Recent results on selfsimilar solutions of degenerate nonlinear diffusion equations

by

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1. Introduction.

In this survey we collect some of the recent results on the porous medium equation,

$$u_t = \Delta(u^m) = \Delta(|u|^{m-1}u), \tag{1.1}$$

and related problems. Here as usual $u = u(x,t)$ with $x \in \Re^N$ and $t > 0$, and m is a fixed positive number. It is also possible to consider $m \le 0$, but then (1.1) has to be rewritten as, after a scaling of t,

$$u_t = \nabla \cdot (|u|^{m-1}\nabla u). \tag{1.2}$$

As far as the physical background of this equation is concerned, let us just mention that originally it was proposed as a model for gas flow through a homogeneous porous medium. In this model u is the density, and by the equation of state, which relates the pressure and the density, the pressure is given by u^{m-1}. The other two ingredients are the conservation of mass principle and Darcy's law, which postulates that the flux is proportional to (minus) the gradient of the pressure, see e.g. [1].

A first basic observation is that for $m \neq 1$, given any two positive numbers λ and μ, the scaling

$$u_{\lambda\mu}(x,t) = \lambda^{\frac{-2}{m-1}}\mu^{\frac{1}{m-1}}u(\lambda x, \mu t) \tag{1.3}$$

leaves (1.1) invariant. Thus we have a two-parameter scaling group, from which we can extract suitable one-parameter scaling groups, e.g. by setting

$$\lambda = \mu^\beta, \tag{1.4}$$

where β is any fixed number. Any solution which is invariant under this particular one-parameter scaling group, satisfies

$$u(x,t) = \mu^\alpha u(\mu^\beta x, \mu t), \quad (m-1)\alpha + 2\beta = 1 \tag{1.5}$$

for all x, t and μ. Substituting $\mu t = 1$, we find

$$u(x,t) = t^{-\alpha}u(xt^{-\beta}, 1) = t^{-\alpha}U(\eta), \quad \eta = xt^{-\beta}. \tag{1.6}$$

For $U(\eta)$ the equation reduces to

$$\Delta(|U|^{m-1}U) + \beta\eta\cdot\nabla U + \alpha U = 0, \tag{1.7}$$

or, for solutions which are also radially invariant,

$$(|U|^{m-1}U)'' + \frac{N-1}{r}(|U|^{m-1}U)' + \beta r U' + \alpha U = 0, \quad r = |\eta|. \tag{1.8}$$

A second basic observation is that integrable (in space) solutions satisfy, after an integration by parts,

$$\frac{d}{dt}\int u(x,t)dx = 0, \tag{1.9}$$

which is a conservation law. For (1.6) this implies, together with (1.5),

$$\alpha = N\beta = \frac{N}{N(m-1)+2}, \tag{1.10}$$

in which case there exists an explicit solution, the Barenblatt-Pattle instantaneous point source solution,

$$u(x,t) = t^{-\frac{N}{N(m-1)+2}}U(\eta), \quad \eta = xt^{-\frac{1}{N(m-1)+2}}, \quad U(\eta) = (C - \frac{m-1}{2m(N(m-1)+2)}|\eta|^2)_+^{\frac{1}{m-1}}, \tag{1.11}$$

which converges to a multiple of the Dirac measure $\delta(x)$ as $t \downarrow 0$. Here and throughout this paper we restrict ourselves to $m > 1$. This solution, which was first published by Zel'dovich and Kompanyeets [30], initiated the development of an extensive theory for equation (1.1) and its generalizations. Without going into to any detail, let us recall that [1]: the Cauchy problem for nonnegative integrable initial data is wellposed; the intermediate asymptotics of (weak) solutions are given by (1.11) with C determined by $\int u(x,0)dx$; compactly supported solutions have supports with a free boundary moving outwards with finite speed.

In this survey we shall discuss a number of related problems, and investigate to what extent the results above remain valid. All these problems will have in common with (1.1) that they allow a scaling similar to (1.3). Consequently similarity solutions will play an important role. The emphasis will be on those similarity solutions, which are fundamental for the theory of the partial differential equation.

2. Compactly supported similarity solutions.

The instanteneous point source solution (1.11) can be thought of as a fundamental solution for (1.1), in much the same way as the heat kernel

$$u(x,t) = E(x,t) = \frac{1}{(4\pi t)^{N/2}}e^{-\frac{|x|^2}{4t}} \tag{2.1}$$

is the fundamental solution for the heat equation ($m = 1$). Note that (2.1) is also selfsimilar, and that its decay as $|x| \to \infty$ is exponential, and that any derivative of (2.1) is again a selfsimilar solution, having the same exponential decay. For $m > 1$ the role of solutions with exponential decay is taken over by solutions with compact support, so it was natural to ask whether compactly supported selfsimilar solutions are as abundant as exponential decay solutions for the heat equation. For the one-dimensional case ($N = 1$), a complete classification was given in [17], the result being

THEOREM 2.1. [17] *Let $m > 1$, $N = 1$, and let $(m-1)\alpha + 2\beta = 1$. Then there exists a strictly increasing sequence*

$$\alpha_1 = \frac{1}{m+1} < \alpha_2 = \frac{1}{m} < \alpha_3 < \alpha_4 < \ldots \uparrow \frac{1}{m-1}, \tag{2.2}$$

such that (1.7) *has (a one-parameter family of) compactly supported solutions if and only if $\alpha = \alpha_k$ for some integer $k \geq 1$. Moreover, $k-1$ equals exactly the number of sign changes of $U(\eta)$, and $U(\eta)$ is symmetric (anti-symmetric) if k is odd (even).*

Note that every compactly supported solution comes with a one-parameter family in view of the scaling

$$U(\eta; \lambda) = \lambda^{\frac{m-1}{2}} U(\frac{\eta}{\lambda}), \tag{2.3}$$

which (1.7) inherits from (1.3). For k odd, the value of the corresponding eigenfunction U_k in $\eta = 0$ can be used as a parameter, and for k even the value of $(|U_k|^{m-1}U_k)'$. The solutions corresponding to α_1 are the explicit instantaneous point source solutions given by (1.11).

For α_2 the solutions in Theorem 2.1 are also explicit and originally due to Barenblatt and Zel'dovich [8]. In fact they belong to a family of explicit solutions parametrized by the dimension N:

$$\alpha = \frac{1}{m}; \quad \beta = \frac{1}{2m}; \quad U(\eta) = \pm|\eta|^{\frac{2-N}{m}} (C - \frac{m-1}{N(m-1)+2}|\eta|^{N+\frac{2-N}{m}})_+^{\frac{1}{m-1}}. \tag{2.4}$$

For $N = 2$ these solutions coincide with (1.11), whereas for $N > 2$ they are singular in $\eta = 0$. In the one-dimensional case (2.4) gives the dipole profiles for (1.1), which describe the intermediate asymptotics for compactly supported solutions of (1.1) with zero integral. This latter fact was established by Kamin and Vazquez [23], who incidentally also showed that compactly supported solutions of (1.1) with nonzero integral eventually loose all their sign changes, in which case the intermediate asymptotics are again given by (1.11). The values of α and β follow from (1.5) and a generalisation of the mass conservation law (1.9) for (compactly supported) solutions of (1.1): integration by parts yields that for every harmonic function $h(x)$

$$\frac{d}{dt} \int h(x)u(x,t)dx = 0. \tag{2.5}$$

With $N = 1$ and $h(x) = x$ this implies $\alpha = 2\beta$. Consequently each of these dipole solutions converges to a multiple of the distributional derivative of the Dirac measure as $t \downarrow 0$. After setting

$$w(x,t) = \int_{-\infty}^{x} u(s,t)ds, \tag{2.6}$$

we obtain functions which have the the Dirac measure itself as initial data. These are (fundamental) solutions of the equation

$$w_t = \nabla \cdot (|\nabla w|^{m-1} \nabla w) \tag{2.7}$$

in dimension $N = 1$. The right hand side of (2.7) is known in the literature as the p-Laplacian ($p = m + 1$). Equation (2.7) was treated in [22]. The theory for compactly supported solutions of this equation is much the same as the theory for (1.1). In particular it allows a family of selfsimilar compactly supported instantaneous point source solutions playing the same role as the Barenblatt-Pattle solutions for the porous medium equation. However in dimension $N > 1$ equation (2.7) cannot be transformed into (1.1) by differentiation.

The first two exponents in Theorem 2.1 are explicit numbers, obtained from combining the relation $(m-1)\alpha + 2\beta = 1$ with a conservation law. Observe that writing

$$k_n = \frac{\alpha_n}{\beta_n}, \tag{2.8}$$

we have $k_1 = 1$ and $k_2 = 2$. The numbers k_n are now commonly refered to as eigenvalues, and the corresponding solution profiles as eigenfunctions. Note that for $\beta > 0$ we may just as well take as similarity variable

$$\eta = \beta^{\frac{1}{2}} x t^{-\beta}, \tag{2.9}$$

which gives

$$\Delta(|U|^{m-1}U) + \eta \cdot \nabla U + kU = 0, \quad k = \frac{\alpha}{\beta}, \tag{2.10}$$

whence the terminology of eigenvalues and eigenfunctions.

In the original russian literature the selfsimilar solutions corresponding to k_1 and k_2 are called selfsimilar solutions of the first kind. Solutions for which there is no physical or dimensional principle to determine a second relation between the exponents, are called selfsimilar solutions of the second kind, and their exponents are called anomalous. For the newly found higher eigenvalues $k_3, k_4, k_5, \ldots$, it was not immediately clear whether they were of the first or the second kind. In the case of the heat equation, when we look at profiles with exponential decay, $k_n = n$ for all n. The corresponding profiles are easily seen to have the property that their first $n-1$ moments are zero, which is related to the fact that the heat equation conserves all these moments equally well. This gave rise to the conjecture that perhaps also for the porous medium equation $k_n = n$ would be true for all n. However the conjecture turned out to be hard to prove, the reason being that some time later Vazquez proved

THEOREM 2.2. [11] *Let $m > 1$, $N = 1$, and let $k_3 = \alpha_3/b_3$, where α_3 and β_3 are as in Theorem 4.1. Then*

$$k_3 > 3. \tag{2.11}$$

The proof was included in [11], which also contains a computation of the limits of all eigenvalues as $m \to \infty$. We have e.g. that

$$k_3 \to 4, \quad k_4 \to 4 + 2\sqrt{2}, \quad k_5 \to 6 + 2\sqrt{5}. \tag{2.12}$$

The limits of the higher order eigenvalues are more complicated algebraic numbers. Numerical experiments indicate that all the eigenvalues are strictly increasing functions of the parameter m, but until now this has not been proved. In particular the inequality $k_n > n$ is still undecided for $n \geq 4$. Another open question is the differentiability of the eigenvalues as functions of m, and, in case of an affirmative answer, what are the explicit values of these derivatives at $m = 1$?

3. The dual porous medium equation.

In Section 2 we already saw that (1.1) can be integrated once with respect to x to give a new equation (2.7) for the primitive of $u(x,t)$. A second integration,

$$z(x,t) = \int_{-\infty}^{x} w(s,t)ds, \tag{3.1}$$

yields the equation

$$z_t = |z_{xx}|^{m-1} z_{xx}, \tag{3.2}$$

which is called the dual porous medium equation [11]. Just as (1.1) and (2.7) this is a degenerate nonlinear diffusion equation, but it does not have a conservation law for $\int z(x,t)dx$. The role played by the instantaneous point source solutions for (1.1) and (2.7), is now taken over by the second primitive of the eigenfunction $U_3(\eta)$ corresponding to k_3.

THEOREM 3.1. [11] *There exists a unique family of nonnegative compactly supported similarity solutions of (3.2), given by*

$$z_3(x,t) = t^{-\alpha_3 + 2\beta_3} Z_3(xt^{-\beta_3}) = t^{(-k_3+2)\beta_3} Z_3(xt^{-\beta_3}), \tag{3.3}$$

with

$$Z_3(\eta) = \int_{-\infty}^{\eta} \int_{-\infty}^{\xi} U_3(\zeta) d\zeta d\xi. \tag{3.4}$$

These solutions describe the intermediate asymptotics of all compactly supported nonnegative solutions.

An immediate consequence of the inequality $k_3 > 3$ is that this solution has the property that

$$\int z_3(x,t)dx = t^{(3-k_3)\beta_3} \int Z_3(\eta)d\eta \tag{3.5}$$

goes to infinity if $t \downarrow 0$. Hence this anomalous "fundamental" solution is very singular in the sense that it does not have a distribution as initial value at $t = 0$.

Theorem 2.2 has another peculiar consequence for the dual porous medium equation. If we take the ratio $k = \alpha/\beta = 3$ and look at the symmetric solutions of (1.6) with $N = 1$, we find they have two sign changes and algebraic decay like $|\eta|^{-3}$ as $|\eta| \to \infty$. Integrating twice, the sign changes disappear, and the decay becomes $|\eta|^{-1}$. This implies that the corresponding similarity solutions, which can then be taken strictly positive, converge to multiples of the function $1/|x|$. The convergence is pointwise in every $x \neq 0$. Note that these similarity solutions are not integrable, and that $1/|x|$ is not locally integrable. Also the similarity exponents are such, that had the similarity profile been integrable, the initial data of the similarity solution would have been a multiple of the Dirac δ-measure. In view of the consequences of this fact later on, we state this as a separate theorem.

THEOREM 3.2. *There exists a strictly positive similarity solution*

$$z(x,t) = t^{-\frac{1}{3m-1}} Z(xt^{-\frac{1}{3m-1}}) \tag{3.6}$$

of equation (4.6) with the property that

$$z(x,t) \to \frac{1}{|x|} \tag{3.7}$$

uniformly on every compact $K \subset \Re - \{0\}$ as $t \downarrow 0$.

4. The dipole in more dimensions.

In this section we discuss the more dimensional case, and in particular the difference between $N = 1$ and $N \geq 2$. To begin with, the asymmetric solutions in Theorem 2.1 disappear. Stated in terms of (2.10) rather than (1.7) we have

THEOREM 4.1. [17] *Let $m > 1$ and $N \geq 2$. Then there exists a strictly increasing sequence*

$$k_1 = N < k_3 < k_5 < k_7 ... \uparrow \infty, \tag{4.1}$$

such that (2.10) *has (a one-parameter family of) compactly supported radially symmetric solutions U_n if and only if $k = k_n$ for some odd integer $n \geq 1$. Moreover, $(k-1)/2$ equals exactly the number of spheres on which U_n changes sign, and $U_n(0)$ can be used as a parameter.*

With the discussion of the one dimensional case in mind, this result raises two questions. First, is there anything in the more dimensional case to replace the number k_2 and the dipole solution? Second, what can be said about k_3? The answers to these two questions are intimately related, as we will point below.

For the heat equation, solutions of dipole type are easily obtained by taking a directional derivative of the heat kernel (2.1). It is well known for instance that

$$u(x,t) = -\frac{\partial}{\partial x_1} E(x,t) = \frac{x_1}{2t} \frac{1}{(4\pi t)^{N/2}} e^{-\frac{x^2}{4t}} \tag{4.2}$$

is a solution of the heat equation with

$$u(x,t) \to -\frac{\partial}{\partial x_1}\delta(x) \tag{4.3}$$

as $t \downarrow 0$. Moreover, this solution decribes the intermediate asymptotics of a large class of solutions on the halfspace $\Re^N_+$ with boundary condition $u(0, x_2, \ldots, x_N, t) = 0$.

We note that (4.2) is a selfsimilar nonradial solution. The symmetry is lost because of the partial derivative with respect to x_1. Thus we cannot reduce the problem of finding a dipole solution for (1.1) to an ordinary differential equation. In stead a partial differential equations approach has to be pursued. This was done in [19], from which we state

THEOREM 4.2. *There exists a selfsimilar weak solution $u_2(x,t)$ to (1.1) of the form (1.6) with*

$$\alpha = \frac{N+1}{(N+1)m+1-N} = (N+1)\beta, \tag{4.4}$$

which satisfies (4.3) as $t \downarrow 0$. The corresponding profile has compact support and is unique in the class of solutions satisfying $u(0, x_2, \ldots, x_N, t) = 0$.

Note that the uniqueness implies that the solution is radially symmetric in the variables $x_2, \ldots, x_N$. Observe also that the exponents are explicit, since they follow from combining the relation $(m-1)\alpha + 2\beta = 1$ with the conservation law for the first moment with respect to x_1.

$$\frac{d}{dt}\int x_1 u(x,t)dx = 0. \tag{4.5}$$

Thus the dipole solution in more space dimensions is a selfsimilar solution of the first kind, although it is unlikely to have an explicit closed formula representation.

The uniqueness of this dipole follows from a rather standard argument based on a comparison principle for solutions of the porous medium equation on a halfspace and the conservation law (4.5) above. More interesting is the construction of the dipole solution. This is done via the dual porous medium equation.

We have seen in the previous sections that (3.2) can be obtained by twice taking the primitive with respect to x. In more dimensions the dual porous medium equation,

$$z_t = |\Delta z|^{m-1}\Delta z, \tag{4.6}$$

can be related to (1.1) by setting

$$u = -\Delta z. \tag{4.7}$$

Here we have put the minus in front of the Laplacian in view of the positivity properties of this differential operator. To invert the relation between u and z, we use the fundamental solution of Laplace's equation

$$\Gamma(x) = \frac{1}{N(2-N)\omega_N}|x|^{2-N} \quad (N=2: \quad \Gamma(x) = \frac{1}{2\pi}\log|x|), \tag{4.8}$$

and write

$$z(x,t) = -\int \Gamma(x-y)u(y,t)dy. \tag{4.9}$$

Thus, since $\delta = \Delta\Gamma$, and $u = -\Delta z$, we have , as $t \downarrow 0$,

$$u(x,t) \to -\frac{\partial}{\partial x_1}\delta(x) \quad \Leftrightarrow \quad z(x,t) \to z_0(x) = \frac{\partial \Gamma}{\partial x_1}(x) = \frac{1}{N\omega_N}\frac{\cos\varphi_1}{r^{N-1}}, \tag{4.10}$$

where we have used polar coordinats $x_1 = r\cos\varphi_1, x_2 = \sin\varphi_1\cos\varphi_2, \ldots$.

The dipole solution is then obtained by first solving the dual porous medium equation with initial data $z_0(x)$ as in (4.10), on the half space $\Re^N_+$, with zero lateral boundary data, and then putting $u = -\Delta z$. The dual porous medium equation is wellposed for continuous initial data with compact support, and the corresponding semigroup is contractive with respect to the supremum norm. Since $z_0(x)$ is unbounded, a limiting argument is needed, to construct a solution of the dual porous medium equation with $z(x,0) = z_0(x)$ as a limit of solutions with bounded initial data. The hard part here is to show that for $t > 0$ this limit is bounded (apriori there is no reason why it should). In particular we need it to be not identically equal to $z_0(x)$, otherwise we would be left with $u = 0$.

For $N = 2$ the proof of this boundedness relies on Theorem 3.2, which is a consequence of the equality $k_3 > 3$. Observing that $z_0(x) < |x|^{1-N}$ we can use the solution in Theorem 3.1 as a supersolution for the limiting problem, provided $N = 2$, forcing the limit solution to be bounded for every positive t.

For $N \geq 3$ the proof relies on a generalization of Theorems 2.2 and 3.2 to the $(N-1)$-dimensional case. Thus information a specific similarity solution in dimension $N-1$, allows a successful construction of the dipole solution in dimension N.

THEOREM 4.3. [19] *Let $m > 1$, $N > 1$, and let $k_3 = \alpha_3/b_3$, where α_3 and β_3 are as in Theorem 4.1. Then*

$$k_3 > N + 2. \tag{4.11}$$

Moreover, there exists a strictly positive similarity solution

$$z(x,t) = t^{-\frac{1}{m(N+2)-N}} Z(xt^{-\frac{1}{m(N+2)-N}})$$

of equation (3.2) with the property that

$$z(x,t) \to |x|^{2-N}$$

uniformly on every compact $K \subset \Re^N - \{0\}$ as $t \downarrow 0$.

5. More anomalous exponents.

Anomalous exponents also appear in the so-called modified heat equation, and in the modified porous medium equation, which read, respectively with $m = 1$ and $m > 1$,

$$u_t + \gamma|u_t| = \Delta u^m. \tag{5.1}$$

Here $\gamma \in (-1,1)$ is a fixed parameter.

With $m = 1$ this is called Barenblatt's equation for elasto-plastic filtration. For $m = N = 2$ it is a model for groundwater flow which takes into account the effect of hysteresis [7]. Denoting the inverse of the function $F(s) = s+\gamma|s|$ by $\Phi(t)$, it can be written as $u_t = \Phi(\Delta u)$, which resembles the dual porous medium equation (4.6), where $F(s) = |s|^{\frac{1}{m}-1}s$ and $\Phi(c) = |c|^{m-1}c$. Kamin, Vazquez and Peletier have observed [KPV] that the Cauchy problem for this equation is wellposed in the class of nonnegative continuous functions which decay to zero at infinity, and that, with respect to the supremum norm, it generates a contraction semigroup in this class. Moreover they prove

THEOREM 5.1. *Let $m = 1$ and $\gamma \in (-1,1)$. There exists a unique family of radially symmetric strictly positive similarity solutions*

$$u(x,t) = t^{-k/2}U(x/t^{1/2}) \tag{5.2}$$

of equation (5.1) with the property that $U(\eta)$ decays exponentially to zero as $|\eta| \to \infty$. These solutions are parametrized by $U(0)$. The exponent k is a strictly increasing smooth function $k(\gamma)$ of γ, with $k(\gamma) \to \max(N-2,0)$ as $\gamma \to -1$, and $k(\gamma) \to \infty$ as $\gamma \to 1$. Moreover, every solution to the Cauchy problem with nonzero nonnegative initial data

$$u(x,0) \le Me^{-Ax^2} \tag{5.3}$$

for some positive numbers 1 and M, has intermediate asymptotics described by one of these similarity solutions.

In other words, these similarity solutions can be thought of as fundamental solutions for Barenblatt's equation. Observe that the solution with $U(0) = A \neq 0$ is simply the solution with $U(0) = 1$, multiplied by 1. This reflects the invariance of the equation under scalar multiplication. Just as in the case of the dual porous medium equation, the exponents are anomalous if $\gamma \neq 0$. For $\gamma > 0$ the solution is very singular, just as (3.4), because its integral blows up at $t = 0$. On the other hand, if $\gamma < 0$, the solution is mildly singular, because its integral goes to zero. Note that for $\gamma = 0$ we have the heat equation with $k = N$.

Later it was shown by Aronson and Vazquez [5], using an implicit function type argument in a dynamical systems setting, that

$$\frac{dk}{\delta\gamma}(0) = \frac{4(N/2)^{N/2}}{e^{N/2}\Gamma(N/2)}. \tag{5.4}$$

This number had also been found by Goldenfeld et al. [13] by means of the formal techniques of renormalization. Finally, Peletier [29] showed that

$$\lim_{\gamma\to 1}(1-\gamma)k(\gamma)^2 = \frac{1}{2}\rho_\nu^2, \tag{5.5}$$

where ρ_ν denotes the first zero of the Bessel function J_ν with $\nu = N/2 - 1$, and that as $\gamma \to 1$, normalizing by $U(0) = 1$,

$$U(\sqrt{k}\eta;\gamma) \to \Gamma(\nu+1)(\frac{1}{4}|\eta|\rho_\nu)^{-\nu} J_\nu(\frac{1}{2}\rho_\nu|\eta|)\chi_{[0,2]}(|\eta|), \tag{5.6}$$

where $\chi_{[0,2]}$ denotes the characteristic function of the interval $[0,2]$. Note that this limit profile is a multiple of the first eigenfunction for the radial Laplacian on the ball, extended by zero outside.

Next we consider (5.1) with $m > 1$. Compactly supported similarity solutions were obtained in [20], from which we state

THEOREM 5.2. [20] *Let $m > 1$ and $\gamma \in (-1,1)$. There exists a unique family of radially symmetric nonnegative similarity solutions*

$$u(x,t) = t^{-\alpha} U(xt^{-\beta}) \tag{5.7}$$

of equation (5.1) with the property that $U(\eta)$ has compact support. These solutions are parametrized by $U(0)$. The exponents satisfy $2\beta + (m-1)\alpha = 1$, and the ratio $k = \alpha/\beta$ is a strictly increasing smooth function $k(\gamma)$ of γ, with $k(\gamma) \to \max(N-2,0)$ as $\gamma \to -1$, and $k(\gamma) \to \infty$ as $\gamma \to 1$. Moreover

$$\frac{\partial k}{\partial \gamma}(0) = \frac{N^{\frac{N}{2}}\ 2^{\frac{m}{m-1}+1}\ (m-1)^{\frac{N}{2}}}{(2+(m-1)N)^{\frac{1}{m-1}+\frac{N}{2}}\ B(\frac{m}{m-1},\frac{N}{2})}, \tag{5.8}$$

where B is the β-function.

Again the derivative agrees with the formal results of renormalization [10].

The limit behaviour of the similarity solutions as $\gamma \to 1$ was investigated by Ducroo de Jongh, who showed in his graduation work at Leiden University, that

PROPOSITION 5.3. *The behaviour of k as $\gamma \to 1$ is given by*

$$\lim_{\gamma \to 1}(1-\gamma)k(\gamma)^{1+\frac{1}{m}} = C = C(m,N) \neq 0. \tag{5.9}$$

In agreement with (5.5), $C(m,N) \to \rho_\nu^2/2$ as $m \to 1$. He also showed that, with the appropiate scalings as $\gamma \to 1$, on its supporting ball the asymptotic similarity profile is a radial solution of the Dirichlet problem with zero boundary data for

$$\Delta U^m + U = 0. \tag{5.10}$$

The similarity solutions in Theorem 5.2 describe the intermediate asymptotics for compactly supported nonnegative solutions of (5.1), see [21]. Note however that (5.1) is not of the form $u_t = \Phi(\Delta u)$, which makes existence and uniqueness results considerably harder to obtain.

We conclude this section with another important example in which anomalous exponents appear, namely the Aronson-Graveleau solution [4] of the porous medium equation. This is a selfsimilar solution of the form

$$u(x,t) = (T-t)^{-\alpha}U(r), \quad r = |x|(T-t)^{-\beta}, \quad (m-1)\alpha + 2\beta = 1. \tag{5.11}$$

Solutions of the corresponding ordinary equation for $U(r)$ are always unbounded. Thus we cannot expect to find any compactly supported solutions of the form (5.11). However there is an equally interesting class of special solutions to be found here, namely the solutions whose pointwise limit as $t \uparrow T$ exists, and is not identically equal to zero. There is only one nonnegative (family of) solution(s) with this property. They are supported on the outside of a focussing ball, and their limit profile at the focussing time is given by $v(x) = u(x)^{m-1} = Cx^{2-1/\beta}$. In dimension $N = 1$ these are explicit colliding travelling wave solutions (whence $\beta = 1$), but in higher dimensions these solutions are of the second kind and have $\beta < 1$. The existence of this solution has important consequences for the regularity theory for solutions of (1.1) in dimension $N \geq 2$, because it shows that in general the pressure ($v = u^{m-1}$) is not Lipschitz continuous. They were announced in [1] after numerical experiments by Graveleau, but only recently their existence and uniqueness was established in [4]. It has also been shown by Angenent and Aronson [2], that these solutions describe the behaviour of radial solutions supported on an annulus as the inner free boundary collapses, see also [3]. Finally, considering β as a function $\beta(N)$ of N, and not restricting to integer values of N, the value of $\beta'(1)$ can be computed explicitly [5].

6. The $k - \epsilon$ model.

The $k - \epsilon$ model for turbulence reads

$$(KE) \quad \begin{cases} k_t = \alpha\Big(\dfrac{k^2}{\varepsilon}k_x\Big)_x - \varepsilon, \\ \varepsilon_t = \beta\Big(\dfrac{k^2}{\varepsilon}\varepsilon_x\Big)_x - \gamma\dfrac{\varepsilon^2}{k}. \end{cases}$$

Here α, β, and γ are positive parameters, and k and ε are unknown nonnegative functions of x and t. This system describes the evolution of turbulent bursts [6], k stands for turbulent energy density, and ε is the dissipation rate of turbulent energy. In the applications α and β are usually different [27,15]. A related scalar problem is discussed in [16] and [24].

We note that (KE) is a coupled system of two quasilinear diffusion-absorption equations. The diffusion coefficients may, depending on k and ε, become degenerate (very small) or singular (very large), and the second absorption term is also singular. In the case of equal diffusion coefficients, (KE) allows a family of explicit selfsimilar compactly supported "source type" solutions found by Barenblatt et al. [9]. To be precise, put

$$k = \frac{1}{t^{2\mu}}f(\eta), \quad \varepsilon = \frac{1}{t^{2\mu+1}}g(\eta), \quad \eta = \frac{x}{t^{1-\mu}}, \tag{6.1}$$

where we restrict our attention to the case $0 < \mu < 1$. If

$$\alpha = \beta = 1, \quad \mu = \frac{\gamma}{3(\gamma-1)}, \quad \gamma > \frac{3}{2}, \tag{6.2}$$

then

$$f(\eta) = \frac{1}{6}\kappa(2-\kappa)(1-\eta^2)_+, \quad g(\eta) = \kappa f(\eta), \quad \kappa = \frac{1}{\gamma-1} \tag{6.3}$$

is an explicit solution of the equations for f and g, which read

$$\begin{cases} \alpha\Big(\dfrac{f^2}{g}f'\Big)' + (1-\mu)\eta f' + 2\mu f - g = 0; & (6.4) \\ \beta\Big(\dfrac{f^2}{g}g'\Big)' + (1-\mu)\eta g' + (1+2\mu)g - \gamma\dfrac{g^2}{f} = 0. & (6.5) \end{cases}$$

In [18] this solution is used to obtain a compactly supported solution for $\alpha \neq \beta$ with $|\alpha-\beta|$ small. The result is

Theorem 6.1. [18] *There exists an open neighbourhood $\mathcal{O}$ of the set*

$$\{(\alpha,\beta,\gamma): \ \alpha=\beta>0, \ \gamma>\frac{3}{2}\},$$

such that for every $(\alpha,\beta,\gamma)\in\mathcal{O}$ there is precisely one $0<\mu<1$ for which equations (6.4) and (6.5) have a solution pair (f,g), with f and g symmetric and positive on $(-1,1)$, and

$$f(\eta)\to 0, \ \ g(\eta)\to 0, \ \ \frac{f(\eta)}{g(\eta)}f'(\eta)\to -\alpha(1-\mu), \ \ \frac{f(\eta)^2}{g(\eta)^2}g'(\eta)\to -\beta(1-\mu), \tag{6.6}$$

as $\zeta\uparrow 1$. Moreover, if we write

$$\kappa = \frac{g(0)}{f(0)}, \quad \lambda = \frac{\alpha}{\beta}, \tag{6.7}$$

then in $\lambda=1$

$$\kappa = \frac{1}{\gamma-1}, \quad \frac{d\mu}{d\lambda} = 0, \quad \frac{d\kappa}{d\lambda} = \frac{\kappa(2-\kappa)}{\kappa+1}\Big(\kappa-1+\frac{2}{B_\kappa}\Big). \tag{6.8}$$

Here B_κ is defined by

$$B_\kappa = \frac{\Gamma(\frac{1}{2})}{\Gamma(a)\Gamma(b)}, \quad a+b=\frac{3}{2}, \quad ab=\frac{3}{2(2-\kappa)}. \tag{6.9}$$

The conditions in (6.6) are the usual free boundary conditions for porous medium type equations. In particular the fluxes at the free boundary are zero.

Theorem 6.1 was the first result for this system with $\alpha\neq\beta$. For $\alpha=\beta$ Bertsch, Dal Passo and Kersner [12] proved an existence result for the Cauchy problem in the class of compactly supported solutions. They also show in their paper that these solutions have intermediate asymptotics described by (6.3), provided $\gamma>3/2$.

7. Quadratic systems.

Our results on similarity solutions depend for a large part on a transformation of the (radial) similarity equation into an autonomous quadratic system. In this section we shall discuss this method. As an illustration we sketch the proof of Proposition 5.3 about the behaviour of the selfsimilar compactly supported solutions of the modified porous medium equation as $\gamma \to 1$.

Let $U(\eta)$ be a radial solution of (2.10). Writing $U(r) = U(|\eta|)$ we let

$$\tau = \log r, \quad x(\tau) = \frac{rU'(r)}{U(r)}, \quad y(\tau) = r^2 |U(r)|^{1-m}. \tag{7.1}$$

This transforms (2.10) into

$$(Q_k) \begin{cases} \dot{x} = (2 - N - mx)x - (x + k)y; \\ \dot{y} = y(2 + (1 - m)x). \end{cases}$$

Note that the scaling (2.3) amounts to a shift in τ. The instantaneous point source solutions correspond to $k = N$ and the straight line $x + y = 0$. It was shown in [H] that for any k the radial solutions are all mapped into a unique orbit $\Gamma_{r,k}$ coming out of the origin into the upper half plane along the eigenvector corresponding to the eigenvalue 2 of the linearisation of (Q_k) around the origin. Also, as long as $k > 0$, there is a unique orbit $\Gamma_{i,k}$ escaping in finite time to infinity in the second quadrant along the straight line $x + y = 0$. This orbit contains the solutions with a free boundary, on which the flux is easily seen to be zero. Using either polar coordinates

$$x = \frac{\rho}{1 - \rho} \cos \phi, \quad y = \frac{\rho}{1 - \rho} \sin \phi, \tag{7.2}$$

or homogeneous coordinates

$$x = \frac{X}{Z}, \quad y = \frac{Y}{Z}, \quad X^2 + Y^2 + Z^2 = 1, \tag{7.3}$$

we find $\Gamma_{i,k}$ is really the unique orbit going into a saddle point at infinity. In terms of (7.2) it is given by $\rho = 1$ and $\phi = 3\pi/4$. We note that the behaviour near infinity is completely determined by the quadratic terms in (Q_k).

Now let $y_r(k)$ and $y_i(k)$ be the intersections of respectively $\Gamma_{r,k}$ and $\Gamma_{i,k}$ with the line $x + k = 0$. Theorem 5.2 is based on the simple observation that for the compactly supported similarity profile of the modified porous medium equation, k and γ are related by

$$\frac{y_i(k)}{y_r(k)} = \frac{1 + \gamma}{1 - \gamma}. \tag{7.4}$$

The left hand side is a real analytic function of k, strictly increasing, defined on the interval $(\max(0, N - 2), \infty)$ with range $(0, \infty)$. Formula (5.8) follows from a computation of the first derivative of this function in $k = N$. In fact one can compute higher order derivatives just as well, in [20] this is done in the case that $m = N = 2$. All these computations are based on solving linear first order inhomogeneous ordinary differential equations with rational coefficients.

Proof of Proposition 5.3. This is based on on a careful inspection of $\Gamma_{i,k}$ and $\Gamma_{r,k}$ as $k \to \infty$. First we consider $\Gamma_{i,k}$. Scaling x, y and τ by

$$A = \frac{x}{k}; \quad B = \frac{y}{k}; \quad t = k\tau, \tag{7.5}$$

we obtain

$$(\tilde{Q}_k) \begin{cases} \dot{A} = (\frac{2-N}{k} - mA)A - (A+1)B; \\ \dot{B} = B(\frac{2}{k} + (1-m)A), \end{cases}$$

in which we can take the limit $k \to \infty$ to obtain

$$(\tilde{Q}_\infty) \begin{cases} \dot{A} = -mA^2 - (A+1)B; \\ \dot{B} = (1-m)AB. \end{cases}$$

The scaling has been chosen to preserve the behaviour at infinity. The orbit $\Gamma_{i,k}$ corresponds to an orbit $\tilde{\Gamma}_{i,k}$ going into a saddle point at infinity of $(\tilde{Q}_k)$ with the same coordinates as the original saddle point. Taking the limit as $k \to \infty$, we have $\tilde{\Gamma}_{i,k} \to \tilde{\Gamma}_{i,\infty}$, which intersects the line $A = -1$ at some $B = B_\infty > 0$. Here we use the Stable Manifold Theorem with parameter dependence, combined with the observation above that the quadratic terms determine the behaviour at infinity. For the original system this implies that

$$\lim_{k\to\infty} \frac{y_i(k)}{k} = B_\infty. \tag{7.6}$$

Next we consider $\Gamma_{r,k}$. Here we have to use a different scaling preserving the behaviour near the origin, which is determined by the linear terms. We leave x and τ unchanged and only set

$$b = ky, \tag{7.7}$$

which transforms (Q_k) into

$$(Q_k^*) \begin{cases} \dot{x} = (2-N-mx)x - (\frac{x}{k}+1)b; \\ \dot{b} = b(2+(1-m)x), \end{cases}$$

in which we can take again the limit $k \to \infty$ to obtain

$$(Q_\infty^*) \begin{cases} \dot{x} = (2-N-mx)x - b; \\ \dot{b} = b(2+(1-m)x), \end{cases}$$

Corresponding to $\Gamma_{r,k}$ we have orbits $\Gamma_{r,k}^*$ and in the limit $\Gamma_{r,\infty}^*$. It was shown in [17] that these orbits escape to infinity in finite time, and that $b(-x)^{(1-m)/m}$ converges to a positive constant C_k. A continuity argument [17] then implies that the intersection height b_k of $x+k=0$ and $\Gamma^*_{r,k}$ satisfies $b_k(-k)^{(1-m)/m} \to C_\infty$ as $k \to \infty$. Consequently we have

$$\lim_{k\to\infty} y_r(k)k^{\frac{1}{m}} = C_\infty. \tag{7.8}$$

Combining (7.4), (7.6) and (7.8) completes the proof of Proposition 5.3. We note in addition that a change of variables similar to (7.1) relates (Q^*_∞) to (5.10), which is why this semilinear elliptic equation describes the asymptotic similarity profile as $k \to \infty$.

We have seen that selfsimilar solutions of scalar diffusion equations with a two-parameter group of scalings are related to two-dimensional quadratic autonomous dynamical systems. For systems however, such scaling groups lead to quadratic systems with dimension equal to twice the number of equations in the system. The $k-\epsilon$ system is an example of such a system. If f and g are solutions of (6.4) and (6.5) we may set

$$t = \log\eta,\ x = \frac{\eta f'}{f},\ y = \frac{\eta g'}{g},\ z = \eta^2\frac{g}{\alpha f^2},\ u = \frac{g}{f}, \tag{7.9}$$

which transforms the two coupled nonautonomous second order equations (6.4) and (6.5) into the four-dimensional quadratic autonomous system

$$(Q_{\lambda,\mu,\gamma})\quad \begin{cases} \dfrac{dx}{dt} = x(1-3x+y) - z(x(1-\mu)+2\mu-u); \\ \dfrac{dy}{dt} = y(1-2x) - \lambda z(y(1-\mu)+2\mu+1-\gamma u); \\ \dfrac{dz}{dt} = z(2+y-2x); \\ \dfrac{du}{dt} = u(y-x). \end{cases}$$

In analogy to what we saw above for scalar equations, compactly supported solutions of (6.4) and (6.5) correspond to intersections of the two-dimensional fast unstable manifold of the u-axis, and the two-dimensional stable manifold of a saddle point at infinity. A global analysis of these intersections is still open, all the results in [H] are based on perturbation arguments in the spirit of [5] and [20]. The classification of the behaviour at infinity is done by using homogeneous coordinates rather than polar coordinates. The transformations are carried out with the help of the symbolic manipulation package Maple. Finally the perturbation argument boils down to solving linear inhomogeneous systems of three coupled first order differential equations with rational coefficients. These systems can be completely solved in terms of hypergeometric functions.

We conclude this section on quadratic systems from a completely different perspective. The basic idea here it not to put the emphasis on the similarity structure of (1.11) but rather on the fact that in (1.11) the pressure, u^{m-1}, is a polynomial. This seems to go back to Kersner who constructed explicit nonselfsimilar solutions of certain porous media equations with sink terms [25]. In [26] this idea is used to construct new solutions for the porous medium equation itself. Since for the normalized pressure,

$$v = \frac{m}{m-1} u^{m-1}, \tag{7.10}$$

the equation reads

$$v_t = (m-1)v\Delta v + |\nabla v|^2, \tag{7.11}$$

this method works for polynomials with degree at most equal to two. As an example consider in two dimensions solutions of the form

$$v(x,y,t) = C(t) - A(t)x^2 - B(t)y^2, \tag{7.12}$$

with $A, B, C > 0$. Substitution leads to the following two dimensional quadratic system for 1 and B

$$\begin{cases} \dfrac{dA}{dt} = -2A((m+1)A + (m-1)B); \\ \dfrac{dB}{dt} = -2B((m-1)A + (m+1)B), \end{cases}$$

coupled with

$$\frac{dC}{dt} = -2(m-1)(A+B)C.$$

Every solution curve of this system with $A, B, C > 0$ defines a compactly supported solution of the porous medium equation with an elliptical free boundary (the instanteneous point source solution corresponds to $A \equiv B$). This example shows that nonlinear equations may have rather nontrivial linear subspaces. For a more general discussion of this phenomenon we refer to [14].

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