MAXIMAL VISCOSITY SOLUTIONS OF THE MODIFIED POROUS MEDIUM EQUATION AND THEIR ASYMPTOTIC BEHAVIOUR

BY

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Abstract

We construct a theory for maximal viscosity solutions of the Cauchy problem for the modified porous medium equation $u_t + \gamma|u_t| = \Delta(u^m)$, with $\gamma \in (-1,1)$ and $m > 1$. We investigate the existence, uniqueness, finite propagation and optimal regularity of these solutions.

As a second main theme we prove that the asymptotic behaviour is given by a certain family of self-similar solutions of the so-called second kind with anomalous similarity exponents.

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Introduction

This paper is devoted to the construction of a theory of viscosity solutions for the *modified porous medium equation* (MPME)

$$u_t + \gamma |u_t| = \Delta(u^m)\,. \tag{0.1}$$

We study the initial-value problem with data

$$u(x,0) = u_0(x)\,, \quad x \in \mathbf{R}^N\,. \tag{0.2}$$

We assume that u_0 is nonnegative, continuous and compactly supported. We look for nonnegative solutions $u(x,t)$ defined in $Q = \mathbf{R}^N \times (0,\infty)$, $N \geq 1$, which are continuous down to $t = 0$.

So-called porous medium equations, i.e. diffusion equations having a diffusivity which is a power of u, are common in filtration theory and have been investigated extensively in recent years. The present model appears for instance as a description of the spreading of a ground-water mound sitting on a horizontal impermeable bed and flowing through a gas-filled porous medium. According to Barenblatt [B1], [BER], if h is the height of the mound as a function of $r = |x|$ and t, then the evolution is described by the following law

$$\partial_t h = \frac{\kappa_1}{r}\partial_r(r\partial_r h^2) \quad \text{for} \quad \partial_t h \leq 0, \tag{0.3.a}$$

$$\partial_t h = \frac{\kappa_2}{r}\partial_r(r\partial_r h^2) \quad \text{for} \quad \partial_t h \geq 0, \tag{0.3.b}$$

with different constants, $\kappa_1 > \kappa_2 > 0$. The difference takes into account the asymmetry in the processes of filling and emptying a pore due to the effect of capillary forces. Take now $u = h$, $m = 2$ and

$$\gamma = \frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2} \tag{0.4}$$

to obtain, after an innocent rescaling of time, equation (0.1) in dimension $N = 2$. More generally, we will assume throughout the paper that $N \geq 1$ and the characteristic parameters of the equation satisfy

$$m > 1\,, \quad \text{and} \quad -1 < \gamma < 1\,. \tag{0.5}$$

The mathematical interest in the MPME stems from the combination of two features. On the one hand, the equation is nonlinear, of parabolic type at the points where $u > 0$, but *degenerate* where $u = 0$. This implies in particular the property of finite speed of propagation with the appearance of free boundaries and the corresponding limited regularity of the solutions, well-known phenomena for porous medium equations. On the other hand, the fact that $\kappa_1 \neq \kappa_2$ implies that it is an equation with its principal part in *non-divergence form*, thus falling out of the standard treatments for degenerate diffusions.

A theory needs therefore to combine the understanding of degenerate parabolic equations with the recent theory of *fully nonlinear* equations, cf. e.g. [CIL], [Ca].

The present study of the MPME uses the concept of *viscosity solutions* as developed in Wang [W1,2] for fully nonlinear parabolic equations. This approach is based on a local comparison of solutions with smooth test functions touching from above and below. Since our (nonnegative) solutions will not be smooth on the free boundary of their support, it may happen that there are no smooth test functions with the desired properties. This already indicates that there is a problem with uniqueness here. How to select the correct solution? In this paper we shall give a rather indirect answer to this question by working with maximal solutions. Specifically we prove that for all compactly supported nonnegative initial data u_0, there exists a (by its nature unique) maximal continuous viscosity solution $u(x,t)$. This solution has compact (eventually) expanding support and obeys the positivity property. The family of maps $S(t): u_0 \to u(\cdot,t)$ is order-preserving and forms a semigroup. Concerning the regularity of solutions we remark that there is a three-level regularity: the solutions are globally C^α, they are $C^{2,\alpha}$ (hence classical solutions) in the positivity set, and C^∞ whenever the solution is positive and has nonzero time derivative. Such behaviour is precisely found in the family of self-similar solutions we have described in [HV].

In fact, these solutions play a prominent role in the second main topic covered by this paper, i.e., the question of asymptotic behaviour of (our class of maximal) viscosity solutions. The self-similar solutions of [HV] belong to the type known as *second kind*, i.e., they are solutions of the form

$$U(x,t) = t^{-\alpha} F(\eta), \quad \eta = |x| t^{-\beta}, \quad (m-1)\alpha + \beta = 1\,. \tag{0.6}$$

where the similarity exponents, α and β, are *not* determined from dimensional considerations. Such exponents are also called *anomalous* (cf. [CGO] for a parallel treatment using renormalization techniques). The profile function F admits an arbitrary parameter c, thus giving rise to a family of solutions. (Of course, we can only perform translations in space and time, thus enlarging again, though in a rather trivial way, the solution family). In Section 7 below we show that these as $t \to \infty$ every one of maximal viscosity solutions constructed before with compactly supported initial data converges to a member of the above self-similar family with a precise value of c. This gives a complete rigorous foundation to the description given in [B1] for the intermediate asymptotics of the ground-water evolution.

We briefly mention related works preceding ours. For the well-established case $\gamma = 0$ (the standard porous medium equation), an excellent reference is still [A]. With $\gamma \neq 0$ and linear diffusion, $m = 1$, Kamin, Peletier and Vazquez [KPV] have studied the theory of equation (0.1) as well as the asymptotic behaviour in terms of similarity solutions of the second kind. See also Aronson and Vazquez, [AV1]. Note however that the latter equation falls into the class of L^∞-semigroups, while the PME generates a semigroup of contractions in $L^1(\mathbf{R}^N)$. Both properties are lost in the general context (0.1),(0.5).

The importance of selfsimilarity with anomalous exponents in continuum mechanics has been known to the physics community for decades, notably after the works of Zeldovich and Barenblatt in the fifties, cf. [B2]. However, only in recent years there has been substantial mathematical progress in the form of a rigorous analysis of a number of typical model cases. This work covers the actual construction of solutions with such a similarity and the description of the asymptotics for wide classes of data (intermediate asymptotics) in terms of them. References to progress in problems in nonlinear diffusion can be found in Aronson-Vazquez [AV2] and Hulshof [H2]. Significant progress has been also achieved from a different point of view, renormalization, by Goldenfeld and his group, cf. [G].

1. Viscosity solutions

We consider the Cauchy problem (0.1)-(0.2) with the aforementioned assumptions on u_0, m and γ. We will use for the nonlinearity in the first member of (0.1) the notation

$$F(s) = s + \gamma|s|\,, \tag{1.1}$$

thus writing the MPME in the form

$$F(u_t) = \Delta(u^m)\,. \tag{1.2}$$

Let also $\Phi = F^{-1}$. Introducing as dependent variable $v = u^m$ equation (0.1) transforms into

$$v_t = mv^{1-\frac{1}{m}}\Phi(\Delta v)\,. \tag{1.3}$$

We shall work with the concept of viscosity solutions for equation (1.3). A parabolic neighbourhood of the point (x_0, t_0) is a set of the form

$$P = P_\varepsilon(x_0, t_0) = \{(x,t) : \ |x - x_0|^2 + |t - t_0| < \varepsilon^2, \ t \le t_0\} \ (\varepsilon > 0)\,. \tag{1.4}$$

Definition. *A nonnegative continuous function v defined in a domain $Q \subset \mathbf{R}^N \times \mathbf{R}^+$ is called a viscosity sub-solution (super-solution) of (1.3) if for any parabolic neighbourhood $P = P_\varepsilon(x_0, t_0)$ and for any smooth nonnegative function $\varphi : P \to \mathbf{R}$ with*

$$v \le (\text{resp. } \ge)\varphi \quad \text{at } P \text{ and } \quad v(x_0, t_0) = \varphi(x_0, t_0)\,,$$

it follows that at the point (x_0, t_0)

$$\varphi_t \le (\text{resp. } \ \ge) m\varphi^{1-\frac{1}{m}}\Phi(\Delta\varphi)\,.$$

If a function is both a viscosity sub- and a viscosity super-solution, it is called a viscosity solution. A viscosity solution of the Cauchy problem is a viscosity solution of (1.3) defined in a strip $Q_T = \mathbf{R}^N \times [0, T)$ which is continuous down to $t = 0$ and satisfies the initial condition pointwise.

In this definition we mean by smooth that $\varphi \in C^{2,1}$. We note that we silently have extended the function s^m to the whole of $\mathbf{R}$ by taking it to be odd in s, i.e. $|s|^{m-1}s$. Like wise the coefficient of Φ in equation (1.3) for v (or φ) is always taken to be extended as an even function.

2. The regularised problem. First estimates

The operator in the second member of (1.3) is not uniformly elliptic because of the degeneracy at the level $u = 0$. Moreover, the function Φ is not differentiable. In order to construct viscosity solutions and obtain convenient a priori estimates for them we eliminate both obstacles. Firstly, we regularise the jump in the derivative of F. We choose a smooth function F_1 with the properties that

$$F_1(s) = \begin{cases} F(s-1) & \text{for} \quad s > 1 \\ F(s) - 1 & \text{for} \quad s < 0\,. \end{cases} \tag{2.1}$$

Here we take F_1 to be convex if $\gamma > 0$ and concave if $\gamma < 0$. Define for $\varepsilon > 0$ the function F_ε by scaling:

$$F_\varepsilon(s) = \varepsilon F_1(\frac{s}{\varepsilon})\,. \tag{2.2}$$

Then F_ε is a strictly increasing sequence of functions which converges uniformly to F. The function Φ_ε is by definition the inverse function of F_ε.

The degenerate diffusion is regularised by lifting the initial data, and the compactness of the support combined with its expected finite speed of propagation leads us to consider the following regularised problem

$$(P_\varepsilon)\begin{cases} F_\varepsilon(u_t) = \Delta(u^m) & \text{in } \ Q_T^A = B_A \times (0,T]\,; \\ u(x,t) = \varepsilon & \text{for } \ 0 < t \le T, |x| = A\,; \\ u(x,0) = u_{0\varepsilon}(x) & \text{for } \ x \in B_A\,. \end{cases}$$

Here $B_A = \{x : |x| < A\}$ and $A > 0$ is chosen in such a way that the support of u_0 is contained in B_A. The functions $u_{0\varepsilon}$ are smooth regularisations of u_0, strictly decreasing as $\varepsilon \downarrow 0$. Moreover, we impose $u_{0\varepsilon}(x) = \varepsilon$ and $\Delta u_{0\varepsilon}^m = -\varepsilon$ for $|x| = A$, the $u_{0\varepsilon}$ converge uniformly to u_0, and the $u_{0\varepsilon}^m$ are uniformly bounded in $H^1(B_A)$.

Restricting our attention to solutions $u \ge \varepsilon$, we note that $P(\varepsilon)$ is a fully nonlinear but uniformly parabolic problem with smooth data. It follows from the results of Lihe Wang [W] that $P(\varepsilon)$ has a smooth ($C^{2,\alpha}$ up to the parabolic boundary) solution $u_\varepsilon \ge \varepsilon$. We shall discuss this in the Appendix. Note that the $u_{0\varepsilon}$ satisfy the compatibility conditions to allow smoothness in the corner points $(x,0)$, $|x| = A$.

Proposition 2.1. *The sequence $\{u_\varepsilon\}$ is strictly decreasing as $\varepsilon \downarrow 0$.*

Proof. Suppose it is not. Then there exist $0 < \varepsilon' < \varepsilon$ such that for the corresponding solutions it is not true that $u'_\varepsilon < u_\varepsilon$. By the construction of (P_ε) this can only happen in the interior of Q, and therefore there must exist a point (x_0, t_0) in Q, for which the solutions coincide for the first time. In this point we then have

$$\Delta((u'_\varepsilon)^m) = F_{\varepsilon'}(u'_{\varepsilon t}) > F_\varepsilon(u'_{\varepsilon t}) \ge F_\varepsilon(u_{\varepsilon t}) \ge \Delta(u_\varepsilon^m)\,, \tag{2.3}$$

a contradiction with the fact that $(u')^m - u^m$ attains a maximum in x at that point. $\square$

Proposition 2.2. *The functions u_ε are uniformly bounded in ε. Precisely,*

$$\varepsilon \le u_\varepsilon(x,t) \le \|u_{0\varepsilon}\|_\infty \,. \tag{2.4}$$

Proof. Choose M larger then all $u_{0\varepsilon}$. Fix $\varepsilon > 0$, and define a solution by

$$(\overline{u}_\varepsilon(x))^m = M^m - \varepsilon(x-A)^2/2N \,.$$

Then similar reasoning as above shows, taking $\varphi = \overline{u}_\varepsilon$, that $u_\varepsilon < \overline{u}_\varepsilon$ for all $\varepsilon > 0$. Likewise, the lower bound ε follows from the fact that $\underline{u} = \varepsilon$ is actually a subsolution, which can be used again as φ in the definition. □

Proposition 2.3. *The functions $\nabla(u_\varepsilon^m)$ are uniformly bounded in $L^2(Q)$ if $\gamma \ge 0$, in $L^2_{loc}(Q)$ if $\gamma < 0$.*

Proof. (i) Let $\gamma \ge 0$. Omitting the subscript ε we multiply by u^m and integrate by parts in $Q_T^A = B \times (0,T)$, $B = B_A$. This yields

$$\int_0^T \int_B (F(u_t)u^m + |\nabla(u^m)|^2)\,dxdt = \int_0^T \int_{|x|=A} u^m\,\partial_\nu(u^m)\,d\sigma dt \le 0\,.$$

Since $F_\varepsilon(s) \ge (1-\gamma)s - \varepsilon$, it follows that

$$\frac{1-\gamma}{m+1}\int_B u^{m+1}dt + \iint |\nabla(u^m)|^2)\,dxdt \le \varepsilon \iint u^m\,dxdt + \frac{1-\gamma}{m+1}\int_B u_0^{m+1}dx\,. \tag{2.5}$$

In view of the previous proposition this completes the proof.

(ii) Let now $\gamma < 0$. Following [KPV] we multiply the equation by $f(u)\zeta(x)^2$ where $f(u)$ is nonnegative and decreasing, and ζ is a cut-off function. Using that $F_\varepsilon(u_t) \le F(u_t) = u_t + \gamma|u_t|$, we obtain

$$\int_0^T \int_B (\zeta^2 u_t f(u) + 2\zeta f(u)\langle\nabla\zeta, \nabla u^m\rangle)\,dxdt \ge$$
$$\int_0^T \int_B (-\gamma\zeta^2|u_t|f(u) - \zeta^2\langle\nabla u^m, \nabla f(u)\rangle)\,dxdt\,.$$

The first term on the left-hand side can be integrated with respect to t and is therefore bounded in view of the uniform bounds on u, whereas the first term on the right-hand side may be dropped from the inequality because it is positive. We now choose $f(u) = M^m - u^m$ and arrive at

$$\int_0^T \int_B \zeta^2|\nabla u^m|^2\,dxdt \le C_1 + C_2\int_0^T \int_B |\zeta\langle\nabla\zeta, \nabla u^m\rangle|dxdt\,.$$

Using Young's inequality the latter integrand can be estimated by

$$\frac{\varepsilon}{2}\zeta^2|\nabla u^m|^2 + \frac{1}{2\varepsilon}|\nabla\zeta|^2\,.$$

Choosing ε sufficiently small, the desired bound follows. □

For simplicity we will also assume in the next estimate that $u_0^m \in H^1(\mathbf{R}^N)$.

Proposition 2.4. *Under the assumption that* $u_0^m \in H^1(\mathbf{R}^N)$ *the functions* u_ε^m *are uniformly bounded in* $L^\infty(0,T;H^1(B))$*, and the functions* $(u_\varepsilon^{\frac{m+1}{2}})_t$ *are uniformly bounded in* $L^2(Q)$.

Proof. We multiply by $(u^m)_t$ and integrate by parts in Q_T^A. This yields

$$\iint F(u_t)(u^m)_t\,dxdt + \frac{1}{2}\int_B |\nabla u(x,T)^m|^2\,dx = \frac{1}{2}\int_B |\nabla u(x,0)^m|^2\,dx\,.$$

It is easy to see that

$$sF_\varepsilon(s) \geq (1-|\gamma|)s^2 - b_\varepsilon|s|\,,\ \text{where } b_\varepsilon \to 0 \text{ as } \varepsilon \to 0\,.$$

Thus

$$\iint F(u_t)(u^m)_t\,dxdt = m\iint u^{m-1}F(u_t)u_t\,dxdt \geq$$
$$m(1-|\gamma|)\iint u^{m-1}u_t^2\,dxdt - mb_\varepsilon\iint u^{m-1}|u_t|\,dxdt\,.$$

Young's inequality can again be applied to absorb the latter integral into the term preceding it. Combining these inequalities with the uniform bound on u we obtain

$$\begin{aligned}&\frac{1}{2}\int_B |\nabla(u(x,T)^m)|^2\,dx + \frac{m(1-|\gamma|)}{2}\iint u^{m-1}u_t^2\,dxdt \leq \\ &\frac{1}{2}\int_B |\nabla(u(x,0)^m)|^2\,dx + C\,. \qquad \square\end{aligned} \tag{2.6}$$

We note that the proof of Proposition 2.4 is valid regardless the sign of γ. If $u_0^m \notin H^1$ we use as a test function $t(u^m)_t$ to get $u_\varepsilon^m \in L^\infty(\tau,\infty : H^1(B))$ (uniformly in ε) for every $\tau > 0$ but not near $t = 0$. Details are similar to the porous medium case, see e.g. [V].

3. Passage to the limit. Maximal viscosity solutions

We note that as a consequence of these propositions it follows that, writing $v = u_\varepsilon^m$,

$$\int_{B_A} |\nabla v|^2 \leq C \text{ and } \int_0^T\int_{B_A} v_t^2 \leq C\,. \tag{3.1}$$

In one space dimension this immediately implies that

$$u_\varepsilon^m \text{ is uniformly bounded in } C^{\frac{1}{2},\frac{1}{4}}(\bar{Q})\,. \tag{3.2}$$

The continuity of u in several space dimensions, more precisely the equi-continuity of the sequence $\{u_\varepsilon\}$, is much more involved and needs serious work which we postpone to Section 4. Accepting it for the moment we continue. It follows then that the functions $v_\varepsilon = u_\varepsilon^m$ converge uniformly to a continuous limit function $v = u^m$.

Theorem 3.1. *The uniform limit v of the functions u_ε^m is a viscosity solution of equation (1.3).*

Proof. Take a smooth function φ on some parabolic neighbourhood P of a point (x_0, t_0) as in the definition of viscosity solution. Assume that $v \le \varphi$ on P and that $\varphi(x_0, t_0) = v(x_0, t_0)$. We replace φ by

$$\varphi(x,t) + \delta|x - x_0|^4 + \delta(t - t_0)^2 \ (\delta > 0). \tag{3.3}$$

This does not change the derivatives of φ in (x_0, t_0) appearing in the definition, and ensures that $\varphi - v$ now has a strict minimum on P in (x_0, t_0).

Now consider $\varphi - v_\varepsilon$. For small ε this function must have a minimum somewhere in P, say in $(x_\varepsilon, t_\varepsilon)$, since the adjustment of φ above, together with the uniform convergence of v_ε excludes the possibility of having a minimum at the parabolic boundary of P. Moreover, it follows that $(x_\varepsilon, t_\varepsilon) \to (x_0, t_0)$. Then we have in $(x_\varepsilon, t_\varepsilon)$ that

$$\varphi_t \le v_{\varepsilon t} = m v_\varepsilon^{1-\frac{1}{m}} \Phi_\varepsilon(\Delta v_\varepsilon) \le m v_\varepsilon^{1-\frac{1}{m}} \Phi_\varepsilon(\Delta \varphi). \tag{3.4}$$

Taking the limit $\varepsilon \to 0$ on both sides we obtain that in (x_0, t_0)

$$\varphi_t \le m \varphi^{1-\frac{1}{m}} \Phi(\Delta\varphi). \tag{3.5}$$

Thus v is a viscosity subsolution, and similarly one shows that it is a viscosity supersolution. □

We remark here that it will follow from Section 5 that for every $T > 0$ we can choose A so large that the support of v is always contained in B_A. Thus no circularity of argument is involved.

Theorem 3.2. *The uniform limit u of the functions u_ε takes on the initial value $u_0(x)$. In other words, $v = u^m$ is a viscosity solution of the Cauchy problem for (1.3).*

Proof. The continuity from above comes from the fact that u is a nonincreasing limit of smooth functions u_ε with initial data $u_{0,\varepsilon}$ which approximate u_0.

The continuity from below is more involved and relies on a barrier argument using the self-similar compactly supported solutions constructed in [HV] that we can write in the form

$$U(x,t) = (t+\tau)^{-\alpha} F(\eta), \quad \eta = \sqrt{\beta}\, x\, (t+\tau)^{-\beta}, \tag{3.6}$$

where the similarity exponents α and β depend on γ and m. As we have said, there is a family of possible profiles F depending on an arbitrary constant c. Also, they are classical solutions as long as they are positive. Consider a point x_0 that without loss of generality we assume to be 0 and where the initial value is positive, $u(0,0) > 0$. Suitably selecting the constants τ and c we can choose U in such a way that $U(x,0)$ is below u_0, hence strictly below $u_{0\varepsilon}$.

We contend that $U \le u_\varepsilon$ for all x and t and all $\varepsilon > 0$. If this is not the case for some $\varepsilon > 0$ there must exist a minimal t_0 and some $x_0 \in B_A$ such that at (x_0, t_0) the functions U and u_ε touch for the first time. At that point $U = u_\varepsilon > 0$ hence using the corresponding v variables, we have $V_t \ge v_{\varepsilon,t}$, $\Delta V \le \Delta v_\varepsilon$.Also from the equation

$$V_t = mV^{1-\frac{1}{m}}\Phi(\Delta V),$$

and

$$v_{\varepsilon t} = mv_\varepsilon^{1-\frac{1}{m}}\Phi_\varepsilon(\Delta v_\varepsilon) > mv_\varepsilon^{1-\frac{1}{m}}\Phi(\Delta v_\varepsilon).,$$

Putting these facts together we arrive at a contradiction. So, $V \le v_\varepsilon$ for every ε, and in the limit $U \le u$. Since we can choose U to approach $u(0,0)$ as much as we like from below the proof is done. $\square$

The argument also proves interesting properties of the solutions.

Corollary 3.3. *Maximal viscosity solutions of (1.3) which are initially above a compactly supported similarity solution, remain above this solution for all times. Maximal viscosity solutions have also the positivity property, that is, if $v(x_0,0) > 0$, then $v(x_0,t) > 0$ for all $t > 0$.*

We now establish the crucial maximality result.

Theorem 3.4. *The uniform limit v of the functions u_ε^m is above every other viscosity solution of the Cauchy problem for (1.3). In other words, v is the maximal viscosity solution of the Cauchy problem.*

Proof. The argument is similar. Suppose there is a viscosity solution $\tilde{v}$ which is not below v. Then for some $\varepsilon > 0$ it must intersect the approximation v_ε. This again means that there exists a minimal t_0 and some $x_0 \in B_A$ such that in (x_0, t_0) the functions v and $\tilde{v}$ touch. By the definition of viscosity solution and using v_ε as test function for $\tilde{v}$ we have

$$v_{\varepsilon t} \le mv_\varepsilon^{1-\frac{1}{m}}\Phi(\Delta v_\varepsilon),$$

while on the other hand v_ε is the solution of (P_ε), hence

$$v_{\varepsilon t} = mv_\varepsilon^{1-\frac{1}{m}}\Phi_\varepsilon(\Delta v_\varepsilon) > mv_\varepsilon^{1-\frac{1}{m}}\Phi(\Delta v_\varepsilon),$$

a contradiction. $\square$

Corollary 3.5. *The maximal viscosity solution is uniquely determined by its initial data. In particular, it is independent of the approximations used in constructing it. The Comparison Principle applies to maximal viscosity solutions.*

4. More estimates. Continuity in several space dimensions

In order to establish a modulus of continuity of the solutions in several dimensions we use the iterative techniques now classical in nonlinear elliptic and parabolic equations. We follow the outline of the treatment by Caffarelli and Friedman [CF] of the porous medium equation as adapted by Gilding and Peletier in [GP]. All we have to do is show that the estimates in [GP] hold for our approximating solutions u_ε, uniformly in ε.

The first step is to obtain a u_t-estimate in the style of Aronson and Bénilan [AB].

Lemma 4.1. *Let u_ε be a solution of (P_ε) as before. We have*

$$u_{\varepsilon t} \geq -\frac{u_\varepsilon}{(m-1)t}\,. \tag{4.1}$$

Proof. Let $v_\varepsilon = u_{\varepsilon t}$. We introduce the function

$$z_\varepsilon = (m-1)tv_\varepsilon + u_\varepsilon\,.$$

Omitting the subscript ε from u_ε and v_ε, and differentiating the equation we find that z satisfies the equation

$$F'_\varepsilon(v)z_t = \Delta(mu^{m-1}z) + m(F'_\varepsilon(v)v - F_\varepsilon(v))\,.$$

The solutions have to be smooth for the differentiation to be justified. This is done in the appendix. Now we use a special property of the functions F_ε, namely that $F'_\varepsilon(s)s \geq F_\varepsilon(s)$. It follows that $z = z_\varepsilon$ is a classical supersolution of a uniformly parabolic equation of the form $a(x,t)z_t = \Delta(b(x,t)z)$. Clearly on the boundary we have $z \geq 0$ so by the Maximum Principle $z \geq 0$ throughout the whole domain. $\square$

Corollary 4.2. *For every $0 < t_1 < t_2 < T$ we have*

$$u(x,t_1) \leq u(x,t_2)(t_2/t_1)^{1/(m-1)}. \tag{4.2}$$

We will also use a representation lemma, cf. [GP, Lemma 2]. We recall the notation

$$\fint_B f(x)dx = \frac{1}{|B|}\int_B f(x)dx = \frac{1}{\omega_N r^N}\int_B f(x)dx,$$

where ω_N is the measure of the unit ball in $\mathbf{R}^N$. We restrict our attention to $N > 2$.

Lemma 4.3. *For every real function $f \in C^2(B)$, B a ball in $\mathbf{R}^N$ with center x_0 and radius r we have*

$$f(x_0) = \fint_B f(x)dx - a_N\int_B G(|x-x_0|;r)\Delta f(x)dx \tag{4.3}$$

where

$$G(\rho, r) = \frac{1}{\rho^{N-2}} - \frac{1}{r^{N-2}} + \frac{N-2}{2}\frac{\rho^2 - r^2}{r^N},$$

and

$$a_N = \frac{1}{N(N-2)\omega_N}.$$

Note that $G > 0$ *for* $0 < \rho < r$.

This representation really follows from an adaptation of the proof of the the mean value theorem for sub- and superharmonic functions, see e.g. [GT].

In [GP] the proof of interior continuity rests on two inequalities derived from the previous lemmas. We will show next that those inequalities can be obtained also for the solutions of our regularised problem. We recall that $M = \sup_Q\{u(x,t)\}$.

Lemma 4.4. *Let* $(x_0, t_0) \in Q$ *and suppose that* $B_0 = B(x_0; r) \subset B_A$. *Then* $v = v_\varepsilon$ *satisfies*

$$v(x_0, t_0) \leq \frac{r^2}{2(N+2)}\left(\frac{M(1-|\gamma|)}{(m-1)t_0} - \varepsilon\right) + \fint_{B_0} v(x, t_0)\, dx. \tag{4.4}$$

Clearly the ε in (4.4) is completely harmless.

Proof. We apply Lemma 4.2 to the function $f(x) = v(x, t_0) + C|x - x_0|^2$. We have, using Lemma 4.1,

$$\Delta v(x, t_0) = F_\varepsilon(u_t) \geq F_\varepsilon\left(-\frac{u}{(m-1)t_0}\right) \geq -\frac{M(1-|\gamma|)}{(m-1)t_0} - \varepsilon,$$

so $\Delta f(x) \geq 0$ if we choose

$$C = \frac{1}{2N}\left(\frac{M(1-|\gamma|)}{(m-1)t_0} - \varepsilon\right).$$

Lemma 4.2 then gives

$$v(x_0, t_0) \leq \fint_{B_0} v(x, t_0)\, dx + C \fint_{B_0} |x - x_0|^2\, dx,$$

whence (4.4) follows. $\square$

The following bound from below needs a more detailed attention to the peculiarities of the MPME.

Lemma 4.5. *Under the same assumptions we have for $x_0 \in B_0 \subset B_A$ and $0 < t < t_0 < T$:*
(4.5)
$$\oint_B u^m(x,t)dx \le u^m(x_0,t_0)\left(\frac{t_0}{t}\right)^{\frac{m}{m-1}} + \left(\frac{1+|\gamma|}{t_0-t} + \frac{2|\gamma|}{(m-1)t}\right)\frac{Mr^2}{2(N+2)}\left(\frac{t_0}{t}\right)^{m/(m-1)}.$$

Again, the new term with γ/t is harmless, provided we keep the ratio t/t_0 away from zero.

Proof. We apply Lemma 4.2 to $u^m(x,t_1)$ in B_0 with $0 < t < t_1 < t_0$ and use Corollary 4.2 to get

$$\left(\frac{t}{t_1}\right)^{m/(m-1)} \oint_{B_0} u^m(x,t)dx \le \left(\frac{t_0}{t_1}\right)^{m/(m-1)} u^m(x_0,t_0)$$
$$+a_N \int_{B_0} G(|x-x_0|,r)\Delta u^m(x,t_1)dx.$$

Integration in t_1 from t to t_0 gives, after some tidying up, and using the mean value theorem,

$$\oint_{B_0} u^m(x,t)dx - \left(\frac{t_0}{t}\right)^{m/(m-1)} u^m(x_0,t_0) \le \tag{4.6a}$$
$$= \frac{a_N}{m-1} t_0^{-m/(m-1)}\left(t^{-1/(m-1)} - t_0^{-1/(m-1)}\right)^{-1}\left(\frac{t_0}{t}\right)^{m/(m-1)} I \le \frac{a_N}{t_0-t}\left(\frac{t_0}{t}\right)^{m/(m-1)} I,$$

where

$$I = \int_t^{t_0}\int_{B_0} G(|x-x_0|;r)F_\varepsilon(u_t(x,t_1))dxdt_1 \tag{4.6.b}$$

still has to be estimated. In [GP, Lemma 4] this integral contains simply the term u_t which can be integrated with respect to time. Here however this is not possible. In order to proceed we observe that for $\gamma \le 0$ we have

$$F_\varepsilon(s) \le F(s) = (1+|\gamma|)s,$$

while for $\gamma > 0$

$$F_\varepsilon \le F(s) = (1+\gamma)s + 2\gamma s^-.$$

Summing up and using Lemma 4.1 we get in both cases

$$F_\varepsilon(u_t) \le (1+|\gamma|)u_t + \frac{2|\gamma|u}{(m-1)t}. \tag{4.7}$$

We can now estimate the integral (4.6.b). Since $0 \le G$, and $0 < u < M$, using (4.7) we get

$$I \le \left(1+|\gamma| + \frac{2|\gamma|(t_0-t)}{(m-1)t}\right)M\int_{B_0} G(|x-x_0|;r)dx.$$

From

$$a_N \int G(|x-x_0|;r)dx = \frac{r^2}{2(N+2)}$$

formula (4.5) immediately follows. $\square$

Once these estimates are obtained the existence of a uniform modulus of continuity for the solutions u_ε is standard from the references [CF], [GP].

Theorem 4.6. *Uniformly bounded families solutions of the problems* (P_ε), $0 < \varepsilon < \varepsilon_0$ *are relatively compact in the uniform norm on compact subsets of* Q.

The techniques in [CF] even imply the uniform Hólder continuity for the sequence of approximating solutions, as well as for its limit, the maximal viscosity solution.

5. Finite speed of propagation

For the standard porous medium equation, the finite speed of propagation property follows from the comparison principle and the choice of an appropiate comparison function with bounded support. Since in our case we have not established uniqueness of viscosity solutions, the comparison argument has to be done via the regularised problems. As comparison functions one can use either the radially symmetric and compactly supported self-similar solution constructed in [HV], or travelling wave solutions. It is more convenient here to work with the latter, because these solutions have no sign change in u_t (indeed, $u_t > 0$ for $u > 0$). Thus, we just have the standard porous medium equation

$$(1+\gamma)u_t = \Delta(u^m)\,, \tag{5.1}$$

which has travelling wave solutions of the form (in terms of v)

$$v(x,t) = V(\eta) = (\frac{m-1}{m}c\eta)_+^{\frac{m}{m-1}}\,,\ \eta = x_i - B + \frac{c}{1+\gamma}t\,, \tag{5.2}$$

where $i = 1, 2, \cdots, N$, and B and c are arbitrary real constants.

Theorem 5.1. *A travelling wave is a maximal viscosity solution. Thus, every viscosity solution* v *of (1.3) which is below a travelling wave* $\bar{v}$ *initially remains below this travelling wave for all positive* t*. Consequently, if the support (of a bounded solution) is bounded initially, it remains bounded.*

Proof. We first approximate the travelling wave solution $\bar{v}$ by a travelling wave solution which stays away from zero. Recall that the differential equation for V reads

$$c(V^{\frac{1}{m}})' = V''\,, \tag{5.3}$$

and that the travelling wave profile follows from integrating this equation setting $V = 0$ at $-\infty$. Here we restrict to $c > 0$. For the approximations we set $V = \varepsilon^m$ at $-\infty$. This yields

$$V_\varepsilon' = c(V_\varepsilon^{\frac{1}{m}} - \varepsilon)\,, \tag{5.4}$$

which we solve with initial condition

$$V_\varepsilon(\eta_M) = M^m\,, \text{where } V(\eta_M) = M^m\,. \tag{5.5}$$

It is easy to see that $V_\varepsilon \downarrow V$ uniformly on $(-\infty, \eta_M)$, and that $V_\varepsilon > \varepsilon$. The corresponding travelling wave is a smooth solution of the modified porous medium equation, and if we replace η by

$$\eta = x_i - B + (\varepsilon + \frac{c}{1+\gamma})t\,, \tag{5.6}$$

where we tacitly assume $c > 0$, it satisfies the equation with equality replaced by "$>$". Consequently, it follows that this travelling wave, which we denote by $\bar{v}_\varepsilon$, and by construction and assumption is strictly above v initially, must stay above v for all $t > 0$. For suppose not, then there must be a minimal t_0 and some $x_0 \in (-A, A)$ such that in (x_0, t_0) the functions v and $\bar{v}_\varepsilon$ touch. Since v is a viscosity (sub-)solution, applying the definition, $\varphi = \bar{v}_\varepsilon$ is forced to satisfy the MPME with equality replaced by "$\leq$", a contradiction. Thus v is below all $\bar{v}_\varepsilon$ if initially it is below $\bar{v}$. Letting $\varepsilon \to 0$ completes the proof. □

6. The semigroup property for maximal solutions

Theorem 6.1. *The maximal viscosity solutions of (1.3) have the semigroup property.*

Proof. Let us be more precise about the statement. Let X be the set of continuous, nonnegative and compactly supported real functions defined in $\mathbf{R}^N$. For $t > 0$ we denote by S_t the map which assigns to an initial function $v_0 \in X$ the maximal viscosity solution $v(x,t)$ (at fixed t). We want to prove that for every pair $t_1, t_2 > 0$ we have $S_{t_1+t_2} = S_{t_1} \circ S_{t_2}$.

Take first t_1 and consider the maximal solution $v^*(x,t)$ of the Cauchy problem with initial data given by $v_1(x) = v(x, t_1)$. The above assertion amounts to show that $v^*(x,t)$ coincides with $v(x, t - t_1)$ for $t > t_1$. To do so we consider again the regularisations v_ε of v. We regularise v_1 as before by smooth functions $v_{1\varepsilon}$, which we choose strictly between $v(x, t_1)$ and $v_\varepsilon(x, t_1)$. This is possible because the functions v_ε are strictly increasing in ε. For initial data v_1 we have now two families of approximating solutions, namely $v_\varepsilon(x, t-t_1)$, and the solution of the regularised equation with initial data $v_{1\varepsilon}$. Denote this solution by $v^*_\varepsilon(x,t)$. By similar reasoning as before it follows that $v(x, t-t_1) < v^*_{\varepsilon'}(x,t) < v_\varepsilon(x, t-t_1)$ for $0 < \varepsilon' < \varepsilon$. Taking the limit $\varepsilon \to 0$ it follows that $v(x, t-t_1) = v^*(x,t)$.

Theorem 6.2. *The compactly supported self-similar solutions constructed in [HV] are maximal viscosity solutions.*

Proof. We first observe that these self-similar solutions are viscosity solutions. Indeed, in points where they are positive this is straightforward, and in points where they are zero one only has to check the sign of φ_t, which is also correct, since $v_t = 0$ there.

Next we approximate the similarity solution from above by a family of self-similar strictly positive strict supersolutions, parametrised by ε. Here, by a strict supersolution we mean that the equation is satisfied with equality replaced by "$>$". Postponing the construction for the moment, let us complete the argument.

Assume that u is our compactly supported self-similar solution, and that $\overline{u}$ is a strictly positive self-similar strict supersolution lying strictly above u. We claim that every viscosity solution $\tilde{u}$ of the Cauchy problem has to be below $\overline{u}$. Suppose not, then again we have that there has to be a minimal $t_0 > 0$ such that $\tilde{u}$ touches $\overline{u}$ from below in a point x_0. We now take $\varphi = \overline{u}$ in the definition of a viscosity subsolution. In terms of v it follows that in this point $\bar{v}$ has to satisfy the equation with equality replaced by "$\leq$", a contradiction.

It remains to show that we can approximate u from above by strictly positive strict supersolutions. Here we have to recall some properties of the ordinary differential equation for the similarity profiles (5.7), which reads

$$(U^m)'' = F(-\eta U' - kU)\,, \tag{6.1}$$

where

$$2\beta + (m-1)\alpha = 1\,, \quad k = \frac{\alpha}{\beta}\,. \tag{6.2}$$

Now suppose, without loss of generality, that we are dealing with the similarity solution with $U(0) = 1$ and $\tau = 1$. Leaving β fixed, we solve (6.1) with

$$U_\delta(0) = 1 + \delta\,, \quad \alpha_\delta = \alpha - \delta\,, \tag{6.3}$$

and define

$$\overline{u}_\delta(x,t) = (t+\tau)^{-\alpha+\delta} U_\delta(\eta)\,, \quad \eta = \sqrt{\beta}\, x\, (t+\tau)^{-\beta}\,. \tag{6.4}$$

It follows from the results in [H1] and [HV] that the profile U_δ is strictly positive, strictly above U, decays as η^{-k_δ}, where $k_\delta = \alpha_\delta/\beta$, and that $U_\delta \downarrow U$ uniformly as $\delta \downarrow 0$. Hence the same holds for $\overline{u}_\delta$, and the δ in the exponent of $(t+\tau)$ guarantees that $\overline{u}_\delta$ is a strict supersolution. This completes the proof. $\square$

7. Asymptotic behaviour

In this section we consider the compactly supported maximal viscosity solution $u(x,t)$ of the Cauchy Problem for the MPME. We will prove that the family of compactly supported self-similar solutions $U_c(x,t) = U(x,t;c) = U(x,t;c,m.\gamma)$ with anomalous exponents as constructed in [HV], describes the large-time behaviour of all such solutions. Note that we may take the parameter c to be given by $c = f(0)$ where f is the similarity profile in (0.6).

Theorem 7.1. *There exists a constant $c_* > 0$ depending on u such that*

$$t^\alpha |u(x,t) - U(x,t;c_*)| \to 0 \quad \text{as} \quad t \to \infty \tag{7.1}$$

uniformly in x. Here $\alpha = \alpha(m,\gamma,N)$ is the anomalous decay exponent of the self-similar solutions $U(x,t;c,m,\gamma)$.

Proof. The basic idea is to fit a solution U_c as closely as possible from below to our solution u. We divide the proof in a series of lemmas.

Lemma 7.2. *There exist constants C and $c > 0$ (depending on u) such that*

$$U(x,t;c) \le u(x,t) \le U(x,t;C) \quad \text{in} \quad Q\,. \tag{7.2}$$

Proof. We compare the functions $U(x,t;C)$, $u(x,t)$ and $U(x,t;c)$, with $C > c > 0$. If C is large and c is small enough we have inequality (7.2) for $t = 0$ and $x \in \mathbf{R}^N$, hence the conclusion follows from the Comparison Principle. $\square$

We can now define for t the optimal lower bound

$$c(t) = \sup\{c > 0 : u(x,t) \ge U(x,t;c) \text{ for } x \in \mathbf{R}^N\}. \tag{7.3}$$

As a consequence of Lemma 7.2 the function $c(t)$ is positive and finite, $0 < c(t) \le C$. It is a simple consequence of the Maximum Principle that $c(t)$ is a nondecreasing function of t. Therefore, there exists the limit

$$c_* = \lim_{t\to\infty} c(t) \tag{7.4}$$

and this limit is positive and finite. Next we introduce the rescaling operator $\mathcal{T}_k$, with parameter $k > 0$, which acts on solutions of the MPME according to the formula

$$(\mathcal{T}_k u)(x,t) = k^\alpha u(k^\beta x, kt) \tag{7.5}$$

to produce a new solution of the MPME. Moreover, U is invariant under such a transformation, $\mathcal{T}_k U(x,t;c) = U(x,t;c)$. Let us abbreviate $\mathcal{T}_k u = u_k$. Given a solution $u(x,t)$ under the assumptions of Theorem 7.1 we consider the family $\{u_k(x,t)\}$ for $k \ge 1$. By the compactness results for bounded solutions of (0.1) we may extract from every sequence in this family a convergent subsequence (on compact subsets of Q). Let

$$\tilde u(x,t) = \lim_{j\to\infty} u_{k_j}(x,t)\,. \tag{7.6}$$

We have

$$U(x,t;c_*) \le \tilde u(x,t) \le U(x,t;C) \quad \text{in} \quad Q\,. \tag{7.7}$$

If we prove that necessarily $\tilde u = U_{c_*}$ the theorem will follow. Unfortunately we cannot conclude at this stage that $\tilde u$ is a maximal viscosity solution. This is due to the open question of uniqueness. Thus any comparison arguments we wish to use have to refer to the solutions u_k.

We know that $\tilde u \ge U_{c_*}$ for all times, see (7.7). Now, since U_{c_*} is a maximal solution, if they coincide identically for some $t = t_1 > 0$ they must coincide for all later times. In this case our theorem is proved. So, to arrive at a contradiction **we may assume that** $\tilde u(\cdot,t) \not\equiv U_{c_*}(\cdot,t)$ for all large times, say $t \ge 1$. For any $\varepsilon > 0$ there exists K_ε such that

$$||u_k - \tilde u||_\infty < \varepsilon\,, \tag{7.8}$$

for all $k = k_j \geq K_\varepsilon$ and fixed $t = 1$. For any sequence $\varepsilon_n \downarrow 0$ we can thus take a sequence $k_n \uparrow \infty$ for which (7.8) holds with $k = k_n$ and $\varepsilon = \varepsilon_n$. Hence

$$u_k(x,1) \geq (U_{c_*}(x,1) - \varepsilon)_+ \,. \tag{7.9}$$

Clearly, taking the positive part on the right-hand side is allowed because all the functions are nonnegative. Now we make a small scaling which leaves the equation invariant, and puts the right-hand side of (7.9) on top of U_{c_*}. This scaling can be taken of the form

$$(T_{\mu_\varepsilon} u)(x,t) = \mu_\varepsilon u(\mu_\varepsilon x, \mu_\varepsilon^{m+1} t)\,, \tag{7.10}$$

where $\mu_\varepsilon > 1$ and $\mu_\varepsilon \to 1$ as $\varepsilon \to 0$. Applying the same scaling to u_{k_n} we obtain from (7.9) that at time $t = 1$

$$v_n \equiv T_{\mu_{\varepsilon_n}} u_{k_n} \geq U_{c_*} \,. \tag{7.11}$$

But now both sides are maximal viscosity solutions, so (7.11) is valid for all $t \geq 1$. Also by construction we have that $v_n \to \tilde{u}$.

We can now choose a function $\psi(x)$ between $v_n(x,1)$ and $U_{c_*}(x,1)$ such that for all n large enough these three functions are strictly ordered on some fixed open set. Let $u_\psi = u_\psi(x,t)$ be the maximal solution of the MPME with initial data $\psi(x)$ at $t = 1$.

Lemma 7.3. *The maximal solutions u_ψ and U_{c_*} with initial data given by ψ and $U_{c_*}(x,1)$ at $t = 1$ are strictly ordered wherever the U_{c_*} is positive and $t > 1$.*

Proof. This follows from the Strong Comparison Principle in the positivity region. See the Appendix. □

If we can also separate the supports of the two solutions in Lemma 7.3, then we can improve the constant c_*, and thereby force a contradiction. Because of Lemma 7.3 this can be done by introducing a small delay.

Lemma 7.4. *There exists $\delta > 0$ and $\tau > 0$ such that*

$$u_\psi(x,t) \geq U(x, t-\tau; c_* + \delta), \quad \text{for } t \geq t_1 > 1. \tag{7.12}$$

Proof. For $t > 1$ we have by Lemma 7.3 that $u_\psi(x,t)$ is strictly larger than $U(x,t;c_*)$ on the support of U_{c_*}. This implies that given $t_1 > 1$ there exists a small $\tau > 0$ such that this strict ordering also holds for $u_\psi(x,t_1)$ and $U_{c_*}(x,t_1-\tau)$ and in addition also the corresponding interfaces are strictly separated. But then we can choose a small $\delta > 0$ such that $u_\psi(x,t_1) \geq U_{c_*+\delta}(x,t_1-\tau)$. By the comparison principle for maximal solutions, this ordering is preserved for all $t \geq t_1$. A similar argument was used in [FK]. □

Lemma 7.5. *If $\tilde{u} \not\equiv U_{c_*}$ then the constant c_* can be improved.*

Proof. We continue from (7.12) which implies that also for the sequence v_n, provided n is large,

$$v_n(x,t) \geq U(x, t-\tau; c_* + \delta), \quad \text{for } t \geq t_1. \tag{7.13}$$

Since the difference between v_n and u_k with $k = k_n$ becomes arbitrarily small for fixed t, we have for some large k that

$$u_k(x,t) \geq U(x, t-\tau; c_* + \delta/2), \quad \text{for } t \geq t_1. \tag{7.13}$$

Undoing the transformation this means that

$$u(k^\beta x, kt_1) \geq U(k^\beta x, k(t_1 - \tau); c_* + \delta/2)\,, \tag{7.14}$$

whence

$$c(k^\beta t_1) > c_* \tag{7.15}$$

By the monotonicity of $c(t)$ we get

$$\lim_{t\to\infty} c(t) > c_*\,. \qquad \Box \tag{7.16}$$

By the very definition of c_* this conclusion is impossible. We must therefore have $\tilde{u} \equiv U_{c_*}$. This proves that u_k converges to U_{c_*} along every sequence $k_j \to \infty$, hence the whole family $\{u_k(x,t)\}$ converges to U_{c_*} as $k \to \infty$. Undoing the rescaling (7.5) we obtain precisely the conclusion of the Theorem.

Remark. The idea of approaching the solutions from below with self-similar solutions was already used in [FK], where the porous medium equation was first treated in all space dimensions. However, the law of mass conservation allows in that case to know from the beginning the value of the asymptotic constant, c_*, a considerable help. This makes our work more related to the study of the asymptotic behaviour of the solutions of the Barenblatt elasto-plastic filtration model (equation (0.1) with $m = 1$) in [KPV]. As explained in detail in [KPV] the existence of an asymptotic constant for these equations is a very important information that follows from the Theorem. In fact, the constant provides us with an invariant quantity to replace the missing conservation of mass. A similar situation occurs in the study of turbulence bursts in [KV].

Appendix. Regularity

In this appendix we first discuss the regularised problem and its solution in Section 2. Because this is not the main issue of this paper we shall be somewhat sketchy. It is convenient to write the regularised equation as

$$v_t = v^\beta \Phi(\Delta v)\,. \tag{A.1}$$

To simplify notation we have written $\beta = 1 - 1/m \in (0,1)$ and absorbed the factor m in the time derivative. Also the subscript ε is omitted. We have to solve (A.1) with the parabolic boundary conditions for v as corresponding to (P_ε). Moreover, we want a smooth solution, because otherwise our manipulations with v_ε throughout this paper cannot be justified. All this will follow from very recent results of the school of Cafarelli, and in particular from the work of Wang. In his thesis he establishes the existence of a viscosity solution between an ordered pair of sub- and supersolutions (see also the work of Crandall, Ishii and Lions). We observe that for our (regularised) problem one can construct such a sub- and a supersolution which also satisfy the parabolic boundary conditions. Indeed, if we bound the function Φ from below or above, depending on the sign of γ, by a linear function with positive slope, we obtain an equation of porous medium type with a strictly positive smooth solution. If $\gamma > 0$ this gives a supersolution and If $\gamma < 0$ this gives a subsolution. The complementary sub- and supersolution can be constructed as suprema or infima of functions of the form $v(x,t) = v_0(x) \pm f(t)$ and equilibrium solutions satisfying the lateral boundary conditions.

It follows from the regularity results of Wang that the solution thus obtained is globally in $C^{2,\alpha}$, i.e. $C^{2+\alpha,1+\alpha/2}_{x,t}$ in standard notation. An intermediate step involves freezing the nonlinear coefficient v^β. Wang then proves his results for the case that Φ is convex. However, by switching to $w = -v$, the results are also valid for concave Φ. Once we know that the regularised solution has this regularity, there are two points which remain to be checked. Firstly, we must justify the equation for z in Section 4, which follows from differentiating (A.1) with respect to t. This can be done by finite time differences. We write

$$w_h(x,t) = v(x,t) - v(x,t-h)\,; \quad z_h = \frac{w_h}{h}\,. \tag{A.2}$$

Then both w_h and z_h satisfy the equation

$$z_t = A\Delta z + Bz\,, \tag{A.3}$$

where

$$A(x,t) = A_h(x,t) = v^\beta \int_0^1 \Phi'(s\Delta v + (1-s)\Delta v_h)ds\,, \tag{A.4}$$

and

$$B(x,t) = B_h(x,t) = \beta \int_0^1 (sv + (1-s)v_h)^{\beta-1} ds \Phi(\Delta v_h)\,. \tag{A.5}$$

As $h \to 0$ the functions A_h, B_h and z_h converge in C^α. Thus v_t is a $C^{2,\alpha}$-solution of (A.3) with coefficients

$$A = A_0 = v^\beta \Phi'(\Delta v)\,; \quad B = B_0 = \beta v^{\beta-1}\Phi(\Delta v)\,. \tag{A.6}$$

Hence it is allowed to differentiate the equation for v_ε with respect to t.

Finally we observe that for the limiting equation the same regularity results hold in the regions where the viscosity solution is positive. The positivity set can thus be divided in a closed set where the time derivative is zero, and an open set where it is nonzero. In the latter set the solution is C^∞. Concerning the Strong Comparison Principle we note that the difference z of two viscosity solutions v and w satisfies equation (A.3) with

$$A = v^\beta \int_0^1 \Phi'(s\Delta v + (1-s)\Delta w)ds\,, \tag{A.7}$$

and

$$B = \beta \int_0^1 (sv + (1-s)w)^{\beta-1} ds \Phi(\Delta w)\,. \tag{A.8}$$

The coefficients are now piecewise continuous. Nevertheless, uniform parabolicity still implies that the Strong Comparison Principle as applied in Section 7, Lemma 7.3, holds.

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