On an ODE from forced coating flow

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1. Introduction

The flow of a film along a dry flat surface is important for applications, but difficult to analyze. The main problem resides in the formulation of the condition at the advancing contact line [MScT,TS]. One way around this difficulty has been to assume that the wall has been prewetted, so that the contact line is replaced by a sharp, but smooth, transition layer in which the surface tension plays an important role. An asymptotic analysis, in which the transition layer is identified as an *inner region*, and the part of the film further upstream as the *outer region* has been given in [MScT]. The local analysis of the film surface leads to the study of a typical class of ordinary differential equations. The study of such equations was stimulated by a survey article by Tuck & Schwartz [TS] and has recently received much attention [BKO, BP,T]. In the present paper we continue the study of these equations. We first present some specific examples.

Let us consider a two-dimensional flow of a thin film of viscous fluid, and denote the free surface of the film $y = h(x,t)$. Here x measures the distance along the plate, in a direction perpendicular to the contact line, and $y(x,t)$ denotes the thickness of the film at the point x, at time t. Then conservation of mass yields the equation

$$\frac{\partial h}{\partial t} + \frac{\partial J}{\partial x} = 0 \tag{1.1}$$

in which J denotes the flux, averaged over the depth of the film.

For a fluid draining down a vertical wall under the influence of gravity, the flux is given by the expression

$$J = \frac{1}{3\mu}\Big\{\rho g h^3 + \sigma h^3 \frac{\partial^3 h}{\partial x^3}\Big\}, \tag{1.2}$$

in which μ denotes the viscosity of the fluid [MScT]. In (1.2) the first term on the right hand side models the effect of gravity, with ρ denoting the density of the fluid and g the acceleration of gravity. The second term models the effect of surface tension, and σ denotes the corresponding coefficient. When we substitute (1.2) into (1.1) we obtain the nonlinear fourth order degenerate diffusion equation

$$\frac{\partial h}{\partial t} = -\frac{1}{3\mu}\frac{\partial}{\partial x}\Big\{\rho g h^3 + \sigma h^3 \frac{\partial^3 h}{\partial x^3}\Big\}. \tag{1.3}$$

If the plate is horizontal, and the flow is caused by shear from a jet of air above the film, we obtain in stead of (1.3)

$$\frac{\partial h}{\partial t} = -\frac{1}{3\mu}\frac{\partial}{\partial x}\Big\{\frac{3T}{2}h^2 + \sigma h^3 \frac{\partial^3 h}{\partial x^3}\Big\}, \tag{1.4}$$

in which T denotes the tangential stress exerted on the film by the external shear flow [MScT].

By a suitable scaling of x and t, equations (1.3) and (1.4) can be brought into the form

$$h_t = -\big(h^p + h^n h_{xxx}\big)_x \tag{1.5}$$

in which the subscripts denote partial differentiation with respect x and t, and $n = 3$ and $p = 2$ or 3.

In this paper we investigate travelling waves and stationary solutions of equations (1.3), (1.4) and (1.5). Thus, we set

$$h(x,t) = \varphi(\xi) \quad \text{and} \quad \xi = x - ct, \quad \xi \in \mathbf{R},$$

in which $c \in \mathbf{R}$ denotes the wave speed, and we substitute into (1.5). This yields the equation

$$c\varphi' = (\varphi^p + \varphi^n \varphi''')' \quad \text{on} \quad \mathbf{R}$$

in which primes denote differentiation with respect to ξ. After one integration we obtain

$$\varphi^n \varphi''' = A - \varphi^p + c\varphi \quad \text{on} \quad \mathbf{R} \tag{1.6}$$

in which A is a constant. We shall consider solutions which tend to a constant value as $\xi \to -\infty$.

If in addition, the wave is to represent a film advancing over a dry surface, then the flux at the front vanishes, and we must set $A = 0$ in (1.6). This yields the problem

$$\begin{cases} \varphi''' = -\varphi^{p-n} + \varphi^{1-n} & \text{(1.7a)} \\ \varphi(-\infty) = 1, & \text{(1.7b)} \end{cases}$$

where we have scaled the function φ so that the wave speed c becomes unity. We then seek a solution φ and a number $a \in \mathbf{R}$ such that

$$\varphi > 0 \quad \text{on} \quad (-\infty, a) \quad \text{and} \quad \varphi(a) = 0, \quad \varphi^n \varphi'''(a) = 0. \tag{1.8}$$

In the specific examples given above, in which $p = 2, 3$ and $n = 3$, we obtain the equations

$$\begin{cases} \varphi''' = -1 + \dfrac{1}{\varphi^2} & \text{(1.3')} \\ \varphi''' = -\dfrac{1}{\varphi} + \dfrac{1}{\varphi^2} & \text{(1.4')} \end{cases}$$

At this point it is interesting to mention yet another example [WJ], where an equation of the form of (1.7a) arises. This is the case of the entry of a falling film into a stationary pool, which gives rise to the equation

$$\varphi''' = -1 + \frac{1}{\varphi^3}. \tag{1.9}$$

In the present paper we study Problem (1.7) for different values of p and n. In fact we shall study the more general problem: we seek a function $u \in C^1(\mathbf{R})$ which satisfies

$$\begin{cases} u''' = f(u) \qquad \text{when} \quad u > 0, & \text{(1.10a)} \\ u(-\infty) = 1, & \text{(1.10b)} \end{cases}$$

in which the function f generalises the right hand side of (1.7a) and satisfies the following hypotheses:

H1. $f \in C^1(0, \infty)$;

H2. $f > 0$ on $(0, 1)$, $f < 0$ on $(1, \infty)$ and $f'(1) < 0$.

H3. $\int_1^\infty s f(s) ds = -\infty$.

In what follows it will be convenient to write

$$F_1(t) = \int_t^1 f(s)\, ds, \qquad F_2(t) = \int_t^1 F_1(s)\, ds = \int_t^1 (s - t) f(s)\, ds.$$

We shall prove the following existence theorem:

Theorem A. *Suppose that f satisfies hypotheses H1-3. Then, if $F_2(0) < \infty$, there exists a solution u of Problem (1.10), which has the properties*

$$u > 0 \quad on \quad (-\infty, a) \quad and \quad u(a) = 0, \quad u'(a) = 0 \tag{1.11}$$

for some $a \in \mathbf{R}$.

If H3 fails, we still have a partial result which says that a solution as in Theorem A exists, provided the number $F_1(\infty)$ is not too small.

The condition on $F_2(0)$ in Theorem A is critical, as the following nonexistence theorem shows.

Theorem B. *Suppose that f satisfies hypotheses H1 and H2. Then, if $F_2(0) = \infty$ there exists no solution u of Problem (1.10) which has the properties (1.11).*

Theorems A and B generalize results obtained by Bernis & Peletier [BP] and by Troy [T]. We also mention a recent paper by Boatto, Kadanoff & Olla [BKO], in which the equation

$$\varphi^n \varphi''' = \pm 1$$

is studied by transforming it to a second order autonomous system. Whereas in [BKO] the emphasis lies on a thorough investigation of the resulting system, we focus here more on solutions of specific two point boundary value problems, which are important for applications.

2. Outline of the existence proof

In proving Theorem A, we follow the method of Troy [T], which involves the matching of two solutions, constructed by forward and backward shooting from the origin.

Thus we study the initial value problem

$$\begin{cases} u''' = f(u) & \text{(2.1a)} \\ u(0) = \alpha, \quad u'(0) = -\beta, \quad u''(0) = 0. & \text{(2.1b)} \end{cases}$$

where we shall choose α and β from the set

$$\mathcal{E} = \{(\alpha, \beta) : 0 < \alpha \leq 1, \ \beta > 0\}. \tag{2.2}$$

Plainly for each $(\alpha, \beta) \in \mathcal{E}$ Problem (2.1) has a unique local solution, which we shall denote by $u(x; \alpha, \beta)$.

The existence proof now consists of three parts.

In Part I we construct a continuum $\Gamma_+ \in \mathcal{E}$ such that if $(\alpha, \beta) \in \Gamma_+$, then the corresponding solution u has the properties

$$0 < u(x) < 1 \quad \text{for} \quad 0 \leq x < a \qquad \text{(2.3a)}$$

$$u(x) \to 0 \ \text{ and } \ u'(x) \to 0 \quad \text{as} \quad x \to a, \qquad \text{(2.3b)}$$

where a is a positive number.

In Part II we construct a continuum $\Gamma_- \in \mathcal{E}$ such that if $(\alpha, \beta) \in \Gamma_-$, then the corresponding solution u has the properties

$$u(x) > 0 \quad \text{for} \quad -\infty < x \leq 0 \qquad (2.4a)$$

$$u(x) \to 1 \quad \text{as} \quad x \to -\infty. \qquad (2.4b)$$

In Part III we show that

$$\Gamma_+ \cap \Gamma_- \neq \emptyset,$$

so that there exists a pair $(\alpha_0, \beta_0) \in \Gamma_+ \cap \Gamma_-$ for which the corresponding solution has the properties asserted in Theorem A.

Part I is dealt with in Section 3, Part II in Sections 4 and 5 and Part III follows in Section 6.

3. The forward problem

As a first observation we note that if u satisfies $0 < u < 1$, then $u' < 0$.

Lemma 3.1. *Let u be a solution of Problem (2.1) such that $u < 1$ as long as $u > 0$. Then $u' < 0$ as long as $u > 0$.*

Proof. Because $\beta > 0$, it follows that $u' < 0$ near $x = 0$. Suppose that u' has a first zero x_0, where u is positive. Then $u''(x_0) \geq 0$ and hence, since $f(u) > 0$ when $0 < u < 1$, we find that

$$u'(x) > 0 \quad \text{and} \quad u''(x) > 0 \quad \text{for} \quad x > x_0,$$

and so $u(x)$ reaches 1 at some finite $x_1 > x_0$, a contradiction.

Lemma 3.1 shows that we are looking for a strictly decreasing solution, and so we may proceed as in [BP] and [BPW] and use u as an independent variable. Let $x(u)$ be the inverse of $u(x)$, which, because $\beta > 0$, is well defined near $u = \alpha$, and set

$$t = u \quad \text{and} \quad y(t) = \{u'(x(t))\}^2. \tag{3.1}$$

Problem (2.1) now becomes

$$\begin{cases} y'' = -\dfrac{2f(t)}{\sqrt{y}} \quad \text{for} \quad t < \alpha & (3.2\text{a}) \\ y(\alpha) = \beta^2 \quad \text{and} \quad y'(\alpha) = 0. & (3.2\text{b}) \end{cases}$$

Let

$$\tau(\alpha, \beta) = \inf\{0 < t < \alpha : y > 0 \text{ on } (t, \alpha)\}.$$

Then, because $0 < \alpha \leq 1$, it follows from (3.2a) that $y'' < 0$ on (τ, α) and hence, in view of (3.2b),

$$y(t) < \beta^2 \quad \text{and} \quad y'(t) > 0 \quad \text{for} \quad \tau \leq t < \alpha. \tag{3.3}$$

To find the desired continuum, we carry out a two-dimensional shooting argument and set

$$\mathcal{S}^+ = \{(\alpha, \beta) \in \mathcal{E} : \tau = 0 \quad \text{and} \quad y(0) = \lim_{t \downarrow 0} y(t) > 0\},$$
$$\mathcal{S}^- = \{(\alpha, \beta) \in \mathcal{E} : \tau > 0\}.$$

To characterise the sets $\mathcal{S}^+$ and $\mathcal{S}^-$ we need the function

$$H(\alpha) = \int_0^\alpha \int_s^\alpha f(r)\, dr\, ds = \int_0^\alpha s f(s)\, ds. \tag{3.4}$$

If $F_2(0) = H(1) < \infty$, then the function H is well defined for all $\alpha \geq 0$ and $H(0) = 0$.

In the following two lemmas we show that if the ratio between β^3 and $H(\alpha)$ is large (small), (α, β) belongs to the set $\mathcal{S}^+$ ($\mathcal{S}^-$).

Lemma 3.2. *We have*

$$\{(\alpha,\beta)\in\mathcal{E}:\beta\geq\beta_1(\alpha)\}\subset\mathcal{S}^+,$$

where $\beta_1(\alpha)=\sqrt{3}\,\{H(\alpha)\}^{1/3}$.

Proof. For each $\alpha\in(0,1]$ we shall show that $\beta\geq\beta_1(\alpha)$ implies in fact that $y(t)>\frac{1}{3}\beta^2$ for all $0<t<\alpha$. The assertion then follows at once.

We may integrate the equation for y twice to obtain

$$y(t)=\beta^2-2\int_t^\alpha\int_s^\alpha\frac{f(r)}{\sqrt{y(r)}}\,dr\,ds. \tag{3.5}$$

Let $\lambda\in(0,1)$. As long as $y(t)>\lambda\beta^2$, we have, because $f>0$ on $(0,1)$,

$$\int_t^\alpha\int_s^\alpha\frac{f(r)}{\sqrt{y(r)}}\,dr\,ds<\frac{1}{\beta\sqrt{\lambda}}\int_0^\alpha\int_s^\alpha f(r)\,dr\,ds=\frac{1}{\beta\sqrt{\lambda}}H(\alpha)$$

and so

$$y(t)\geq\beta^2-\frac{2}{\beta\sqrt{\lambda}}H(\alpha). \tag{3.6}$$

Therefore $y(t)>\lambda\beta^2$ for all $0<t<\alpha$ if we take

$$\beta^3\geq\frac{2}{(1-\lambda)\sqrt{\lambda}}H(\alpha).$$

If we take the optimal value $\lambda=1/3$, this proves the assertion.

Lemma 3.3. *We have*

$$\{(\alpha,\beta)\in\mathcal{E}:\beta\leq\beta_2(\alpha)\}\subset\mathcal{S}^-,$$

where $\beta_2(\alpha)=\{2H(\alpha)\}^{1/3}$.

Proof. From (3.5) and the monotonicity of $y(t)$ we have

$$\beta^2>y(t)+\frac{2}{\beta}\int_t^\alpha\int_s^\alpha f(r)\,dr\,ds. \tag{3.7}$$

Thus, if

$$\beta^3\leq 2H(\alpha),$$

then it follows from (3.7) that $y(t)$ cannot be positive on $(0,1]$, so that $(\alpha,\beta)\in\mathcal{S}^-$.

Lemma 3.4. *The sets* $\mathcal{S}^+$ *and* $\mathcal{S}^-$ *are relatively open in* $\mathcal{E}$.

Proof. Using standard arguments it is easy to see that the conditions on f guarantee the continuous dependence of y on α and β, from which the assertion follows.

By a proposition of plane point set topology, see [McLS] for a precise statement, proof and the first application in the context of two-dimensional shooting arguments, this implies the existence of a continuum $\Gamma^+\subset\mathcal{E}\backslash(\mathcal{S}^+\cup\mathcal{S}^-)$ which joins the origin and the line $\{\alpha=1,\ \beta>0\}$.

In the lemma below we characterize solutions of Problem 2.1 with initial data (α,β) on Γ^+.

Lemma 3.5. *If $(\alpha,\beta) \in \Gamma^+$, then there exists a point $a \in (0,\infty)$ such that*

$$u(x) \to 0 \text{ and } u'(x) \to 0 \text{ as } x \to a.$$

Proof. Since $(\alpha,\beta) \notin S^-$ it follows that

$$\lim_{x\to a} u(x) = 0,$$

where a may be finite or infinite. Since $(\alpha,\beta) \notin S^+$ we may conclude that

$$\lim_{x\to a} u'(x) = 0,$$

as well.

It remains to show that $a < \infty$. Suppose this is not so. Then $u''(x) > 0$ and $u'''(x) > 0$ for all $x > 0$, whence $u'(x) \to \infty$ as $x \to \infty$, a contradiction.

We conclude this section with a result when $F_2(0) = \infty$.

Lemma 3.6. *Suppose that $F_2(0) = \infty$. Then $\mathcal{S}^- = \mathcal{E}$.*

Proof. Suppose to the contrary that there exists a point $(\alpha_0,\beta_0) \in \mathcal{E}$ which does not lie in $\mathcal{S}^-$. Then the corresponding solution $y_0(t) = y(t;\alpha_0,\beta_0)$ is positive for all $t \in (0,\alpha_0)$. Hence, since $y_0' > 0$ on $(0,\alpha_0)$, we have, as in (3.7)

$$\beta_0^3 > \beta_0\, y_0(t) + 2\int_t^{\alpha_0}\int_s^{\alpha_0} f(r)\,dr\,ds > 2\int_t^{\alpha_0}\int_s^{\alpha_0} f(r)\,dr\,ds.$$

If we now let $t \to 0$, we find that the right hand side becomes infinite, whilst the left hand side does not change and remains finite. Thus, we have a contradiction, and the assertion is proved.

It follows from Lemma 3.6 that if $F_2(0) = \infty$, then there exists no solution of the form as described in Lemma 3.5 and Theorem B is proved.

4. The backward problem

For simplicity we shall assume throughout most of this section that $F_1(\infty) = \infty$. In a final remark, we shall indicate how this assumption can be relaxed.

It is convenient to set $x = -\tilde{x}$ and omit the tilde again. Then Problem (2.1) becomes

$$\begin{cases} u''' = -f(u) & \text{(4.1a)} \\ u(0) = \alpha, \quad u'(0) = \beta > 0, \quad u''(0) = 0. & \text{(4.1b)} \end{cases}$$

Lemma 4.1. *Let $(\alpha,\beta) \in \mathcal{E}$. If there exists a first $x_0 > 0$ such that $u(x_0) = \alpha$ and $u(x) > \alpha$ for $x \in (0,x_0)$, then $u'(x_0) < 0$.*

Proof. It follows from the definition of x_0 that $u'(x_0) \le 0$. Suppose that $u'(x_0) = 0$. Then the energy functional defined by

$$\Phi = u'u'' - F_1(u),$$

satisfies

$$\Phi(x_0) = \Phi(0).$$

Since

$$\Phi' = (u'')^2,$$

this implies that $u''(x) = 0$ for x in $[0, x_0]$, which is impossible.

For our next result we introduce the set

$$Q = \{(u, u', u'') : u > 1, u' > 0, u'' > 0\}. \tag{4.2}$$

In the following lemma we observe that Q is invariant.

Lemma 4.2. *If $(u(\hat{x}), u'(\hat{x}), u''(\hat{x})) \in Q$ for some $\hat{x}$, then $(u(x), u'(x), u''(x)) \in Q$ for all $x > \hat{x}$ where the solution u exists.*

Proof. From equation (4.1a) and the fact that $-f(u) > 0$ for $u > 1$ it follows that $u''' > 0, u'' > 0, u' > 0$ and $u > 1$ for all $x > \hat{x}$ where the solution u exists.

To find the desired continuum Γ_- we carry out again a two-dimensional shooting argument and define the sets

$$\mathcal{T}^+ = \{(\alpha, \beta) \in \mathcal{E} : (u, u', u'') \in Q \text{ for some } \hat{x} > 0 \text{ and } u > \alpha \text{ on } (0, \hat{x})\},$$
$$\mathcal{T}^- = \{(\alpha, \beta) \in \mathcal{E} : \exists\, x_0 > 0 \text{ such that } u(x_0) = \alpha \text{ and } u > \alpha \text{ on } (0, x_0)\}.$$

From Lemmas 4.1 and 4.2 and continuity of solutions on initial data, it follows that $\mathcal{T}^+$ and $\mathcal{T}^-$ are both open in $\mathcal{E}$, that $\mathcal{T}^+ \cap \mathcal{T}^- = \emptyset$ and that $(1, \beta) \in \mathcal{T}^+$ for any $\beta > 0$. In the following two lemmas we will show that the sets $\mathcal{T}^+$ and $\mathcal{T}^-$ are nonempty.

In the proof of the next lemma we shall investigate increasing solutions of Problem (4.1) and it will be convenient to use u as an independent variable. We denote the inverse of $u(x)$ by $x(u)$ and set

$$t = u \quad \text{and} \quad y(t) = \{u'(x(t))\}^2. \tag{4.3}$$

This leads to Problem (3.2) which we now analyze for $t > \alpha$. Since $y' = 2u'u''(u')^{-1} = 2u''$, in these variables, points in Q are characterized by

$$t > 1, \; y > 0 \text{ and } y' > 0. \tag{4.4}$$

Lemma 4.3. *We have*

$$\{(\alpha, \beta) \in \mathcal{E} : \beta \geq \beta^+(\alpha)\} \subset \mathcal{T}^+,$$

for some explicitly given function $\beta^+(\alpha)$ with

$$\beta^+(\alpha) \to 0 \quad \text{as} \quad \alpha \to 1.$$

Proof. Before embarking on the analysis of Problem (4.3) we make one further transformation. To simplify notation we set

$$z(t) = \beta^{-2} y(t).$$

Then Problem (3.2) can be written as

$$\begin{cases} z'' = -2\varepsilon \dfrac{f(t)}{\sqrt{z}} \quad \text{for} \quad t > \alpha \quad (\varepsilon = \beta^{-3}) & (4.5a) \\ z(\alpha) = 1 \quad \text{and} \quad z'(\alpha) = 0 & (4.5b) \end{cases}$$

We need to show that for β large (i.e. ε small) the solution $z(t)$ does not hit zero and is eventually increasing. This is done in two steps. First we let t run from α to 1.

As long as $z(t) > 0$ and $t > \alpha$ we have

$$z(t) = 1 - 2\varepsilon \int_\alpha^t (t-s)\frac{f(s)}{\sqrt{z(s)}}ds.$$

Note that $f(t) > 0$ for $t \in (\alpha, 1)$ and so, as long as $z(t) > \frac{1}{2}$, we obtain

$$\int_\alpha^t (t-s)\frac{f(s)}{\sqrt{z(s)}}ds < \sqrt{2}\int_\alpha^1 f(s)\,ds \quad \text{for} \quad \alpha \le t \le 1.$$

Therefore

$$z(t) > 1 - 2\sqrt{2}\,\varepsilon\, F_1(\alpha) \quad \text{for} \quad \alpha \le t \le 1.$$

when $z(t) > \frac{1}{2}$. This will be so if

$$1 - 2\sqrt{2}\,\varepsilon\, F_1(\alpha) \ge \frac{1}{2}$$

or

$$\beta \ge \left(4\sqrt{2}\,F_1(\alpha)\right)^{\frac{1}{3}}. \tag{4.6}$$

At $t = 1$ we have

$$1 > z(1) > \frac{1}{2} \quad \text{and} \quad z'(1) > -2\sqrt{2}\,\varepsilon\, F_1(\alpha), \tag{4.7}$$

Next we let t increase from 1 to the point $T = T(\alpha) > 1$ defined by

$$F_1(T(\alpha)) = \sqrt{2}\,F_1(\alpha). \tag{4.8}$$

Since $F_1(t)$ is a strictly increasing function on $(1, \infty)$ and $F_1(\infty) = \infty$, it has range $(0, \infty)$, and so the point $T(\alpha)$ is well defined.

If $z(t)$ happens to reach the value 1 again in the interval $[1, T]$ it means that $u(x)$ enters Q and the assertion is proved.

Thus suppose that $z(t) < 1$ for $1 \le t \le T$. We integrate (4.5a) over $(1, T)$ and assume that

$$z(t) \ge \frac{1}{4} \quad \text{for all} \quad t \in [1, T]. \tag{4.9}$$

Then, since $z(t) < 1$,

$$z'(T) > z'(1) + 2\varepsilon\, F_1(T) > 2\varepsilon\left(-\sqrt{2}\,F_1(\alpha) + F_1(T)\right) = 0,$$

so that $z'(t) > 0$ at some $t \in [1, T]$ and again the corresponding solution $u(x)$ enters Q.

To ensure that (4.9) holds, we need to choose ϵ sufficiently small: since $z'' > 0$ on $(1, \infty)$, it follows from (4.7) that

$$z(t) > \frac{1}{2} + z'(1)(t-1) > \frac{1}{2} - 2\sqrt{2}\,\varepsilon\, F_1(\alpha)\,(t-1).$$

Thus, (4.9) will hold if

$$\frac{1}{2} - 2\sqrt{2}\,\varepsilon\, F_1(\alpha)\,(T-1) \ge \frac{1}{4},$$

or if

$$\beta \geq \left(8\sqrt{2}\,F_1(\alpha)\{T(\alpha)-1\}\right)^{\frac{1}{3}}.$$

Remembering (4.6) we conclude that if

$$\beta \geq \beta^+(\alpha) = \{4\sqrt{2}\,F_1(\alpha)\ \max(2\{T(\alpha)-1\},1)\}^{1/3}, \tag{4.10}$$

then (4.4) holds at $u = T$.

Remark. We note that if we drop the assumption that $F_1(\infty) = \infty$, the function $\beta^+(\alpha)$ is defined for α sufficiently close to 1. If there exists a value α_+ such that

$$F_1(\infty) = \sqrt{2}\,F_1(\alpha_+), \tag{4.11}$$

then as $\alpha \downarrow \alpha_+$ we have $T(\alpha) \to \infty$ and therefore $\beta^+(\alpha) \to \infty$. Also, the factor $\sqrt{2}$ in (4.11) is not optimal and can be removed at the expense of a less explicit choice of the function $\beta^+(\alpha)$. This can be seen by differentiating (4.5a) with respect to ε in $\varepsilon = 0$.

We next show that $\mathcal{T}^-$ is nonempty.

Lemma 4.4. *We have*

$$\{(\alpha,\beta) \in \mathcal{E} : \beta \leq \beta^-(\alpha)\} \subset \mathcal{T}^-,$$

where

$$\beta^-(\alpha) = \left(2\int_\alpha^1 (1-s)f(s)\,ds\right)^{1/3} > 0.$$

Proof. We first observe that if for some $(\alpha,\beta) \in \mathcal{E}$ there exists some first point $x_1 > 0$ where $u(x_1) < 1$ and $u'(x_1) = 0$ then, since $-f(t) < 0$ for $t \in (0,1)$ it follows that u descends to α, so that $(\alpha,\beta) \in \mathcal{T}^-$. Hence, again we may look at increasing solutions of (4.1) and use the change of variables (4.3).

As long as $y(t) > 0$ and $\alpha < t < 1$ we have that $y(t) < \beta^2$ since $-f(t) < 0$, so that

$$y(t) \leq \beta^2 - \frac{2}{\beta}\int_\alpha^t (t-s)f(s)\,ds.$$

Therefore y vanishes on $(\alpha,1)$ if

$$\beta \leq \left(2\int_\alpha^1 (1-s)f(s)\,ds\right)^{1/3},$$

from which our claim follows.

As in the previous section it follows that there is a continuum $\Gamma^- \subset \mathcal{E}\backslash(\mathcal{T}^+ \cup \mathcal{T}^-)$ starting at the point $\{\alpha = 1,\ \beta = 0\}$ which can be continued to the left as long as $\beta^+(\alpha)$ exists.

In the following lemma we characterize the limiting bahaviour of the solutions with initial data (α,β) in Γ^- (in fact for all initial data (α,β) in $\mathcal{E}\backslash(\mathcal{T}^+ \cup \mathcal{T}^-)$).

Lemma 4.5. *Suppose that f satisfies H1 and H2 and that $(\alpha,\beta) \in \Gamma^-$. If one of the following conditions on f holds:*

(a) *f satisfies hypothesis H3;*

(b) *α and the potential F_1 satisfy the relation $F_1(\alpha) \leq F_1(\infty)$.*

Then the solution of the backward problem (4.1) exists for all $x > 0$, is oscillating with minima increasing and maxima decreasing, and converges to 1 as $x \to \infty$.

The proof is given in the next section. If $F_1(\infty) = \infty$ the proof of this lemma can be found in [T, Lemma 7].

5. Proof of Lemma 4.5

The proof proceeds along a series of lemmas.

Since the solution will prove to be oscillating, it is convenient to define the sequences of points (ξ_k) where u has a local maximum and of points (η_k) where u has a local minimum, just as in [PT]. Thus, define

$$\xi_1 = \sup\{x > 0 : u' > 0 \quad \text{on} \quad [0, x)\}. \tag{5.1}$$

Since $u'(0) = \beta > 0$ the point ξ_1 is well defined.

Lemma 5.1. *Let f satisfy the conditions imposed in Lemma 4.5. If $(\alpha, \beta) \in \Gamma^-$, then*

(a)
$$\xi_1 < \infty \quad \textit{and} \quad u'(\xi_1) = 0,$$

and

(b)
$$u(\xi_1) > 1 \quad \textit{and} \quad u''(\xi_1) < 0.$$

Proof. It is convenient to analyse the solution $y(t)$ of the transformed problem

$$\begin{cases} y'' = -\dfrac{2f(t)}{\sqrt{y}} \quad \text{for} \quad t > \alpha & (5.2\text{a}) \\ y(\alpha) = \beta^2 \quad \text{and} \quad y'(\alpha) = 0. & (5.2\text{b}) \end{cases}$$

Suppose that $\xi_1 = \infty$. We distinguish two cases:

(i) $u(x) \leq 1$ for all $x > 0$;

(ii) there exists a point $x_1 > 0$ such that $u(x) > 1$ for $x > x_1$.

In Case (i), we have $y'' < 0$ and so y vanishes in finite time, with $y' < 0$ and bounded away from zero. Since $y' = 2u''$, it is clear that $\xi_1 < \infty$, a contradiction.

In Case (ii) we know, because $(\alpha, \beta) \notin \mathcal{T}^+$, that $y' = 2u'' < 0$ as long as $y > 0$. Hence

$$y(t) > 0, \quad y'(t) < 0 \quad \text{and} \quad y''(t) > 0 \quad \text{on} \quad [1, \infty),$$

and so

$$y(\infty) = \lim_{t \to \infty} y(t) \geq 0 \quad \text{and} \quad y'(\infty) = \lim_{t \to \infty} y'(t) = 0. \tag{5.3}$$

This implies that when we integrate the equation for y twice over (t, ∞), we obtain

$$y(t) = y(\infty) - \int_t^\infty \int_r^\infty \frac{f(s)}{\sqrt{y(s)}}\, ds\, dr = y(\infty) - \int_t^\infty (s-t) \frac{f(s)}{\sqrt{y(s)}}\, ds.$$

Therefore, if we take $t > 1$, we obtain

$$\int_t^\infty (s-t)|f(s)|\, ds \leq \sqrt{y(t)} \int_t^\infty (s-t) \frac{|f(s)|}{\sqrt{y(s)}}\, ds = \sqrt{y(t)}\{y(t) - y(\infty)\},$$

whch implies that $s \to sf(s) \in L^1(1,\infty)$. This contradicts hypothesis H3.

If H3 is not satisfied, we use the montonicity of the function Φ which, in the transformed variables becomes

$$\Phi^*(t) = \frac{1}{2}\sqrt{y}\, y' - F_1(t). \tag{5.4}$$

Note that by (5.3),

$$\Phi^*(t) \to -F_1(\infty) \quad \text{as} \quad t \to \infty.$$

Hence, because

$$(\Phi^*)' = \frac{(y')^2}{4\sqrt{y}} > 0$$

it follows that

$$F_1(\alpha) > F_1(\infty),$$

which contradicts condition (b) of Lemma 4.5.

Thus, in both cases

$$\tau = \sup\{t > \alpha : y > 0 \ \text{ on } \ [\alpha, t)\} < \infty \quad \text{and} \quad y(\tau) = 0.$$

Returning to the original variables we find that $u(x) \to \tau$ and $u'(x) \to 0$ as $x \to \xi_1$ for some $\xi_1 \leq \infty$. Since $f(\tau) \neq 0$ it follows that $\xi_1 < \infty$ and thus we have proved Part (a).

Suppose by way of contradiction that

$$u(\xi_1) \leq 1.$$

Then, since $u''(0) = 0$ and $u''' < 0$ as long as $u < 1$, we find that $u''(\xi_1) < 0$ and decreasing. Hence, there exists a point $x_0 > \xi_1$ such that $u(x_0) = \alpha$, and $u > \alpha$ on $(0, x_0)$, which implies that $(\alpha, \beta) \in \mathcal{T}^-$, a contradiction. Thus $u(\xi_1) > 1$.

It is now clear that $u''(\xi_1) \leq 0$. Suppose that $u''(\xi_1) = 0$. Then, since $u''' > 0$ when $u > 1$, it follows that $u'' > 0$ in a right-neighbourhood of ξ_1, so that we would have $(\alpha, \beta) \in \mathcal{T}^+$. This is not the case and so $u''(\xi_1) < 0$.

This completes the proof of Lemma 5.1.

We can now define the first point where $u(x)$ has a local minimum:

$$\eta_1 = \sup\{x > \xi_1 : u' < 0 \ \text{ on } \ (\xi_1, x)\}. \tag{5.5}$$

By Part (b) of Lemma 5.1, η_1 is well defined.

Lemma 5.2. *Let f satisfy the conditions imposed in Lemma 4.5. If $(\alpha, \beta) \in \Gamma^-$, then*

(a)
$$\eta_1 < \infty \quad \textit{and} \quad u'(\eta_1) = 0,$$

and

(b)
$$\alpha < u(\eta_1) < 1 \quad \textit{and} \quad u''(\eta_1) > 0.$$

Proof. Suppose that $\eta_1 = \infty$. Then

$$u'(x) < 0 \quad \text{for} \quad \xi_1 < x < \infty$$

and because $(\alpha,\beta) \notin \mathcal{T}^-$, we also know that

$$u(x) > \alpha \quad \text{for} \quad 0 < x < \infty.$$

Hence

$$\lim_{x\to\infty} u(x) \text{ exists} \stackrel{\text{def}}{=} u^*.$$

Plainly, we must have $f(u^*) = 0$ and so $u^* = 1$. Thus, $u(x)$ tends monotonically to 1 as $x \to \infty$.

The linearization of equation (4.1a) at $u = 1$ yields the equation

$$v''' - A\,v = 0, \qquad A = -f'(1) > 0,$$

and so the spectrum consists of the eigenvalues $\{\lambda_k = A^{1/3}e^{2k\pi i/3} : k = 0, 1, 2\}$. Hence, any solution which converges to $u = 1$ as $x \to \infty$, must eventually be oscillatory. Since $u(x)$ is monotone, we have a contradiction. Thus $\eta_1 < \infty$ and $u'(\eta_1) = 0$. This proves Part (a).

To prove Part (b) we first suppose that $u(\eta_1) > 1$. Then $u'''(\eta_1) > 0$ and hence, because $u''(\eta_1) \geq 0$, it follows that $u' > 0$ and $u'' > 0$ in a right-neigbourhood of η_1. This would mean that $(\alpha,\beta) \in \mathcal{T}^+$, a contradiction.

Next suppose that $u(\eta_1) = 1$. Since $u'(\eta_1) = 0$ and $u = 1$ is a solution of (4.1a), which however does not satisfy the initial conditions, it follows from a uniqueness argument that $u''(\eta_1) > 0$. Hence $u' > 0$ and $u'' > 0$ in a right-neigbourhood of η_1 and we find again that $(\alpha,\beta) \in \mathcal{T}^+$, a contradiction. Thus $u(\eta_1) < 1$.

It remains to prove, that $u''(\eta_1) > 0$. Suppose that $u''(\eta_1) = 0$. Then, since $u''' < 0$ whenever $u < 1$ it follows that $u''(x) < 0$ for $x > \eta_1$ and since $u'(\eta_1) = 0$, also $u'(x) < 0$ for $x > \eta_1$. This means that $u(x)$ drops down to α and so, that $(\alpha,\beta) \in \mathcal{T}^-$, a contradiction.

Continuing this argument inductively, we construct an increasing sequence of local maximimum points (ξ_k) and an increasing sequence of local minimum points (η_k), such that $\xi_k < \eta_k < \xi_{k+1}$ and

$$\alpha < u(\eta_k) < 1 < u(\xi_k) \quad \text{for all} \quad k \geq 1 \tag{5.6}$$

and by the monotonicity of $\Phi(u)$:

$$u(\eta_{k+1}) > u(\eta_k) \quad \text{and} \quad u(\xi_{k+1}) < u(\xi_k) \quad \text{for all} \quad k \geq 1 \tag{5.7}$$

Thus, $u(x)$ oscillates around $u = 1$ with decreasing amplitude. In the following lemma we show that it tends to one.

Lemma 5.3. *We have*

$$u(\eta_k) \nearrow 1 \quad \textit{and} \quad u(\xi_k) \searrow 1 \quad \textit{as} \quad k \to \infty.$$

Proof. It follows from (5.6) and (5.7) that the sequences $(u(\xi_k))$ and $(u(\eta_k))$ are convergent:

$$u(\eta_k) \nearrow u^- \leq 1 \quad \textit{and} \quad u(\xi_k) \searrow u^+ \geq 1 \quad \textit{as} \quad k \to \infty.$$

Hence

$$\Phi(u(\eta_k)) = -F_1(u(\eta_k)) \to -F(u^-) \quad \text{as} \quad k \to \infty$$

and

$$\Phi(u(\xi_k)) = -F_1(u(\xi_k)) \to -F(u^+) \quad \text{as} \quad k \to \infty$$

Since $\Phi(u)$ is increasing this implies that

$$F_1(u^-) = F_1(u^+). \tag{5.8}$$

Let

$$u_k(x) = u(\eta_k + x), \qquad k \geq 1.$$

Then, since

$$\alpha < u_k(x) < u(\xi_1) \quad \text{for} \quad x \in \mathbf{R}^+,$$

it follows that there exists a positive constant M such that

$$|u_k'''(x)| \leq M \quad \text{for} \quad x \in \mathbf{R}^+.$$

Hence, by compactness arguments and using the differential equation, there exists a subsequence, which we denote again by (u_k), and a function $U \in C^3([0,\infty))$ such that

$$u_k \to U \quad \text{as} \quad k \to \infty$$

uniformly in $C^3([0,L])$ for any $L > 0$. About U we see at once that

$$U''' = -f(U) \quad \text{on} \quad \mathbf{R}^+ \quad \text{and} \quad U(0) = u^-, \quad U'(0) = 0. \tag{5.9}$$

In the remainder of the proof we shall show that if $u^- < 1$, then

$$U'''(x) = 0.$$

This implies by (5.9) that $U = u^- = 1$, which in turn, by (5.8) implies that $u^+ = 1$.

Set

$$\ell_k = \xi_{k+1} - \eta_k.$$

By the uniform boundedness of u'' on compact sets, it follows that there exists a $\delta > 0$, such that

$$\ell_k > \delta \qquad \text{for all} \quad k \geq 1.$$

Integration of $(\Phi(u))'$ over (η_k, ξ_{k+1}) yields

$$-F_1(u(\xi_{k+1})) + F_1(u(\eta_k)) = \int_{\eta_k}^{\xi_{k+1}} (u'')^2\, dx = \int_0^{\ell_n} (u_k'')^2\, dx.$$

and hence

$$\int_0^{\delta} (u_k'')^2\, dx \leq -F_1(u(\xi_{k+1})) + F_1(u(\eta_k)).$$

If we now let $k \to \infty$ along the convergent subsequence, we obtain

$$\int_0^{\delta} (U'')^2\, dx \leq -F_1(u^+) + F_1(u^-) = 0.$$

Therefore, $U'' = 0$ on $[0,\delta]$. This means that $U''' = 0$ on $[0,\delta]$ and hence, by (5.8), that $U(x) = u^-$. This completes the proof of Lemma 4.5.

6. Conclusion of the proof of Theorem A.

The continuum Γ^+ connects the lower left endpoint of the strip $\mathcal{E}$ to a segment on the right boundary not containing the lower right endpoint, while vice versa, assuming $F_1(\infty) = \infty$, the continuum Γ^- connects the lower right endpoint to the positive β-axis. Thus it is clear that the interior of $\mathcal{E}$ contains an intersection point of Γ^+ and Γ^-, i.e. a point for which the conclusions of Lemma 3.5 and Lemma 4.5 hold. This completes the proof for the case that $F_1(\infty) = \infty$.

We observe that for the construction of a point in $\mathcal{E}$ belonging to the complement of $\mathcal{S}^+ \cup \mathcal{S}^- \cup \mathcal{T}^+ \cup \mathcal{T}^-$ we do not really need the second continuum. Indeed, if we vary the point (α, β) along Γ^+, it is clear from Lemma 4.4, that near the origin it is contained in $\mathcal{T}^-$. As we approach the right boundary $\{\alpha = 1\}$, β is bounded away from 0 in view of Lemma 3.2, so that the argument in the proof of Lemma 4.3 (with or without the assumption that $F_1(\infty) = \infty$) shows that eventually (α, β) is contained in $\mathcal{T}^+$. Thus, since $\mathcal{T}^+$ and $\mathcal{T}^-$ are open sets, Γ^+ must contain a point belonging neither to $\mathcal{T}^+$ nor $\mathcal{T}^-$. In other words, the shooting argument works with just assumptions H1 and H2. Clearly, by Lemma 4.5, the constructed solution satisfies $u(x) \to 1$ as $x \to -\infty$, which completes the proof of Theorem A.

For the case that H3 fails to be true, the conclusion of Theorem A still holds provided the value of α obtained in the shooting argument satisfies the relation $F_1(\alpha) \le F_1(\infty)$. Since this value is larger than the value α^* where β_1 and β_- intersect, i.e.

$$\int_0^{\alpha^*} sf(s)\,ds = \frac{2}{3^{\frac{3}{2}}} \int_{\alpha^*}^1 (1-s) f(s) ds, \tag{6.1}$$

this relation is certainly satisfied if the assumption H3* below holds.

H3*. $F_1(\alpha^*) \le F_1(\infty)$ where α^* is implicitly defined by (6.1).

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