N) = N$,

$$P''(N) = \frac{y_i''(N)}{N} - 2\frac{y_i'(N)y_r'(N)}{N^2} + 2\frac{y_r'(N)^2}{N^2} - \frac{y_r''(N)}{N}\,. \tag{4.5}$$

Finally

$$\frac{\partial^2 k}{\partial \epsilon^2}(0) = \frac{\partial^2 k}{\partial p^2}(1) = -\frac{P''(N)}{P'(N)^3}\,, \tag{4.6}$$

and

$$\frac{\partial^2 \alpha}{\partial \epsilon^2}(0) = \frac{\partial^2 \alpha}{\partial p^2}(1) = \frac{2}{(2+(m-1)N)^2}\frac{\partial^2 k}{\partial \epsilon^2}(0) - \frac{4(m-1)}{(2+(m-1)N)^3}(\frac{\partial k}{\partial \epsilon}(0))^2\,. \tag{4.7}$$

Next we evaluate the formulas above for the case $m = N = 2$. We then find for the functions z_r and z_i that

$$z_r(y;2) = -(y+2)^2 \int_0^y \frac{1}{(\eta+2)^3} d\eta = -\frac{1}{8} y(4+y)\,, \tag{4.8}$$

while

$$z_i(y;2) = (y+2)^2 \int_y^\infty \frac{1}{(\eta+2)^3} d\eta = \frac{1}{2}\,. \tag{4.9}$$

The computation of w_r and w_i from (4.2) is similar and yields

$$w_r(y;2) = -(y+2)^2 \int_0^y \frac{1}{8} \frac{\eta(\eta+4)}{(\eta+2)^3} d\eta = \frac{1}{16} y(y+4) + \frac{1}{8}(y+2)^2 \log \frac{2}{y+2}\,, \tag{4.10}$$

while

$$w_i(y;2) = (y+2)^2 \int_y^\infty \frac{1}{\eta(\eta+2)^3} d\eta = \frac{1}{8}(y+2)^2 \log \frac{2+y}{y} - \frac{1}{4}(y+3)\,. \tag{4.11}$$

Substituting $y=2$ we find

$$z_r(2;2) = -\frac{3}{2}\,, \quad z_i(2;2) = \frac{1}{2}\,, \quad w_r(2;2) = \frac{3}{4} - 2\log 2\,, \quad w_i(2;2) = 2\log 2 - \frac{5}{4}\,, \tag{4.12}$$

and hence, evaluating (3.16-17) and (4.4-7),

$$P'(2) = 1\,, \quad \frac{\partial \alpha}{\partial \varepsilon}(0) = \frac{1}{8}\,, \quad y_r'(2) = \frac{7}{4} - 2\log 2\,, \quad y_r'(2) = 2\log 2 - \frac{5}{4}\,. \tag{4.13}$$

so that

$$P''(2) = 2\log 2 - 1 = -\frac{\partial^2 k}{\partial \varepsilon^2}(0)\,, \quad \frac{\partial^2 \alpha}{\partial \varepsilon^2}(0) = \frac{1}{16} - \frac{1}{4}\log 2 = -0.11078667952\,. \tag{4.14}$$

Conclusion. We have established the existence of the compactly supported similarity solutions to the modified porous medium equation, and we have shown that these solutions are unique upto scaling. All this was done using a reduction to a quadratic two-dimensional dynamical system. By means of the Implicit Function Theorem and dynamical systems methods, we have proved the formal results concerning the behaviour of the anomalous exponents in the nonanomalous limit. In a forthcoming paper [HV] we shall prove that the Cauchy problem is well posed, and that the selfsimilar solution obtained in this paper describe the intermediate asymptotics. In another forthcoming paper [H2], the ideas in this paper will be applied to the $k-\varepsilon$ model from turbulence.

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