A local analysis of similarity solutions of the thin film equation

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Abstract

We classify singularities of the equation $(f^n f''')' = \beta\eta f' + \alpha f$. Solutions of this equation are similarity profiles of the equation $u_t + (u^n u_{xxx})_x = 0$, which arises in the description of thin film viscosity driven flow. The classification is based upon a transformation to a 4-dimensional quadratic system and a local critical point analysis.

Key Words and Phrases. Similarity solutions, 4-dimensional quadratic dynamical systems, Poincaré transformation, critical points.

1 Introduction

In this paper we study similarity solutions of the the following 4th order degenerate diffusion equation.

$$u_t + (u^n u_{xxx})_x = 0. \tag{1.1}$$

Here $u = u(x,t)$ and $n > 0$. This equation arises in the modelling of thin film slow viscous flows such as painting layers. Let us briefly go through the derivation of this model. We follow [10].

Consider a two-dimensional thin flow on a horizontal plate. The flow domain is described by $\{(x,y) : \ 0 \le y \le h(x,t)\}$. Assuming the inertia terms to be dominated by the viscosity terms, the Navier-Stokes equations reduce to

$$u_x + v_y = 0; \quad -\frac{1}{\rho}p_x + \nu u_{xx} = 0; \quad -\frac{1}{\rho}p_y + \nu v_{yy} = g. \tag{1.2}$$

Here u and v are respectively the velocities in the horizontal x- and vertical y-direction, p is pressure, g gravity, ν viscosity and ρ density. Assuming that also νv_{yy} is neglible one obtains upon integration of the last equation with respect to y that $p = \rho g(h-y) - h_{xx}\sigma$, where $\sigma > 0$ reflects surface tension. If this term is dominant, then the pressure is simply $p = -h_{xx}\sigma$. Going back to (1.2) this leads to

$$u_{yy} = -\frac{\sigma}{\mu}h_{xxx}, \quad \mu = \nu\rho. \tag{1.3}$$

On the surface of the flow we have a no-stress condition $u_y = 0$, while on the horizontal plate $u = v = 0$ (no-slip condition). Thus

$$u = -(y^2 - 2hy)\frac{\sigma}{2\mu}h_{xxx} \quad \Rightarrow \quad v_y = -u_x = (y^2-2hy)\frac{\sigma}{2\mu}h_{xxxx} - y\frac{\sigma}{\mu}h_x h_{xxx}. \tag{1.4}$$

Integration with respect to y then yields

$$v = (\frac{y^3}{3} - hy^2)\frac{\sigma}{2\mu}h_{xxxx} - \frac{y^2}{2}\frac{\sigma}{\mu}h_x h_{xxx}. \tag{1.5}$$

Finally we must have at the surface of the flow that the convective derivative of $y - h(x,t)$ vanishes, so that on this surface

$$h_t = -uh_x + v, \tag{1.6}$$

in which we substitute the above expressions with y replaced by h to obtain

$$h_t = -h^2 \frac{\sigma}{\mu} h_{xxx} h_x - \frac{h^3}{3} \frac{\sigma}{\mu} h_{xxxx} = -\frac{\sigma}{\mu} (h^3 h_{xxx})_x. \tag{1.7}$$

This is (1.1) with $n = 3$. There are several variants of this derivation which include second and first order terms corresponding respectively to gravity and an inclination of the plate, see [9]. Also there are physical applications for other values of n, for references see [1] and [6].

In all cases the interest is in nonnegative solutions and there are several rigorous mathematical results for various ranges of the parameter n in this direction, notably by Bernis and Friedman [3], Beretta, Bertsch and Dal Passo [1], Bertozzi and Pugh [5], all using a weak formulation of (1.1). For $n \geq 4$ an approximation with regularized uniformly parabolic 4th order equations leads to solutions which remain positive if they are positive initially. This can then be used to construct nonnegative solutions for all $n > 0$, approximating the nonlinearity with a function which behaves locally like u^4. We refer to [1] for a precise description. By means of ordinary differential equation techniques, Bernis, Peletier and Williams [4] have obtained selfsimilar, compactly supported nonnegative source type solutions for $0 < n < 3$, satisfying $u = u_x = u^n u_{xxx} = 0$ at the boundary of the support. These source type solutions are weak solutions in the sense of the usual definition, where the integral of the space derivatives is taken only over de positivity set. This is closely related to the remarkable difference between the results for $n > 0$ and the classical theory for the linear equation, which does not have any positivity property. Indeed, for $n = 0$, the (selfsimilar) fundamental solution has a profile which is the inverse Fourier transform of $\exp(-x^4)$ and changes sign. Therefore, as $n \to 0$, the source type solutions in [4] cannot be expected to converge to solutions of the linear equation, and in fact they do not. What one obtains in the limit is a selfsimilar solution of a free boundary problem for the linear equation, the free boundary conditions being $u = u_x = u_{xxx} = 0$.

It seems to be fair to say that the concept of nonnegative weak solutions is not yet understood. This has motivated researchers to do a local analysis of solutions of ordinary differential equations related to (1.1). This was done for travelling wave solutions in [6]. A more global analysis was recently carried out in [2]. It is the modest aim of this paper to carry out the local analysis for selfsimilar solutions. It is hoped that this analysis will be of benefit to the development of a complete theory for (1.1). The approach is based on the techniques in [7] and [8]. The computations have been carried out using Maple 5 release 2.

We consider selfsimilar solutons of the form

$$u(x,t) = t^{-\alpha} f(xt^{-\beta}), \tag{1.8}$$

where necessarily

$$n\alpha + 4\beta = 1, \tag{1.9}$$

so that

$$(f^n f''')' = \beta\eta f' + \alpha f. \tag{1.10}$$

This ordinary differential equation is singular when $f = 0$. When $f > 0$ it is locally well posed when f, f', f'' and f''' are prescribed at a given $\eta = \eta_0$.

In order to find the possible behaviours of solutions near points where $f = 0$ we employ a transformation of (1.10) into a four-dimensional quadratic dynamic system. We have in fact two of such transformations, one which is better suited for the behaviour at $\eta = 0$, while the other is used for the classification of moving fronts.

2 First transformation: behaviour as $\eta \to 0$

Following [7] we set

$$x = \frac{\eta f'}{f}; y = \frac{\eta^2 f''}{f}; z = \frac{\eta^3 f'''}{f}; u = \eta^4 f^{-n}; t = \log \eta. \tag{2.1}$$

This leads to

$$\begin{aligned} \dot{x} &= x(1-x) + y \\ \dot{y} &= y(2-x) + z \\ \dot{z} &= z(3-(n+1)x) + u(\alpha + \beta x) \\ \dot{u} &= u(4-nx) \end{aligned} \tag{2.2}$$

Any positive solution of (1.10) defined on some open interval in $(0,\infty)$ is mapped into an orbit of this system. Note that (1.10) has a scaling invariance which amounts to a shift in t along orbits of (2.2).

We shall attempt to classify the behaviour of such orbits in forward and backward time. Note that $t \to -\infty$ and $t \to +\infty$ correspond to $\eta \to 0$ and $\eta \to +\infty$ respectively. If an orbit is unbounded it may and indeed does happen that the time interval of existence is bounded from above or below. This corresponds to solutions f which exist one some η-interval bounded away from $\eta = \infty$ or $\eta = 0$.

Orbits of dynamical systems may have very complicated ω-limit sets if the dimension is larger than 2. Below we only examine the critical points of (2.2) and in particular their linear structure. This yields information about singular solutions of (1.10). The critical points are

$$[0,0,0,0], \quad [1,0,0,0], \quad [2,2,0,0], \tag{2.3}$$

$$[\frac{3}{n+1}, -\frac{-6+3\,n}{(n+1)^2}, \frac{(6\,n-3)\,(-2+n)}{(n+1)^3}, 0], \tag{2.4}$$

and

$$[\frac{4}{n}, -\frac{4\,n-16}{n^2}, \frac{(-16+8\,n)\,(n-4)}{n^3}, \frac{(8\,n+32)\,(-2+n)\,(n-4)}{n^3\,(\alpha\,n+4\,\beta)}]. \tag{2.5}$$

Orbits coming out of these critical points into the set $\{u > 0\}$ correspond to solutions $f(\eta)$ defined on some interval $(0, a)$ with $0 < a \leq \infty$, while orbits going into these critical points out of $\{u > 0\}$ correspond to solutions $f(\eta)$ defined on some interval $(b, +\infty)$ with $0 \leq b < +\infty$. Also, a critical point with a positive u-coordinate, corresponds to an explicit solution, namely a power with exponent $n/4$.

2.1 The critical point $[0,0,0,0]$, $f(0) > 0$

The linearization of this point is

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & \alpha \\ 0 & 0 & 0 & 4 \end{bmatrix} \tag{2.6}$$

Thus the eigenvalues are $1, 2, 3$ and 4 and this point is a source. It is not difficult to see that every (regular) solution of (1.10) with $f(0) > 0$ is mapped into such an orbit. These solutions are parametrized by 4 parameters, namely $f(0), f'(0), f''(0)$ and $f'''(0)$. Throughout this paper we shall denote this number by d. So here $d = 4$. This d is in fact the dimension of the unstable manifold $\mathcal{U}$, provided this is not contained in the set $\{u = 0\}$.

2.2 The critical point $[1, 0, 0, 0]$, linear behaviour

The linearization of this point is

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2-n & \alpha+\beta \\ 0 & 0 & 0 & 4-n \end{bmatrix} \tag{2.7}$$

Eigenvalues are $-1, 1, 2-n$ and $4-n$. The eigenvector of the eigenvalue $4-n$ is needed to obtain orbits in the unstable manifold $\mathcal{U}$ with $u > 0$. These contain solutions with linear behaviour near zero in the sense that

$$\lim_{\eta \downarrow 0} \frac{\eta f'(\eta)}{f(\eta)} = 1. \tag{2.8}$$

For the linear equation ($n = 0$) these solutions have 3 degrees of freedom. From the matrix we see that for $0 < n < 2$ we have $d = 3$, and for $2 < n < 4$ we have $d = 2$. For $n > 4$ the unstable manifold $\mathcal{U}$ has dimension 1 but is contained in $\{u = 0\}$ and therefore $d = 0$.

2.3 The critical point $[2, 2, 0, 0]$, quadratic behaviour

The linearization of this point is

$$\begin{bmatrix} -3 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 1-2n & \alpha+2\beta \\ 0 & 0 & 0 & 4-2n \end{bmatrix} \tag{2.9}$$

Orbits in $\mathcal{U}$ contain solutions with quadratic behaviour near zero in the sense that

$$\lim_{\eta \downarrow 0} \frac{\eta f'(\eta)}{f(\eta)} = 2. \tag{2.10}$$

The eigenvalues are $-2, -1, 1-2n, 4-2n$. Thus for $0 < n < 1/2$ we have $d = 2$, just as in the linear case $n = 0$. For $1/2 < n < 2$ we have $d = 1$, and $d = 0$ for $n > 2$.

2.4 $[\frac{3}{n+1}, -\frac{-6+3\,n}{(n+1)^2}, \frac{(6\,n-3)(-2+n)}{(n+1)^3}, 0]$, exponent $\frac{3}{n+1}$

The linearization of this point is

$$\begin{bmatrix} \frac{n-5}{n+1} & 1 & 0 & 0 \\ \frac{-6+3\,n}{(n+1)^2} & \frac{2\,n-1}{n+1} & 1 & 0 \\ -\frac{(6\,n-3)(-2+n)}{(n+1)^2} & 0 & 0 & \frac{\alpha\,n+\alpha+3\,\beta}{n+1} \\ 0 & 0 & 0 & \frac{n+4}{n+1} \end{bmatrix} \tag{2.11}$$

The eigenvalues are $\frac{n+4}{n+1}$, -1 and

$$\frac{4\,n-5 \pm \sqrt{-8\,n^2+20\,n+1}}{2\,n+2} \tag{2.12}$$

These last two are both negative real for $0 \leq n < 1/2$, while for $1/2 < n < 2$ one is negative and one is positive. For $2 < n < (5+\sqrt{27})/4$ both are positive, and for $n > (5+\sqrt{27})/4$ both are complex with positive real part. Thus we find solutions which behave like $\eta^{3/(n+1)}$ near zero in the sense that

$$\lim_{\eta \downarrow 0} \frac{\eta f'(\eta)}{f(\eta)} = \frac{3}{n+1}. \tag{2.13}$$

We have $d = 1$ for $0 \leq n < 1/2$, $d = 2$ for $1/2 < n < 2$ and $d = 3$ for $n > 2$. Unlike the previous two cases, we gain degrees of freedom as we increase n.

2.5 $[\frac{4}{n}, -\frac{4\,n-16}{n^2}, \frac{(-16+8\,n)(n-4)}{n^3}, \frac{(8\,n+32)(-2+n)(n-4)}{n^3(\alpha\,n+4\,\beta)}]$, exponent $\frac{4}{n}$

This is the only point with a nonzero u-coordinate. Note that we have not used (1.9). If we do, we find that for $0 < n < 2$ and $n > 4$ this point corresponds to an explicit solution namely

$$f(\eta) = C\,\eta^{\frac{4}{n}}, \qquad C^n = \frac{n^3\,(\alpha\,n+4\,\beta)}{(8\,n+32)\,(-2+n)\,(n-4)} \tag{2.14}$$

This solution is invariant under the scaling of (1.10).

The linearization in this point is

$$\begin{bmatrix} \frac{n-8}{n} & 1 & 0 & 0 \\ \frac{4\,n-16}{n^2} & \frac{2\,n-4}{n} & 1 & 0 \\ -\frac{(-16+8\,n)(n-4)(\alpha\,n+\alpha+3\,\beta)}{n^2(\alpha\,n+4\,\beta)} & 0 & -\frac{n+4}{n} & \frac{\alpha\,n+4\,\beta}{n} \\ -\frac{(8\,n+32)(-2+n)(n-4)}{n^2(\alpha\,n+4\,\beta)} & 0 & 0 & 0 \end{bmatrix} \tag{2.15}$$

Since the computation of the eigenvalues of this matrix is rather involved we shall use now the relation between α and β. The determinant of this matrix equals

$$\frac{(8\,n+32)\,(-2+n)\,(n-4)}{n^3} \tag{2.16}$$

but a more detailed analysis is not so easy. However for $n = 2$ the eigenvalues are $-3, -2, -1, 0$ and for $n = 4$ they are $-2, -1, 0, 1$. In view of the determinant this means that for $n < 2$ sufficiently close to 2 we have 4 negative eigenvalues, and for $n > 4$ sufficiently close to 4 we have 2 negative eigenvalues and 2 positive eigenvalues. In the latter case we find a two-parameter family of solutions ($d = 2$) behaving like the explicit solution as $\eta \to 0$ (same exponent *and* same constant).

3 Second transformation: $f(\eta) \to 0$ as $\eta \to \eta_0 > 0$

Solutions of (1.10) with this behaviour correspond to orbits becoming unbounded, because they cannot exist for $t > \log \eta_0$. A way to find such orbits is by looking at the critical points at infinity. This is done using the Poincaré transformation. Unfortunately (2.2) is not really suited for this approach, because it contains to many linear terms. This is due to the specific form of the transformation (2.1) from (1.10) to (2.2). If f goes to zero, then it is unlikely that x, y, z, u will be of the same order. There is however a variant of the transformation which is better suited for this case. We set

$$x = \frac{\eta f'}{f}; y = \frac{\eta f''}{f'}; z = \frac{\eta f'''}{f''}; u = \frac{\eta^2 f^{1-n}}{f''}; t = \log \eta. \tag{3.1}$$

This leads to

$$\begin{aligned}
\dot{x} &= x(1 - x + y)\\
\dot{y} &= y(1 - y + z)\\
\dot{z} &= z(1 - z - nx) + u(\alpha + \beta x)\\
\dot{u} &= u(2 + (1 - n)x - z)
\end{aligned} \tag{3.2}$$

Note that (3.2) has more quadratic terms than (2.2). Before we examine the critical points of (3.2) at infinity, we examine the finite critical points and check that the results are in agreement with what we saw in the previous section. These points are now

$$[0,0,0,0], [0,1,0,0], [0,0,1,0], [0,0,2,\frac{2}{\alpha}], [0,2,1,0], [0,3,2,\frac{2}{\alpha}], \tag{3.3}$$

$$[1,0,0,0],[2,1,0,0],[1,0,1-n,0],[\frac{3}{1+n},-\frac{-2+n}{1+n},-\frac{-1+2\,n}{1+n},0], \quad (3.4)$$

$$[1,0,3-n,-\frac{-6+2\,n}{\alpha+\beta}],[\frac{4}{n},-\frac{n-4}{n},-\frac{2\,n-4}{n},-\frac{(2\,n+8)\,(-2+n)}{n\,(\alpha\,n+4\,\beta)}], \quad (3.5)$$

Solutions of (1.10) with $f(0)>0$ are still contained in orbits coming out of critical points with x-coordinate equal to zero. Next we look at critical points with x-coordinate equal to 1. In the case of (2.2) this was only one point, but now there are 3 of these points, namely $[1,0,0,0]$ with eigenvalues $-1,1,1-n,3-n$, $[1,0,1-n,0]$ with eigenvalues $-1,n-1,2-n,2$, and $[1,0,3-n,-\frac{-6+2\,n}{\alpha+\beta}]$ with eigenvalues $-2,-1,n-3,4-n$. We leave it to the reader to check that this is in agreement with the previous section. Having discussed the points with x-coordinate equal to 0 or 1, we are left with 3 points with x-coordinates equal to 2, $3/(n+1)$ and $4/n$ respectively. The local analysis of these is the same as that of the corresponding points in the previous section.

Next we look at the points at infinity, we follow Poincaré and set

$$x=\frac{X}{V};y=\frac{Y}{V};z=\frac{Z}{V};u=\frac{U}{V}, \qquad X^2+Y^2+Z^2+U^2+V^2=C^2, \quad (3.6)$$

where C is an arbitrary nonzero number. Rewriting (3.2) as

$$\begin{aligned}\dot{x}&=p(x,y,z,u)\\ \dot{y}&=q(x,y,z,u)\\ \dot{z}&=r(x,y,z,u)\\ \dot{u}&=s(x,y,z,u)\end{aligned} \quad (3.7)$$

and setting

$$P(X,Y,Z,U,V)=V^2p(\frac{X}{V},\frac{Y}{V},\frac{Z}{V},\frac{U}{V}),\ldots \quad (3.8)$$

we find that the orbits of (3.2) correspond to orbits of

$$\begin{aligned}\dot{X}&=\left(Y^2+Z^2+U^2+V^2\right)P-X\,(YQ+ZR+US)\\ \dot{Y}&=\left(X^2+Z^2+U^2+V^2\right)Q-Y\,(XP+ZR+US)\\ \dot{Z}&=\left(X^2+Y^2+U^2+V^2\right)R-Z\,(XP+YQ+US)\\ \dot{U}&=\left(X^2+Y^2+Z^2+V^2\right)S-U\,(XP+YQ+ZR)\\ \dot{V}&=-V\,(XP+YQ+ZR+US)\end{aligned} \quad (3.9)$$

Critical point at infinity are critical points of (3.9) with $V = 0$. We substitute this in (3.9) and obtain the critical points by solving for X, Y, Z, U. We find two circles of critical points, namely

$$\{X = Y = 0, Z^2 + U^2 = C^2\}, \quad \{X = 0, Y = 2Z, Y^2 + Z^2 + U^2 = C^2\}, \tag{3.10}$$

and the following isolated critical points which we list up to a multiple.

$$[0,1,0,0], [0,-1,0,0], [-1,0,n-1,0], [1,0,1-n,0], \tag{3.11}$$

$$[1,0,0,0], [-1,0,0,0], [2,1,0,0], [-2,-1,0,0], \tag{3.12}$$

$$[-3, n-2, 2n-1, 0], [3, 2-n, 1-2n, 0], \tag{3.13}$$

$$[-\beta, 0, \beta(n-2), n-2], [\beta, 0, \beta(2-n), 2-n], \tag{3.14}$$

$$[-3\beta, \beta(n-3), \beta(2n-3), 2n-3], [3\beta, \beta(3-n), \beta(3-2n), 3-2n]. \tag{3.15}$$

Before we go into further detail, we observe that all solutions with the property that for some $\eta_0 > 0$ not f but the first or second derivative of f becomes zero, are mapped into orbits going into one of the critical points at infinity with $X = 0$. Thus we only consider the points with $X \neq 0$. We also observe that, since we are looking for solutions which become zero in some η_0 at the moment, we restrict to the case $X < 0$. Now, suppose then we find that Y is a multiple of X, say $Y = aX$ in this point. Then the equation for $x(t)$ will asymptotically be (here we only give the formal analysis):

$$\dot{x} = (a-1)x^2 \tag{3.16}$$

so

$$x^{-1} = (1-a)(t - t_0) \qquad \Rightarrow \qquad \frac{f}{\eta f'} = (1-a)(\log \eta - \log \eta_0) \tag{3.17}$$

whence

$$\log f(\eta) = B + \frac{1}{1-a} \log \log \frac{\eta}{\eta_0} \qquad \Rightarrow \qquad f(\eta) = C(\log(\frac{\eta}{\eta_0})^{1/(1-a)} \tag{3.18}$$

Thus these solutions come down to zero with a powerlike behaviour, the exponent being $1/(1-a)$. Note that the equalities are only asymptotic, so some work still has to be done to rigorize this.

We also observe that if

$$\frac{z(t)}{u(t)} = \frac{1}{\eta} f(\eta)^{n-1} f'''(\eta) \to \beta, \tag{3.19}$$

this is a direct analogue of the interface condition for solutions of the porous medium equation [7]. We shall encounter this condition a number of times.

The exponents we find are 1, 2, $3/(n+1)$ and $3/n$. Below we compute the degrees of freedom, which now coincide with the dimension of the stable manifold $\mathcal{S}$, unless this is contained in $\{XYZUV = 0\}$. Note that one degree of freedom is in the choice of the point η_0.

3.1 The critical point $[-1, 0, n-1, 0]$, exponent 1

The linearization of this point is

$$\begin{bmatrix} -(n-1)^3 & -(n-1)^2 & -(n-1)^2 & -(n-1)\,\beta & 0 \\ 0 & (n-2)\,a_n & 0 & 0 & 0 \\ -(n-1)^2 & 1-n & 1-n & -\beta & 0 \\ 0 & 0 & 0 & -a_n & 0 \\ 0 & 0 & 0 & 0 & -a_n \end{bmatrix} \tag{3.20}$$

The eigenvalues are a positive multiple, namely $a_n = 2 - 2n + n^2$, of

$$0, 1-n, n-2, -1, -1. \tag{3.21}$$

so it seems we find $d = 3$ for $0 < n < 1$, $d = 4$ for $1 < n < 2$, and $d = 3$ for $n > 2$. However, the only eigenvector with a nonzero y-component is that of $n-2$. Thus for $n > 2$ the stable manifold $\mathcal{S}$ is contained in $y = 0$ so that in fact there are no solutions $f(\eta)$ with this behaviour. For $n < 2$ this eigenvalue is negative, and its eigenvector is $((n-1)^2, (3-2n)(2-2n+n^2), n-1, 0, 0)$, which implies that indeed $d = 3$ for $0 < n < 1$ and $d = 4$ for $1 < n < 2$. Note that for $n = 3/2$ two negative eigenvalues coincide, but still a generalized eigenvector with nonzero y-component exists.

3.2 The critical point $[-2, -1, 0, 0]$, exponent 2

The linearization of this point is

$$\begin{bmatrix} 2 & -4 & 2 & 0 & 0 \\ -4 & 8 & -4 & 0 & 0 \\ 0 & 0 & -5+10\,n & -10\,b & 0 \\ 0 & 0 & 0 & 10\,n-15 & 0 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix} \tag{3.22}$$

with eigenvalues 5 times

$$0, -1, 2, 2n-1, 2n-3 \tag{3.23}$$

so $d = 3$ for $0 < n < 1/2$, $d = 2$ for $1/2 < n < 3/2$, and $d = 0$ for $n > 3/2$ (because the stable manifold $\mathcal{S}$ is one-dimensional but contained in $\{U = 0\}$).

The eigenvectors for -1 and $2n-3$ are respectively $(0,0,0,0,1)$ and $(-2\beta, 4\beta, 5(5-2n)\beta, 5(5-2n)$. Thus, if $1/2 < n < 3/2$, every solution in $\mathcal{S}$ satisfies the "interface condition" (3.19). For $0 < n < 1/2$ there is another negative eigenvalue $2n-1$ with eigenvector $(2, -4, 5(2n-3), 0, 0)$. This is the negative eigenvalue closest to zero, which means that the orbits we already had for $1/2 < n < 3/2$ are joined by a new family of orbits in $\mathcal{S}$ tangent to $(2, -4, 5(2n-3), 0, 0)$. These do no not have (3.19). We conclude that for all $0 < n < 3/2$ there is a two-parameter (one of the parameters being η_0) family of solutions satisfying the "interface condition". Note that for $n = 0$ this argument is not complete because of two coinciding eigenvalues.

3.3 The critical point $[-3, n-2, 2n-1, 0]$, exponent $3/(n+1)$

The linearization of this point has left upper part

$$\left[\begin{array}{cc} -3\,(4n-5)\,(n^2-n+1) & 36n-27-18n^2 \\ (n-2)\,(n+1)\,(4n^2-8n+9) & -(n-2)\,(4n^2-4n+19) \\ -(2n-1)\,(n^3-4n^2+13n-9) & (n-5)\,(2n-1)\,(n+1) \end{array}\right],$$

in the 3th and 4th column

$$\left[\begin{array}{cc} -9\,(-1+n)\,(n+1) & -9\,(2n-1)\,b \\ (n-2)\,(8n^2-8n+11) & 3\,(n-2)\,(2n-1)\,b \\ -(2n-1)\,(2n^2-8n+17) & -3b\,(13+n^2-4n) \\ 0 & -15n^2+24n-42 \\ 0 & 0 \end{array}\right],$$

and bottom right $-(n+1)(5n^2-8n+14)$. All other entries are zero.

The eigenvalues are $(5\,n^2-8\,n+14)$ times

$$0, -(n+1), -3, -\frac{4\,n-5 \pm \sqrt{1+20\,n-8\,n^2}}{2}$$

The last two we already encountered in the analysis of the behaviour near $\eta = 0$, where the same exponent occurred. Here they are positive for $0 < n < 1/2$, so $d = 2$, have negative product for $1/2 < n < 2$ so $d = 3$, and are both negative for $n > 2$ (real part negative as n crosses $5/4 + \frac{3\sqrt{3}}{4}$), so $d = 4$. In fact the analysis gives exactly the same results as for the same exponent in the previous section. The fact that d is is one higher here is because of the freedom in the choice of η_0. We conclude that for this exponent there is no difference between $\eta_0 = 0$ and $\eta_0 > 0$.

3.4 The critical point $[-\beta, 0, \beta(n-2), n-2]$, exponent 1, $\beta > 0$

The linearization of this point has in its first two columns

$$\begin{bmatrix} -(n-2)^3\beta\,(\beta^2+1) & -\beta\,(n-2)^2\,(\beta^2+1) \\ 0 & \beta\,(n-3)\,(\beta^2n^2+n^2-4\,\beta^2n-4\,n+5\,\beta^2+4) \\ -\beta^3\,(n-2)^2 & -\beta^3\,(n-2) \\ -\beta^2\,(n-2)^2 & -\beta^2\,(n-2) \\ 0 & 0 \end{bmatrix}.$$

The second two columns are

$$\begin{bmatrix} -\beta\,(n-2)\,(-3\,\beta^2+\beta^2n+n-2) & -\beta^4\,(n-2) \\ 0 & 0 \\ -\beta\,(\beta^2n-3\,\beta^2-4-n^2+4\,n) & -\beta^2\,(4+n^2-4\,n+\beta^2) \\ -(n-2)\,\beta^2\,(-1+n) & \beta^3\,(n-2)^2 \\ 0 & 0 \end{bmatrix}.$$

The last column is

$$\begin{bmatrix} \beta\,(16\,\beta^2-4\,\beta^2n+4\,\beta\,\alpha-4\,\beta+\beta\,n+4+n^2-4\,n) \\ 0 \\ \alpha-40\,\beta+12+12\,\beta\,n-6\,n+2\,\beta\,n^2+n^2-4\,\beta^3+\beta^2-\beta\,n^3 \\ \beta\,(38\,\beta-11\,\beta\,n-2\,\beta\,n^2+\beta\,n^3-12+6\,n-n^2+8\,\alpha) \\ -\beta\,(\beta^2n^2+n^2-4\,\beta^2n-4\,n+5\,\beta^2+4) \end{bmatrix}.$$

The eigenvalues are $(\beta^2+1)(n-2)^2+\beta^2$ times

$$\beta, -\beta, \beta\,(2-n)\,, \beta\,(n-3)\,. \tag{3.24}$$

At first sight we find $d = 2$ for $0 < n < 2$, $d = 3$ for $2 < n < 3$ and $d = 2$ for $n > 3$, however, as in the other critical point with this exponent, all eigenvectors are contained in $\{y = 0\}$, except the eigenvector of $\beta(n-3)$. Thus for $n > 3$ there are no solutions $f(\eta)$ with this behaviour. However, for $0 < n < 2$ and $2 < n < 3$, these solutions do exist, and moreover they satisfy the "interface condition" (3.19). In that sense they are different from the other solutions with exponent 1.

3.5 The critical point $[-3\beta, \beta(n-3), \beta(2n-3), 2n-3]$, exponent $3/n$, $\beta > 0$

The matrix has first column is

$$\left[\begin{array}{c} -3\,\beta\,\left(-18+33\,n+39\,\beta^2n-20\,n^2-21\,\beta^2n^2+4\,n^3+4\,\beta^2n^3-27\,\beta^2\right) \\ \beta\,(n-3)\left(21\,\beta^2n-16\,\beta^2n^2+4\,n^3-16\,n^2+21\,n+4\,\beta^2n^3-9\right) \\ -\beta^3\,(2\,n-3)\left(n^3-7\,n^2+24\,n-27\right) \\ -\left(2\,n-3\right)\beta^2\left(n^3-7\,n^2+24\,n-27\right) \\ 0 \end{array}\right],$$

the second column

$$\left[\begin{array}{c} -3\,\beta\,\left(-12\,\beta^2n+9\,\beta^2+3\,\beta^2n^2+4\,n^2-12\,n+9\right) \\ -\beta\,(n-3)\left(4\,\beta^2n^2+4\,n^2-12\,\beta^2n-12\,n+27\,\beta^2+9\right) \\ \beta^3n\,(2\,n-3)\,(n-6) \\ (2\,n-3)\,\beta^2n\,(n-6) \\ 0 \end{array}\right],$$

the third column

$$\left[\begin{array}{c} -9\,\beta^4\,(2\,n-3) \\ 2\,\beta\,(n-3)\left(4\,\beta^2n^2-15\,\beta^2n+18\,\beta^2+4\,n^2-12\,n+9\right) \\ -\beta\,\left(108\,\beta^2n-135\,\beta^2+4\,\beta^2n^3-33\,\beta^2n^2-12\,n^2+36\,n-27\right) \\ -2\,\left(2\,n-3\right)\beta^2\left(-3\,n+n^2+9\right) \\ 0 \end{array}\right],$$

the fourth column

$$\begin{bmatrix} -9\,\beta^4\,(2\,n-3) \\ 3\,\beta^4\,(n-3)\,(2\,n-3) \\ -3\,\beta^2\,\left(4\,n^2-12\,n+9+18\,\beta^2+\beta^2 n^2-6\,\beta^2 n\right) \\ 3\,\beta^3\,(2\,n-3)^2 \\ 0 \end{bmatrix},$$

and the fifth column is

$$\begin{bmatrix} 3\,\beta\,\left(48\,\beta^2-16\,\beta^2 n+9\,\beta\,\alpha-12\,\beta+4\,\beta\,n+4\,n^2-12\,n+9\right) \\ \beta\,(180\,\beta^2-96\,\beta^2 n+16\,\beta^2 n^2-45\,\beta-4\,\beta\,n^2 \\ \qquad +24\,\beta\,n+27\,\beta\,\alpha+27-45\,n+24\,n^2-4\,n^3) \\ -54\,\beta^2\alpha-27\,\alpha-189\,\beta+54+90\,\beta\,n-36\,n+4\,\beta\,n^2+8\,n^2 \\ \qquad -216\,\beta^3+54\,\beta^2+60\,\beta^3 n-15\,\beta^2 n-8\,\beta^3 n^2+2\,\beta^2 n^2-8\,\beta\,n^3 \\ \beta\,\left(135\,\beta-36\,\beta\,n-19\,\beta\,n^2+10\,\beta\,n^3-54+36\,n-8\,n^2+27\,\alpha\right) \\ -\beta\,n\,\left(5\,\beta^2 n^2+4\,n^2-18\,\beta^2 n-12\,n+27\,\beta^2+9\right) \end{bmatrix}.$$

Eigenvalues are $\beta\,\left(5\,\beta^2 n^2+4\,n^2-18\,\beta^2 n-12\,n+27\,\beta^2+9\right)$ (which is positive) times

$$0, 3, -n, -\frac{4\,n-9\pm\sqrt{-8\,n^2+36\,n-27}}{2} \tag{3.25}$$

The last two have negative product for $\frac{3}{2}<n<3$ whence $d=2$. For $n<\frac{3}{2}$ they have positive real part (changing from real to complex if n drops below $\frac{3}{4}(3-\sqrt{3})$) so $d=1$. For $n>3$ they have negative real part (changing from real to complex if n crosses $\frac{3}{4}(3+\sqrt{3})$) so that $d=3$. All these solutions satisfy the "interface condition" (3.19).

4 Discussion

Our analysis confirms that the values $n=\frac{1}{2}$, $n=\frac{3}{2}$, $n=2$, $n=3$ and $n=4$ are special for the equation. At these values the dimensions of solution sets with prescribed singularities change.

The exponent $\frac{4}{n}$ only occurs for $\eta=0$, whereas the exponent $\frac{3}{n}$ only occurs at moving fronts. The exponents 1, 2 and $\frac{3}{n+1}$ occur in both cases. For moving fronts the results in [4] and [1] suggest that the appropiate exponent is 2 for $0 < n < \frac{3}{2}$ and $\frac{3}{n}$ for $\frac{3}{2} < n < 3$. We expected beforehand that the corresponding number of degrees of freedom would always be $d = 2$, in agreement with what we have seen in [8], where free boundaries correspond with $d = 2$.

Recalling the porous medium equation (PME) $u_t = (u^n u_x)_x$, it seems tempting to compare the exponents which for PME (see [7]) are $\frac{1}{n}$ (free boundary, compare to $\frac{3}{n}$ for (1.1)), $\frac{1}{n+1}$ (sign change), $\frac{2}{n}$ (waiting time, compare to $\frac{4}{n}$ for (1.1)) and $\frac{-\alpha}{\beta}$ (algebraic decay at infinity). For the latter we have not found evidence of an analogue for (1.1).

Finally, we have found solutions which satisfy (3.19) as $\eta \to \eta_0$. Note that this "interface condition" implies zero flux. Within this class exponent 2 occurs for $0 < n < 3/2$ with $d = 2$, while exponent $3/n$ occurs with $d = 2$ for $3/2 < n < 3$. However the last exponent also occurs with $d = 1$ for $0 < n < 3/2$ and with $d = 3$ for $n > 3$. Surprisingly we also found exponent 1, with $d = 2$ for $0 < n < 2$ and $d = 3$ for $2 < n < 3$.

References

[1] E.Beretta, M. Bertsch & R. dal Passo, *Nonnegative solutions of a fourth order nonlinear degenerate diffusion equation*, to appear in Arch. Rat. Mech. & Anal.

[2] E. Beretta, J. Hulshof & L.A. Peletier, *On a third order equation from forced coating flow*, in preparation.

[3] F. Bernis & A. Friedman , *Higher order nonlinear degenerate parabolic equations*, J. Differential Equations, 83 (1990), 179-206.

[4] F. Bernis, L.A. Peletier & S. Williams, *Source type solutions of a fourth order nonlinear degenerate parabolic equation*, Nonlinear Analysis T. M. A., 18 (1992), 217-233.

[5] A. Bertozzi & M. Pugh, *The lubrication approximation for thin viscous films: regularity and long time behaviour of weak solutions*, preprint (1994).

[6] S. Boatto, L. Kadanoff & P. Olla, *Travelling wave solutions to thin film equations*, Phys. Rev. E, 48 (1993).

[7] J. Hulshof, *Similarity solutions of the porous media equation with sign changes*, J. Math. Anal., 157 (1991), 75-111.

[8] J. Hulshof, *Selfsimilar solutions for the $k-\varepsilon$ model in turbulence*, Leiden preprint W93-11 (1993).

[9] L.W. Schwartz & R.R. Eley, *Numerical Simulation of Post-Application Coating Flows Arising from Surface Tension Gradients*, preprint.

[10] S.M. Williams, *Similarity Methods with Reference to a High Order Nonlinear Diffusion Equation*, Thesis, Oxford (1989).