EXTINCTION AND FOCUSING BEHAVIOUR OF SPHERICAL AND ANNULAR FLAMES DESCRIBED BY A FREE BOUNDARY PROBLEM

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Abstract

We consider a free-boundary problem for the heat equation which arises in the description of premixed equi-diffusional flames in the limit of high activation energy. It consists of the heat equation

$$u_t = \Delta u, \quad u > 0,$$

posed in an apriori unknown set $\Omega \subset Q_T = \mathbf{R}^N \times (0, T)$ for some $T > 0$ with boundary conditions on the free lateral boundary $\Gamma = \partial\Omega \cap Q_T$ (the flame front):

$$u = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu} = -1.$$

We impose initial condition $u_0(x) \geq 0$ on the known initial domain $\Omega_0 = \overline{\Omega} \cap \{t = 0\}$. The paper establishes a theory of existence, uniqueness and regularity for radial symmetric solutions having bounded support. We remark that such solutions vanish on finite time (extinction phenomenon). In the paper we analyze the different types of possible extinction behaviour. We also investigate the focusing behaviour for solutions whose support expands in finite time to fill a hole. In all the cases the asymptotic behaviour is shown to be self-similar.

AMS (MOS) Subject Classification: 35K55, 35K65.

Key Words. Heat equation, flame propagation, free boundary problem, extinction, focusing, asymptotic behaviour.

Introduction

In this paper we study a free boundary problem for the heat equation which arises in combustion theory to describe the propagation of premixed equi-diffusional flames in the limit of high activation energy. The problem is to find a function $u(x,t)$ which solves the heat equation

$$u_t = \Delta u, \quad u > 0 \quad \text{in} \;\; \Omega, \tag{0.1}$$

where $\Omega \subset Q_T = \mathbf{R}^N \times (0,T)$ for some $T > 0$ is an *a priori* unknown domain with lateral boundary $\Gamma = \partial\Omega \cap Q_T$, called the *free boundary,* which represents in combustion the *flame front.* For convenience of presentation u is taken to be positive, though in the combustion application the temperature in the fresh zone, T, is lower than the constant temperature at the front, T_c; accordingly, we have set $u = T_c - T$. The region Ω represents the *fresh* or *unburnt zone.* On the free boundary Γ we impose the conditions

$$u = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu} = -1, \tag{0.2}$$

where $\partial/\partial\nu$ denotes the derivative with respect to the outward spatial normal ν to Γ. The first says that the (normalized) temperature is continuous across the front, while the second expresses the heat production at the front. The fixed-slope condition makes this problem very different from the Stefan problem. The lateral boundary is also assumed to start from the initial position $\Gamma_0 = \partial\Omega_0$ with $\Omega_0 \neq \emptyset$ and the initial condition is given by

$$u(x,0) = u_0(x) > 0 \quad \text{in} \;\; \Omega_0. \tag{0.3}$$

A precise statement of the problem was given in [CV], where a weak form of these conditions was introduced and existence and regularity of such solutions was established in the general N-dimensional setting under suitable conditions on the data. The approach in [CV] is based on approximation by problems of the form

$$u_t = \Delta u_\varepsilon - \beta_\varepsilon(u_\varepsilon) \quad \text{in} \;\; Q_T, \tag{0.4}$$

and then taking $\varepsilon \to 0$. Such approach comes from the derivation of the model for flame propagation at the high activation energy limit under a number of simplifying assumptions, as proposed by Zeldovich and Frank-Kamenetskii in 1938, [ZFK]. It is called the thermo-diffusive model. High-activation energy asymptotics has become an important tool of practical investigation in combustion problems and much progress has been achieved, cf. [BL], [W], but a fully rigorous mathematical study of the area is still necessary.

The same type of free-boundary problem in one space dimension was proposed by Florin [Fl] in 1951 in the study of groundwater filtration. Existence and uniqueness of classical solutions for a particular setting in a half-line was proved first by Ventsel' [Ve] and then extended to local-in-time existence for a problem in a half strip in two space dimensions by Meirmanov [M]. His solutions are periodic in the transversal variable. Recently, Andreucci and Gianni [AG] constructed local-in-time classical solutions of a two-phase version of the above problem, $N \geq 1$. There are also a series of papers devoted to the global study of travelling waves for our model and other more general systems of combustion, done by Berestycki and collaborators, cf. [BNS] and [BCN] among others. For a more detailed overview of the different models related to our equation and the current mathematical situation we refer the reader to the work of one of the authors [V2].

The study of global weak solutions proposed in [CV] faces mathematical difficulties, for instance uniqueness does not hold in general. It is therefore convenient to consider simpler geometries. Thus, in one space dimension, $N = 1$, a different and mathematically simpler approach can be used based on transforming the problem into a problem of mixed, elliptic-parabolic, type. This was studied in [H], where global existence and uniqueness are proved for Ω_0 an interval. Regularity of the solution and of the free boundary, also called interface, for $N = 1$ has been investigated in [HH1], [HH2]. Meirmanov [M] proves global existence and uniqueness of classical solutions for his two-dimensional problem under assumptions which imply monotonicity in the direction of the strip.

In this work we perform a general investigation of global solutions for the multi-dimensional problem under the assumption of radial spatial symmetry. The study has two parts. In the first one we build a theory for the problem starting in Section 1 with existence, uniqueness and comparison by means of an elliptic-parabolic approach. This is essentially a one-dimensional technique. The adaptation to several dimensions performed here is not standard and leads to curious nonlinear boundary conditions of the third type. We check in Section 2 that the solution thus obtained coincides with the high-activation energy limit solution of [CV]. Let us remark that our present formulation allows for expanding and contracting supports (which correspond respectively to receding or advancing flames in the application we started from). Section 3 establishes the analytic regularity of the interfaces. Here we straighten the free boundary via the classical von Mises transformation and borrowing some ideas from [A2].

The second part of the paper is devoted to a detailed study of the phenomena labeled as *extinction* and *focusing*. It was shown in [CV] that if u_0 is a compactly supported function then the solution vanishes in a finite time $T = T(u_0)$ called the *extinction time,* i.e.,

$$u(x,T) \equiv 0 \quad \text{and} \quad u(x,t) \not\equiv 0 \text{ for all } t \in (0,T). \tag{0.5}$$

In the combustion application the flame front advances towards the fresh domain which is completely burnt in finite time. We will study here the asymptotic behaviour of the solution as $t \to T^-$ near an *extinction point,* x_0, i.e., a point of the set

$$E = E(u_0) = \{x \in \mathbf{R}^N : \exists \ \{x_n\} \to x \text{ and } \{t_n\} \to T^- \text{ such that } u(x_n,t_n) > 0\}, \tag{0.6}$$

called the *extinction set*, which is nonempty under our hypotheses. We describe the finite-time extinction process in the following cases:

(i) Single point extinction of radial symmetric solutions, $u = u(r,t)$, $r = |x|$, with support in a ball, Sections 4 - 6. The precise extinction is described in Theorem 6.6. The technique of proof combines a priori estimates for the rescaled solution and its interface, different comparison techniques and mass estimates, to finish with a Lyapunov functional and usual dynamical systems approach.

(ii) Single point extinction of $u(r,t)$ when the support (i.e., the fresh zone) is an annulus, given by a self-similar solution, Section 5.

(iii) One-dimensional extinction with general initial data, Section 7. A symmetric self-similar asymptotic behaviour is obtained even without the assumption of radial symmetry on the data, cf. Theorem 7.1.

(iv) Extinction on a sphere for annular support in several dimensions, Section 8. The asymptotic profile is now found as the self-similar solution of a one-dimensional problem (so-called *approximate self-similarity* [SGKM, Chapter 4] or *asymptotic simplification* [V1]). Since the solution concentrates on a tiny annulus around a sphere $|x| = r_*$ the effect of the curved geometry is lost in first approximation. Cf. Theorem 8.2.

Finally, we discuss in Section 9 the phenomenon of *focusing* at the origin in finite time. By focusing we mean the phenomenon whereby a solution supported in the complement of a ball advances to fill the ball in finite time, and the flame front moves accordingly until it disappears at the origin. The reader will readily observe that this is a kind of extinction phenomenon converse to the previous ones, whereby the combustion process is switched off by freezing. We show that also in this case the asymptotic behaviour is described by a unique self-similar solution, cf. Theorems 9.1 and 9.2.

Let us recall that in the symmetric one-dimensional case the extinction behaviour was proved to be asymptotically self-similar in [HH2]. Let us also mention that general solutions in $N = 1$ for a related problem with convection were studied in [BHS] and the stability of travelling waves is studied in [BLS].

Concerning the applications suggested above, the reader should bear in mind that our asymptotic results must be understood in the sense of intermediate asymptotics, as explained in [B]. In the last stage, when we are too close to the focusing or extinction time, the curvature of the free boundary tends to infinity and then the basic idea of replacing a thin reaction zone by a surface is not justified and the problem ceases to faithfully represent the physical process.

We end this Introduction by a more general reflection. The main mathematical points underlying the asymptotic study are on one hand the occurrence of self-similar singularities for the different problems of extinction and focusing, on the other hand the methods of analysis which allow to establish exact asymptotic behaviour. We contend that these methods can have wide applicability to different free boundary problems for other semilinear and quasilinear heat equations admitting finite time extinction or blow-up. Finally, it has to be said that a full understanding of these phenomena for asymmetrical configurations is still pending.

PART I

1. The elliptic-parabolic approach. Uniqueness and comparison

In this section we establish two well-posedness results for solutions of the combustion problem under the assumption of spatial radial symmetry. We establish the well-posedness in a rather weak class of solutions (Theorems 1.8 and 1.10) and then show that the solution has in fact a much higher regularity (Section 3). This section is based on an adaptation of the theory of elliptic-parabolic initial and boundary value problems to the case of nonlocal boundary conditions. The basic ideas are explained next. We shall restrict our attention to the case $N \geq 3$. The case $N = 2$ is left to the reader for space reasons. We consider in detail the case when the initial function is supported on a ball. The case of an annular support offers small variations. We make a brief comment at the end. When discussing focusing in Section 9 we will consider the opposite case where the support is the complement of a ball.

In order to keep the correspondence with previous results for elliptic-parabolic problems we will work in this section with nonpositive solutions, just changing u into $-u$, which is is anyway closer to the proposed physical model where temperatures in the fresh zone are below the critical temperature. In doing this we have to replace the radial free-boundary condition by

$$u(\zeta(t), t) = 0, \quad u_r(\zeta(t), t) = 1, \tag{1.1}$$

where $r = \zeta(t)$ denotes the interface.

Formulation. Our results are based upon the observation that radial (negative) solutions can be naturally extended as solutions to an *elliptic-parabolic* problem on a fixed ball containing the supporting balls for all $0 \leq t \leq T$, thus arriving at a problem of mixed type in a fixed domain, which is easier to solve. Formally, this is done as follows. Consider the equation

$$(c(u))_t = \Delta u, \tag{1.2}$$

where $c(s) = \min\{0, s\}$, and suppose that u is a solution. Then u solves the heat equation if $u < 0$, while for $u > 0$ it is harmonic in x. We can extend u to the region $r > \zeta(t)$ by setting $\Delta u = 0$ there, which by (1.1) implies that

$$(r^{N-1} u_r(r,t))_r = 0 \quad \Rightarrow \quad u_r(r,t) = (\frac{\zeta(t)}{r})^{N-1}, \tag{1.3}$$

hence

$$u(r,t) = \int_{\zeta(t)}^{r} \left(\frac{\zeta(t)}{\rho}\right)^{N-1} d\rho = \frac{\zeta(t)}{N-2}\left[1 - \left(\frac{\zeta(t)}{r}\right)^{N-2}\right]. \tag{1.4}$$

This turns u into a radial solution of (1.2) which has a jump in u_t and Δu across the free boundary. Fixing a ball with radius R containing the support of the original solution for all $t \in [0,T]$, we obtain nonlocal boundary conditions on ∂B_R of the form

$$u_r(R,t) = (\zeta(t)/R)^{N-1} \quad \text{(Neumann)}, \tag{1.5}$$

and

$$u(R,t) = \frac{\zeta(t)}{N-2}\left[1 - \left(\frac{\zeta(t)}{R}\right)^{N-2}\right] \equiv F(\zeta(t),R) \quad \text{(Dirichlet)}. \tag{1.6}$$

Here $r = \zeta(t)$ is now the a priori unknown level set of $u = 0$. We can eliminate $\zeta(t)$ from these two conditions to obtain

$$u(R,t) = \frac{R}{N-2}(u_r(R,t)^{\frac{1}{N-1}} - u_r(R,t)) = \mathbf{G}(u_r(R,t)), \tag{1.7}$$

which however is not completely straightforward to work with, in view of the fact that $\mathbf{G}$ is a nonmonotone function of u_r. Both functions F and $\mathbf{G}$ depend on N. We shall therefore first base our approach on (1.5) and (1.6). Later on we shall use (1.7) for an alternative approach in order to get more regularity. Both functions F and $\mathbf{G}$ depend on N.

We note that as a function of ζ this function $F(\zeta,R)$ is increasing for $0 \le \zeta < R(N-1)^{-1/(N-2)}$ and decreasing for $R(N-1)^{-1/(N-2)} < \zeta \le R$. We shall choose R appropriately to ensure that we will be in the latter situation, for then, bearing in mind that larger solutions have smaller interfaces, we can set up a monotone iteration scheme.

Before we do this however it is necessary to recall and adapt some aspects of the elliptic-parabolic theory in the radially symmetric case. To this end, we consider the following auxiliary Dirichlet problem:

$$(P_D) \qquad \begin{cases} c(u)_t = \Delta u & \text{in } Q_T = B_R \times (0,T]\,, \\ u(R,t) = f(t) > 0 & \text{for } 0 < t \le T\,, \\ c(u(x,0)) = v_0(x) & \text{for } |x| \le R\,. \end{cases}$$

Here $v_0(r) = c(u_0(r))$, which is understood to be zero for $r > \zeta_0$.

Definition 1.1. *A function $u \in L^2(0,T;H^1(B_R))$ with $c(u) \in C(\overline{Q}_T)$ is called a solution of (P_D) on $[0,T]$ if*

$$\int\int_{Q_T} (-\varphi_t c(u) + \nabla u \nabla \varphi)dxdt + \int_{B_R} \varphi(x,T)c(u(x,T))dx = \int_{B_R} \varphi(x,0)v_0(x)dx$$

for all $\varphi \in H^1(Q_T)$ which vanish at ∂B_R, and if u satisfies the Dirichlet boundary condition in the a.e. sense.

We also consider another auxiliary problem, the Neumann problem:

$$(P_N) \qquad \begin{cases} c(u)_t = \Delta u & \text{in } Q_T = B_R \times (0,T]\,, \\ u_r(R,t) = g(t) > 0 & \text{for } 0 < t \le T\,, \\ c(u(x,0)) = v_0(x) & \text{for } |x| \le R\,, \end{cases}$$

Definition 1.2. *A function* $u \in L^2(0,T;H^1(B_R))$ *with* $c(u) \in C(\overline{Q_T})$ *is called a (weak) solution of* (P_N) *on* $[0,T]$ *if*

$$\int\int_{Q_T}(-\varphi_t c(u) + \nabla u \nabla\varphi)dxdt + \int_{B_R}\varphi(x,T)c(u(x,T))dx$$

$$= \int_{B_R}\varphi(x,0)v_0(x)dx + \int_0^T\int_{\partial B_R}\varphi(x,t)g(t)dxdt$$

for all $\varphi \in H^1(Q_T)$.

Usually, one assumes that $f(t)$ and $g(t)$ have some smoothness, e.g. H^1. This is needed for the L^2-estimate for $c(u)_t$ which follows from testing with u_t. In the one-dimensional or radial case, f is often assumed to be Lipschitz continuous [H], so as to construct barrier functions for the regularized problem. Here we really do not want to use, apart from the measurability of f, more than the following bound on f and g:

$$0 < \delta \le f(t),\ g(t) \le M < \infty. \tag{1.8}$$

Clearly with this assumption we expect a zero-set of u to be of the form $r = \zeta(t)$.

We can derive the following results about the solvability of the Dirichlet and Neumann problems.

Proposition 1.3. *Suppose that* v_0 *is continuous and radially symmetric on* B_R*, negative on* $\{|x| < \zeta_0\}$*, zero on* $\{|x| \ge \zeta_0\}$*, where* $\zeta_0 < R$*. Also assume that* $r^{N-1}v_0'(r)$ *is bounded. Suppose that* f *and* g *are measurable and satisfy (1.8). There exist* $T > 0$ *(depending on the initial data) and unique solutions on* $[0,T]$ *to* (P_D) *and* (P_N)*, each having a continuous interface* $0 < |x| = \zeta(t) < R$ *separating the regions where* $u < 0$ *and* $u > 0$*. The time* T *depends only on the numbers* ζ_0*, and* R*, the bounds on* f *and* g*, and the initial data.*

The solution depends monotonically on v_0 *and* f *resp.* g *(weak comparison principle). If* $f_n(t) \to f(t)$ *and* $g_n(t) \to g(t)$ *then the corresponding sequences of solutions converge weakly in* $L^2(0,T;H^1(B_R))$ *to solutions with Dirichlet boundary data* $f(t)$ *and Neumann boundary data* $g(t)$*.*

Proof. For smooth data v_0, f and g these results follow from standard methods, cf. e.g. [H1], [H2]. A standard limit argument extends them to our case. Crucial is the observation that the "flux", i.e., $r^{N-1}u_r$ satisfies a maximum principle. This gives a uniform bound in $L^\infty(0,T;W^{1,\infty}(B_R))$, and together with the weak formulation a uniform bound on the function $r^{N-1}c(u(r,t))$ in $C^{0+1}(\overline{Q}_T)$. □

Moreover, adapting the results in [H1], [H2], we have that these solutions are related in an interesting way.

Proposition 1.4. *The solution to the Dirichlet problem solves the Neumann problem with boundary condition*

$$u_r(R,t) = \frac{(2-N)R^{1-N}}{R^{2-N} - \zeta(t)^{2-N}}f(t), \tag{1.9}$$

and the solution to the Neumann problem solves the Dirichlet problem with boundary condition

$$u(R,t) = \frac{R^{2-N} - \zeta(t)^{2-N}}{(2-N)R^{1-N}} g(t). \tag{1.10}$$

The solution to the Neumann problem has $r^{N-1}u_r(r,t) = R^{N-1}g(t)$ *for* $r > \zeta(t)$.

Also the function $r^{N-1}u_r(r,t)$ *is uniformly bounded by a constant with the same dependence as* T. *Finally, there exists a continuous function* $\psi(t) \geq 0$ *with* $\psi(0) = 0$ *such that*

$$0 \leq t \leq s \leq T \quad \Rightarrow \quad \zeta(s) \leq \zeta(t) + \psi(s-t). \tag{1.11}$$

Again this function depends on the same data as T.

Next we start the *iteration process*. Suppose u is a weak solution as in Propositions 1.3, 1.4. Then its interface $r = \zeta(t)$ is continuous. We consider the following problem for w:

$$(P_i) \qquad \begin{cases} c(w)_t = \Delta w & \text{in } Q_T = B_R \times (0,T]\,, \\ w(R,t) = F(\zeta(t), R) > 0 & \text{for } 0 < t \leq T\,, \\ c(w(,0)) = v_0(x) & \text{for } |x| \leq R\,. \end{cases}$$

Then we have

Proposition 1.5. *Suppose that* v_0 *is continuous and radially symmetric on* B_R, *negative on* $\{|x| < \zeta_0\}$, *zero on* $\{|x| \geq \zeta_0\}$, *where* $\zeta_0 < R$. *Also assume that* $r^{N-1}v_0'(r)$ *is bounded. For every small* $\epsilon > 0$ *and every continuous function* $\zeta_0 - \epsilon \leq \zeta(t) \leq \zeta_0 + \epsilon$ *there exists* $T > 0$ *and a weak solution* w *on* $[0,T]$ *of* (P_i) *with a continuous interface* $\zeta_0 - \epsilon < |x| = \chi(t) < \zeta_0 + \epsilon$ *separating the regions where* $u < 0$ *and* $u > 0$. *The number* T *depends only on the numbers* ζ_0, $\epsilon > 0$ *and* R, *and the function* v_0.

This proposition allows us to construct two monotonic sequences of approximating solutions, one from below, and one from above. Let $\underline{w}_1$ be the solution from Proposition 1.5 corresponding to $\zeta(t) = \zeta_0 + \epsilon$, and $\overline{w}_1$ the solution corresponding to $\zeta(t) = \zeta_0 - \epsilon$. Denote the corresponding interfaces by χ_1^+ and χ_1^-, and let $\underline{w}_2$ and $\overline{w}_2$ be the solutions from Proposition 1.5 corresponding respectively to $\zeta(t) = \chi_1^+(t)$ and $\zeta(t) = \chi_1^-(t)$. And so on. Provided we choose $\epsilon > 0$ and R such that the interval $[\zeta_0 - \epsilon, \zeta_0 + \epsilon]$ is contained in the monotonicity interval $(R(N-1)^{-1/(N-2)}, R)$ we obtain an increasing sequence $(\underline{w}_k)_{k=1}^\infty$ and a decreasing sequence $(\overline{w}_k)_{k=1}^\infty$ with

$$\underline{w}_1 \leq \underline{w}_2 \leq \cdots \leq \overline{w}_2 \leq \overline{w}_1, \tag{1.12}$$

with free boundaries

$$\zeta_0 + \epsilon \geq \chi_1^+(t) \geq \chi_2^+(t) \geq \cdots \geq \chi_2^-(t) \geq \chi_1^-(t) \geq \zeta_0 - \epsilon. \tag{1.13}$$

Consequently, the pointwise limits of these free boundaries exist:

$$\chi^-(t) = \lim_{k\to\infty} \chi_k^-(t) \text{ and } \chi^+(t) = \lim_{k\to\infty} \chi_k^+(t). \tag{1.14}$$

Also the limits

$$\overline{w} = \lim_{k\to\infty} \overline{w}_k \text{ and } \underline{w} = \lim_{k\to\infty} \underline{w}_k \tag{1.15}$$

exist as solutions with the corresponding boundary conditions. Thus by Propositions 1.3, 1.4 we find that the limits are solutions to the Dirichlet problem with boundary data

$$\underline{w}(R,t) = F(\chi^+(t), R) \quad \text{and} \quad \overline{w}(R,t) = F(\chi^-(t), R), \tag{1.16}$$

and to the Neumann problem with boundary data

$$\underline{w}_r(R,t) = (\chi^+(t)/R)^{N-1} \quad \text{and} \quad \overline{w}_r(R,t) = (\chi^-(t)/R)^{N-1}. \tag{1.17}$$

Lemma 1.6. *The functions χ^- and χ^+ are the (continuous) free boundaries of $\overline{w}$ and $\underline{w}$.*

Proof. Suppose this is false. Clearly the real (continuous) free boundary of the limit is then to the left of respectively χ^- and χ^+. These latter functions still satisfy the uniform estimates in Propositions 1.3, 1.4. The only way for χ^- and χ^+ to be discontinuous is to have a jump to the left. But then we can construct a rectangle on which the sequence, but not its limit, is strictly negative, while on a part of the parabolic boundary of this rectangle the sequence is bounded away from zero. This is impossible in view of the Strong Maximum Principle for the heat equation. □

Corollary 1.7. *The convergence in (1.14) is uniform.*

Proof. This follows from Dini's Theorem. □

We summarize our results so far in the following theorem.

Theorem 1.8. *Suppose that v_0 is continuous and radially symmetric on B_R, negative on $\{|x| < \zeta_0\}$, zero on $\{|x| \geq \zeta_0\}$, where $\zeta_0 < R$. Assume that ζ_0 lies in the interval $(R(N-1)^{-1/(N-2)}, R)$ and that $r^{N-1}v_0'(r)$ is bounded. Then there exists $T > 0$ and a unique function $u \in L^2(0,T;H^1(B_R))$ which has a continuous interface $r = \zeta(t)$ such that u is a weak solution to the Neumann problem with (1.5) and to the Dirichlet problem with (1.6). The weak formulation is equivalent to*

$$\int_0^T \int_0^{\zeta(t)} (-r^{N-1}\varphi_t c(u) + r^{N-1}u_r\varphi_r)dr\,dt + \int_0^{\zeta(T)} r^{N-1}\varphi(r,T)c(u(r,T))dr$$
$$= \int_0^{\zeta_0} r^{N-1}\varphi(r,0)u_0(r)dr + \int_0^T \varphi(\zeta(t),t)\zeta(t)^{N-1}dt,$$

for all $\varphi \in H^1(Q_T)$. In this sense the pair (u,ζ) is the unique solution. Moreover, the comparison principle holds: if we have two solutions (u_i,ζ_i), $i = 1,2$ and their initial data v_{0i} and ζ_{0i} are ordered, $v_{01} \leq v_{02}$ and $\zeta_{01} \geq \zeta_{02}$, then the solutions (and interfaces) are ordered in the same way.

Proof. We already know that $\underline{w}$ and $\overline{w}$ are solutions. Let us show that these two coincide. By construction we have $\underline{w} \leq \overline{w}$. Now, for any two solutions u and $\tilde{u}$ we set $w = u - \tilde{u}$ and

$$\varphi(r,t) = \int_t^T (u(r,s) - \tilde{u}(r,s))ds, \tag{1.18}$$

and observe that since

$$
\begin{aligned}
(1.19)\qquad & \int_0^T\int_0^R(\varphi_r u_r-\varphi_t c(u))r^{N-1}dr\,dt\\
&=\int_0^R\varphi(r,0)c(u(r,0))r^{N-1}dr+\int_0^T\varphi(R,t)\zeta(t)^{N-1}dt,
\end{aligned}
$$

and a similar formula for $\tilde{u}$, subtraction gives

$$
\begin{aligned}
(1.20)\qquad & \int_0^R\int_0^T w_r(r,t)\int_t^T w_r(r,s)ds dt\, r^{N-1}dr+\int_0^T\int_0^R(u-\tilde{u})(c(u)-c(\tilde{u}))r^{N-1}dr\,dt\\
&=\int_0^T\int_t^T w(R,s)ds(\zeta(t)^{N-1}-\tilde{\zeta}(t)^{N-1})dt,
\end{aligned}
$$

or

$$
\begin{aligned}
(1.21)\qquad & \frac{1}{2}\int_0^R\left(\int_0^T w_r dt\right)^2 r^{N-1}dr+\int_0^T\int_0^R(u-\tilde{u})(c(u)-c(\tilde{u}))r^{N-1}dr\,dt\\
&=\int_0^T\int_t^T[F(\zeta(s),R)-F(\tilde{\zeta}(s),R)]ds(\zeta(t)^{N-1}-\tilde{\zeta}(t)^{N-1})dt.
\end{aligned}
$$

Now suppose that (u,ζ) and $(\tilde{u},\tilde{\zeta})$ are ordered, say $u\le\tilde{u}$. Since we are looking at nonpositive solutions, this implies that $\zeta\ge\tilde{\zeta}$ (reversed order). Clearly the left-hand side of (1.21) is strictly positive, unless the two solutions are identical. Thus we can force a contradiction by making the right-hand side negative. In order to do so we choose $R>0$ such that $\zeta(0)$ lies between $R(N-1)^{-1/(N-2)}$ and R, which is the range in which $F(\cdot,R)$ is decreasing. Since both $\zeta(t)$ and $\tilde{\zeta}(t)$ are continuous, we can then choose $T>0$ so small that

$$
\forall\, t\in[0,T]:\quad \zeta(t),\tilde{\zeta}(t)\in(R(N-1)^{-1/(N-2)},R).
$$

But then the integral in the right hand side is clearly negative, which is a contradiction.

In particular, this argument implies that $\overline{w}=\underline{w}$. It remains to show that any other solution lies between the approximations from below and above. Suppose that u is such another solution. We know then that u is a fixed point of the iteration map. Also, given $\epsilon>0$ there exists $T_\epsilon>0$ such that $\zeta_0-\epsilon<\zeta(t)<\zeta_0+\epsilon$ for $0\le t\le T_\epsilon$. But then, in view of the monotonicity, we have for all $t\le\min(T,T_\epsilon)$ and for all n that u lies between $\underline{w}_n$ and $\overline{w}_n$. Hence $\overline{w}=u=\underline{w}$.

¿From the construction of the solution it follows that a solution in B_R is also a solution on every B_ρ provided $\rho>\zeta(t)$. The final assertion in Theorem 1.8 follows again from setting $r^{N-1}u_r(r,t)=\zeta(t)^{N-1}$ for $r>\zeta(t)$. Observe that the uniqueness is really in the class of solutions for which ζ is continuous in $t=0$. Comparison follows easily from the construction process with $\underline{w}$ and $\overline{w}$. $\square$

At this stage we do not yet have the regularity $\Delta u \in L^2(Q_T)$. This would give the boundary condition for u_r in the a.e. sense. As already observed, this result usually follows from testing the classical solutions of the approximating uniformly parabolic problems with test function u_t. We now use the nonlinear boundary condition condition (1.7). The function $\mathbf{G}$, which depends on the choice of R and N, is not invertible: it has a maximum

$$\mathbf{G}(a_N) = \mathbf{G}((N-1)^{\frac{N-1}{2-N}}) = (N-1)^{\frac{N-1}{2-N}} R = b_N R > 0, \ 0 < a_N < 1. \tag{1.22}$$

Thus, if we want to write (1.7) as a nonlinear boundary condition of the form $u_r = F(u)$ we have to choose a monotone branch to invert. Choosing $R > \zeta_0$ close to ζ_0 it is clear that we have to take the decreasing branch containing the values $(u, u_r) = (0, 1)$. Thus, we define the function

$$F : [0, b_N R] \ \to \ [a_N, 1] \quad \text{by} \quad \mathbf{G}(F(u)) = u. \tag{1.23}$$

For $u > b_N R$ we simply set $F(u) = F(b_N R)$, and consider the following elliptic-parabolic problem with a nonlinear Neumann boundary condition:

$$(P_{NL}) \qquad \begin{cases} c(u)_t = \Delta u & \text{in } Q_T = B_R \times (0, T]\,, \\ u_r(R, t) = F(u(R, t)) & \text{for } 0 < t \le T\,, \\ c(u(x, 0)) = v_0(x) & \text{for } |x| \le R\,. \end{cases}$$

Here the function v_0 is defined by $v_0 = c(u_0)$, where u_0 is any extension of the given initial profile u_0 on $[0, \zeta_0]$ such that $u_0 \ge 0$ on $[\zeta_0, R]$. The precise choice of this extension does not enter in the concept of a weak solution. However, there is only one choice which leads solutions u which are continuous down to $t = 0$ over the whole of $[0, R]$. This is done by making the extension consistent with the expected behaviour $u > 0$. Thus for $r > \zeta_0$ the function u_0 is defined by (1.4) with $\zeta(t)$ replaced by ζ_0.

Definition 1.9. *A function $u \in L^2(0, T; H^1(B_R))$ is called a (weak) solution of (P_{NL}) on $[0, T]$ if $c(u) \in C(Q_T)$, and if*

$$\begin{aligned} &\int\!\!\int_{Q_T} (-\varphi_t c(u) + \nabla u \nabla \varphi) dx dt + \int_{B_R} \varphi(x, T) c(u(x, T)) dx \\ &= \int_{B_R} \varphi(x, 0) v_0(x) dx + \int_0^T \int_{\partial B_R} \varphi(x, t) F(u(R, t)) dx dt \end{aligned}$$

for all $\varphi \in H^1(Q_T)$.

Theorem 1.10. *Suppose that v_0 is continuous and radially symmetric on B_R, negative on $\{|x| < \zeta_0\}$, zero on $\{|x| \ge \zeta_0\}$, where $\zeta_0 < R$. Then for R close to ζ_0 and T small, there exists a solution u of (P_{NL}) which coincides with the solution constructed in Theorem 1.8. This solution has $\Delta u \in L^2(Q_T)$, $u \in L^\infty(0, T; H^1(B_R))$ and it satisfies the free boundary conditions in the almost everywhere sense.*

Proof. We approximate the problem with a sequence of uniformly parabolic problems in the usual way. Note that F is positive and bounded away from zero and infinity. Thus, we already have the estimates for $r^{N-1}u_r$ and u from the treatment of the standard Neumann problem in [H]. Omitting the subscripts and multiplying by u_t we obtain the following estimate:

$$\int\int_{Q_T} u_t c(u)_t + \int_{B_R} [\frac{1}{2}|\nabla u|^2]_{t=0}^{t=T} dx = \int_0^T \int_{\partial B_R} u_t(R,t)F(u(R,t))dxdt. \tag{1.24}$$

The last term can be integrated and is bounded.

We can now pass to the limit and obtain a weak solution. Again using the a priori bound on $F(u)$ away from zero and infinity, the continuity of the interface follows. Also, the free boundary conditions are easily derived from solving the two-point boundary valued problem for $(r^{N-1}u_r)_r = 0$ on $\zeta(t), R)$. Thus this new solution is also a solution in the sense of Theorem 1.8. Therefore it is unique. Note that we do not give a direct uniqueness proof for (P_{NL}).

Solutions in an annulus. We conclude this section with a discussion of the combustion problem in an annulus. We adjust the previous proof by also extending the solution to the inside. We call the outer free boundary $\zeta(t)$ and the inner free boundary $\xi(t)$. Since $u_r = -1$ on the inner free boundary, imposing $\Delta u = 0$ for $r < \xi(t)$ implies

$$r^{N-1}u_r(r,t) = -\xi(t)^{N-1}, \tag{1.25}$$

so that

$$u(r,t) = \int_r^{\xi(t)} \left(\frac{\xi(t)}{\rho}\right)^{N-1} d\rho = \frac{\xi(t)}{N-2}\left[\left(\frac{\xi(t)}{r}\right)^{N-2} - 1\right], \tag{1.26}$$

and at a fixed $R_0 < \xi(t)$ we obtain the boundary conditions

$$-u_r(R_0,t) = (\frac{\xi(t)}{R_0})^{N-1}; \quad u(R_0,t) = \frac{\xi(t)}{N-2}\left[\left(\frac{\xi(t)}{R_0}\right)^{N-2} - 1\right] \equiv G(\xi(t),R_0). \tag{1.27}$$

In the case of the ball we had to make sure that F was decreasing in ζ. Because for the inner free boundaries the ordering is the same as that of the solution, we need G to be increasing in ξ. Thus the same arguments as in the case of the ball apply, for indeed the function G is increasing in ξ for $\xi > R_0$. Finally we can again eliminate the free boundary from (1.27) to obtain

$$u(R_0,t) = \frac{R_0}{N-2}[-u_r(R_0,t) + (-u_r(R_0,t))^{\frac{1}{N-1}}], \tag{1.28}$$

which can be treated in a similar fashion as above.

2. Comparison with the high-activation limit

In [CV] solutions to the problem have been obtained as limits of the solutions to the approximate semilinear equations (E_ε),

$$u_t = \Delta u - \beta_\varepsilon(u)\,, \tag{2.1}$$

where $\beta_\varepsilon : \mathbf{R} \to \mathbf{R}$ is C^∞-smooth, nonnegative and bounded, with $\beta_\varepsilon(s) = 0$ for $s \le 0$ and support in a small neighbourhood of $s = 0$. It is convenient to define the functions β_ε by scaling

$$\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta(\frac{s}{\varepsilon})\,. \tag{2.2}$$

in terms of a fixed $\beta : \mathbf{R} \to \mathbf{R}$ which is asked to satisfy the following assumptions:

(i) β is positive in the interval $I = \{0 < s < 1\}$ and 0 otherwise,

(ii) it is a C^∞-function in $[0, \infty)$,

(iii) it is increasing for $0 \le s < 1/2$, decreasing for $1/2 < s \le 1$,

(iv) the integral of β, $\int \beta(s)\, ds$, equals $1/2$.

The normalizing factor $1/2$ is needed to obtain in the limit the second of the free boundary conditions (0.2) with slope precisely 1, cf. [BCN], [CV]. We also take initial conditions

$$u_\varepsilon(x, 0) = u_{0\varepsilon}(x)\,, \tag{2.3}$$

where the $u_{0\varepsilon}$ are C^∞-smooth and nonnegative approximations of u_0.

We define a *weak formulation* for problem (P), valid for N-dimensional problems with general geometry, as follows: we look for a domain Ω in $\mathbf{R}^N \times \mathbf{R}_+$ with Lipschitz continuous lateral boundary Γ, and a positive function $u \in C(\Omega \cup \Gamma)$ such that:

(i) for every test function $\phi \in C_0^\infty(\mathbf{R}^N \times [0, T))$

$$\iint_\Omega u\,(\phi_t + \Delta\phi)\, dx\, dt + \int_{\Omega_0} u_0\, \phi\, dx = \int_\Gamma \phi\, d\Sigma\, \cos\alpha\,, \tag{2.4}$$

(ii) u vanishes on Γ, and

(iii) the free boundary Γ starts from $\Gamma_0 = \partial\Omega_0$, i.e., the section Γ_t at time t converges to Γ_0 as $t \to 0$ in some suitable sense.

In (2.4) $d\Sigma$ is the area element on Γ and α is the angle formed by the exterior normal $\nu(x, t)$ at a point $(x, t) \in \Gamma$ and the hyperplane $t =$ constant, so that $dS = d\Sigma\, \cos\alpha$ is the space projection of the element $d\Sigma$.

Here we will call this solution a *limit solution* to make the difference with the approach of the previous section. As a consequence of the results of [CV] we have the following existence result for our solutions. In keeping with the restriction of this paper we will assume that the initial data satisfy:

(ii) $u_0(x) = u_0(r)$ is radially symmetric and $u_0(r)$ is a positive C^1-function in an interval $0 \le r < r_0$, with $u_0(r) = 0$ for $r \ge r_0$.

(ii) We have $u_0'(r)$ bounded and $|u'(r_0)| < 1$. Also $\Delta u_0 < 0$ in $B_{r_0}(0)$.

Assumption (ii) can be weakened, see (H1), (H2) in Section 8 of [CV]. Then the results of [CV] imply:

Theorem 2.1. *Under the above conditions problem (0.1)-(0.3) has a radially symmetric limit solution with a Lipschitz-continuous interface* $t = \theta(r)$ *with* $\theta(r_0) = 0$*, which satisfies the boundary conditions (0.2) for a.e.* t*. Such solution vanishes in a finite time. Moreover,* $u_t \le 0$ *and the interface* $t = \theta(r)$ *is monotone nonincreasing.*

In our conditions $u_t < 0$ means an advancing flame. Let us show that this solution coincides with the weak solution obtained in the previous section from the elliptic-parabolic approach. We need the following result, which is a standard consequence of the Strong Maximum Principle for the heat equation.

Lemma 2.2. *The function* $t = \theta(r)$ *does not have any horizontal segments.*

Therefore, we can write the interface as $r = \zeta(t)$ with a continuous and decreasing ζ defined in $[0, T]$. Then we have

Theorem 2.3. *Once properly extended, the limit solution is a solution in the sense of Definition 1.9, hence it is unique.*

Proof. In order to obtain from u defined in $\Omega = \{(x,t) : 0 < |x| = r < \zeta(t), 0 \le t < T\}$ a weak solution v of problem (PL) of the previous section we put $v = -u < 0$ in Ω and perform the harmonic extension (1.3)-(1.4) in the region $Q_T \setminus \Omega$. We take R large enough. Observe that in this way relation (1.7) is satisfied, i.e., $v = \mathbf{G}(v_r)$ for $|x| = R$. It is then an exercise to integrate by parts equations (2.4) to get the identity in Definition 1.9. $\square$

3. Analyticity

In this section we establish the analyticity of our solutions and their free boundaries. The method we use is due to Angenent (e.g. [A3]) who showed that a uniformly parabolic quasilinear equation with autonomous analytic coefficients and boundary data generates a nonlinear analytic semiflow in a suitable interpolation space between the domain and the range of the linearized operator. This is based on Da Prato's and Grisvard's so-called maximal regularity theory for linear semigroups.

Thus, it suffices to transform our free boundary problem into such an equation with fixed boundary data. Let us first consider the case that the initial support is a ball and that the initial function is radially symmetric and monotone, continuously differentiable and satisfies the boundary conditions.

As in Section 1 we consider solutions $u \le 0$ with free boundary condition $\partial_\nu u = 1$. We begin with choosing a real analytic diffeomorphism $(X(\xi, \eta), Y(\xi, \eta))$ between a rectangle

$$R = \{(\xi, \eta) : \xi_1 \le \xi \le \xi_2, \eta_1 \le \eta \le \eta_2\} \tag{3.1}$$

in the (ξ, η)-plane and a closed neighbourhood of the graph of the initial profile u_0 in the (r, u)-plane, with some additional properties. We need

$$X(\xi_1, \eta) = Y(\xi_2, \eta) = 0, \tag{3.2}$$

and also that the inverse image of the the graph of u_0 is itself the graph of a continuously differentiable function $\eta = v_0(\xi)$. This ensures that a free boundary starting at ζ_0 corresponds to the fixed boundary at $\xi = \xi_2$. Finally, it will be convenient to have that, up to a positive multiple, the Jacobian matrix of $(X(\xi,\eta), Y(\xi,\eta))$ is the identity matrix at $\xi = \xi_1$ and a rotation of angle $+\pi/2$ at $\xi = \xi_2$. Next we define a function $\eta = v(\xi, t)$ by

$$u(X(\xi, v(\xi,t)), t) = Y(\xi, v(\xi,t)), \tag{3.3}$$

where u is the as of yet unknown solution to the free boundary problem. In other words, the graph $\eta = v(\xi,t)$ is the inverse image of the graph of u.

In the case of monotone initial data such a diffeomorphism is easily constructed using the complex exponential function. We choose

$$(X(\xi,\eta), Y(\xi,\eta)) = (e^{-\eta}\sin\xi, -e^{-\eta}\cos\xi), \quad \xi_1 = 0,\, \xi_2 = \frac{\pi}{2}, \tag{3.4}$$

so that

$$\frac{\partial(X,Y)}{\partial(\xi,\eta)} = \begin{pmatrix} e^{-\eta}\cos\xi & -e^{-\eta}\sin\xi \\ e^{-\eta}\sin\xi & e^{-\eta}\cos\xi \end{pmatrix}, \tag{3.5}$$

whence

$$\left|\frac{\partial(X,Y)}{\partial(\xi,\eta)}\right| = e^{-2\eta}. \tag{3.6}$$

In sufficiently small (with respect to the norm in the space of continuously differentiable functions) neighbourhoods of u_0 and v_0, there is a one to one correspondence between u and v, and $u \to u_0$ is equivalent to $v \to v_0$. If we take X and Y defined by (3.4), our mapping is in fact conformal and it is easy to see that any nonincreasing function $u \in C^1([0,\zeta])$ with $u'(0) = 0$ and $u'(\zeta) = 1$ corresponds to a function $v \in C^1([0,\frac{\pi}{2}])$ with $v'(0) = 0$ and $v'(\frac{\pi}{2}) = -1$.

Using (3.3) we derive an equation for v. Differentiating with respect to ξ we obtain

$$u_r(X_\xi + X_\eta v_\xi) = Y_\xi + Y_\eta v_\xi \quad \Rightarrow \quad u_r = \frac{Y_\xi + Y_\eta v_\xi}{X_\xi + X_\eta v_\xi}. \tag{3.7}$$

Since both u_0 and v_0 are continuously differentiable, the denominator in (3.7) is nonzero if we replace v by v_0. Hence the same is true for $v(\xi)$ in C^1-neighbourhood of v_0. Note that the formula for u_r is really the ratio between the Y- and the X-increments along the graph of the function $v(\cdot,t)$. Next, differentiating with respect to t, using (3.7),

$$u_r X_\eta v_t + u_t = Y_\eta v_t \quad \Rightarrow \quad v_t = \frac{u_t}{Y_\eta - u_r X_\eta} = \frac{X_\xi + X_\eta v_\xi}{X_\xi Y_\eta - X_\eta Y_\xi} u_t. \tag{3.8}$$

Note that the denominator in the last expression of (3.8) is the positive Jacobian of the diffeomorphism (X, Y). Multiplying the second equation in (3.7) by $r^{N-1} = X^{N-1}$ and differentiating once more with respect to ξ gives

$$(3.9) \qquad (r^{N-1}u_r)_r(X_\xi + X_\eta v_\xi) = \big(X^{N-1}\frac{Y_\xi + Y_\eta v_\xi}{X_\xi + X_\eta v_\xi}\big)_\xi,$$

so that the radial heat equation for u transforms into

$$(3.10) \qquad \begin{aligned} &(X_\xi Y_\eta - X_\eta Y_\xi)v_t = X^{1-N}\big(X^{N-1}A(\xi, v, v_\xi)\big)_\xi = \\ &A(\xi, v, v_\xi)_\xi + (N-1)\frac{X_\xi + X_\eta v_\xi}{X}A(\xi, v, v_\xi), \end{aligned}$$

with

$$(3.11) \qquad A(\xi, v, p) = \frac{Y_\xi + Y_\eta p}{X_\xi + X_\eta p} \quad \Rightarrow \quad \frac{\partial A}{\partial p} = \frac{X_\xi Y_\eta - X_\eta Y_\xi}{(X_\xi + X_\eta p)^2} > 0.$$

The boundary conditions become, in view of the assumptions on the Jacobian matrix at $\eta = 0$ and $\eta = \pi/2$,

$$(3.12) \qquad v_\xi(0, t) = 0, \quad v_\xi(\pi/2, t) = -1.$$

Instead of the combustion problem we look at equation (3.10) with boundary conditions (3.12), and initial data v_0. It follows from our construction that this equation is uniformly parabolic in a C^1-neighbourhood of the initial profile $\eta = v_0(\xi)$. Note however that the first-order terms contain a singularity similar to the that in the radial laplacian.

With (3.4) the transformed equation (3.10) reads

$$(3.13) \qquad v_t = e^{2v}\Big[\frac{v_{\xi\xi} + v_\xi^2 + 1}{(\cos\xi - \sin\xi\, v_\xi)^2} + (N-1)\left(1 + \frac{\cos\xi}{\sin\xi}\, v_\xi\right)\Big].$$

The corresponding linearized equation is

$$\begin{aligned} z_t = &\frac{e^{2v}}{(\cos\xi - \sin\xi\, v_\xi)^2}\, z_{\xi\xi} \\ &+ e^{2v}\Big[\frac{2v_\xi}{(\cos\xi - \sin\xi\, v_\xi)^2} + \frac{2\sin\xi\,(v_{\xi\xi} + v_\xi^2 + 1)}{(\cos\xi - \sin\xi\, v_\xi)^3} + (N-1)\frac{\cos\xi}{\sin\xi}\Big]\, z_\xi \\ &+ 2e^{2v}\Big[\frac{v_{\xi\xi} + v_\xi^2 + 1}{(\cos\xi - \sin\xi\, v_\xi)^2} + (N-1)\left(1 + \frac{\cos\xi}{\sin\xi}\, v_\xi\right)\Big]\, z \\ &= a(\xi)\, z_{\xi\xi} + \frac{b(\xi)}{\xi}\, z_\xi + c(\xi)\, z, \end{aligned}$$

where the coefficients $a(\xi)$, $b(\xi)$ and $c(\xi)$ belong to the little Hölder spaces, $a(\xi) > 0$ on $[0, \frac{\pi}{2}]$ and $b(0) > 0$. It can be shown that such operators have the maximal regularity property, and therefore (3.13) generates an analytic semiflow [A3]. Thus, the corresponding solution for the combustion problem is analytic in space and time. In particular the interface is analytic.

We summarize this in part (i) of the following theorem.

Theorem 3.1. *(i) Suppose Ω_0 is a ball B, and u_0 is radially symmetric, continuously differentiable, zero on the boundary, with normal derivative equal to one. Then the combustion problem has a unique continuous solution on some interval $(0,T]$, which is real analytic for $t>0$.*

(ii) Let $(0,T_m)$ be the maximal open time interval on which a unique analytic solution exists. Then

$$\lim_{t\to T_m}\zeta(t)=0. \tag{3.14}$$

Proof of part (ii). Suppose that (3.14) is false. We claim that then $\zeta(t)$ converges to a positive finite number $\zeta(T_m)$ as $t\to T_m$. Indeed, in view of the one-sided bound on the Laplacian (see a semiconvexity argument in Section 8),

$$\zeta'(t)=-\frac{u_t}{u_r}=-\Delta u\le K. \tag{3.15}$$

Hence $\zeta(t)$ remains bounded and cannot be oscillatory as $t\to T_m$.

Next we observe that consequently also the limit profile exists, and that it is strictly negative, continuously differentiable up to $\zeta(T_m)$, and that $u_r(\zeta(T_m),T_m)=1$. The latter equality follows from the fact that in view of (3.15) we can put a straight line through the point $(\zeta(T_m),T_m)$ which for $t<T_m$ lies in $\Omega(t)$. Thus we can construct a solution starting at $t=T_m$ using the method above. In order to derive a contradiction with the maximality of T_m, it remains to show that the extended solution is analytic in a neighbourhood of $t=T_m$. It follows from the construction however, that with the diffeomorphism constructed from $v(\cdot,T_m)$, we can also handle $v(\cdot,T_m-\epsilon)$ as initial data, provided $\epsilon>0$ is small. The corresponding analytic solution of the free boundary problem persists as long as it lies in R, which is certainly still true at $t=T_m$, because it coincides with the solution already constructed. Hence it is also true for t in some interval $(T_m-\varepsilon,T_m+\delta)$ with $\delta>0$. Thus the solution is analytic on $(0,T_m+\delta)$, contradiction. $\square$

The same technique applies also to continuously differentiable radial inital data on an annulus.

Theorem 3.2. *(i) Suppose Ω_0 is an annulus, and u_0 is radially symmetric, continuously differentiable, zero on the boundary, with nonzero gradient. Then the combustion problem has a unique annular continuous solution on some interval $(0,T]$, which is real analytic for $t>0$.*

(ii) Let $(0,T_m)$ be the maximal open time interval on which a unique analytic annular solution exists. Then both interfaces converge to nonnegative numbers as $t\to T_m$. If the smaller limit is positive then the two limits coincide (extinction on a sphere). If the smaller limit is zero then the larger limit may be zero (annular extinction in a point), or it may be positive. In the latter case we may continue the solution for $t>T$ as a solution supported on a ball, to which the previous theorem applies.

Proof. (i) This follows along the same lines as for radial solutions on a ball. The only difference is that we have to choose a diffeomorphism which is a rotation over $-\pi/2$ at $\xi=0$. Consequently the ("radial") singularity at $\xi=0$ does not occur. (ii) This is similar to the proof of Theorem 3.1 (ii).

Remark. We note that when only the inner sphere collapses to a point at $t = T_m$, then the outer free boundary is smooth (C^∞) at $t = T = T_m$. We suspect that it is also analytic.

PART II

We proceed next with the detailed description of the phenomena of extinction and focusing.

4. Scaling and first estimates

We begin in this section the study of the asymptotic properties by introducing the natural scaling and change of variables and deriving the first bounds. We assume that the initial function u_0 is bounded, nonnegative and supported in a ball $B_R \in \mathbf{R}^N$, $N > 1$. We consider the free-boundary value problem for a radial solution $u(r,t)$,

$$\text{(P)} \qquad \begin{cases} u_t = \frac{1}{r^{N-1}}(r^{N-1}u_r)_r \quad \text{for } \ 0 < r < \zeta(t), \ t > 0; \\ u_r(0,t) = 0 \quad \text{for } \ t \ge 0 \ \ \text{(symmetry condition)}, \\ u(\zeta(t),t) = 0, \ \ u_r(\zeta(t),t) = -1 \quad \text{for } \ t \ge 0, \\ u(r,0) = u_0(r) > 0 \quad \text{for } \ 0 \le r < R = \zeta(0), \end{cases}$$

where $\{r = \zeta(t)\}$ is the unknown free boundary (interface). By standard regularity we may assume that the data are smooth. As we have proved Problem (P) has a unique analytic solution for $t \in (0,T)$ (Sections 2, 3), where the extinction time $T = T(u_0)$ is finite (see [CV]).

4.1. Rescaled equation, self-similar solution. We recall that in [CV] a self-similar solution was constructed of the form

$$u(r,t) = \sqrt{T-t}\, f_0(\eta), \quad \eta = r/\sqrt{T-t}, \tag{4.1}$$

where the profile $f_0(\eta)$ satisfies the stationary problem

$$\begin{cases} \mathbf{A}(f_0) = 0 \quad \text{for } \ 0 < \eta < \eta_0; \\ f_0'(0) = 0, \ \ f_0(\eta) > 0 \quad \text{for } 0 \le \eta < \eta_0, \\ f_0(\eta_0) = 0 \quad \text{and } \ f_0'(\eta_0) = -1, \end{cases} \tag{4.2}$$

and the operator $\mathbf{A}$ is defined as

$$\mathbf{A}(f) \equiv \frac{1}{\eta^{N-1}}(\eta^{N-1} f_\eta)_\eta - \frac{1}{2}\eta\, f_\eta + \frac{1}{2} f\,. \tag{4.3}$$

We will show that solution (4.1) is the model for the extinction behaviour of the solutions to problem (P) and this will take some work. For the moment it motivates the change of variables

$$w(\eta,\tau) = (T-t)^{-1/2} u(r,t), \quad \eta = r/\sqrt{T-t}, \tag{4.4}$$

where $\tau = -\log(T-t) \to \infty$ as $t \to T^-$ is the new time. Then the interface becomes

$$\psi(\tau) = e^{\tau/2}\zeta(t). \tag{4.5}$$

Denote $\Omega = \{0 \le \eta < \psi(\tau)\} \times (\tau_0, \infty)$, $\tau_0 = -\log T$, and let $\partial\Omega$ be the lateral boundary of Ω. Then we arrive at the problem

(4.6.a) $\quad w_\tau = \mathbf{A}(w) \quad \text{in } \Omega,$

(4.6.b) $\quad w = 0, \quad w_\eta = -1 \quad \text{on } \partial\Omega,$

(4.6.c) $\quad w(\tau_0, \eta) = w_0(\eta) \equiv T^{-1/2}u_0(T^{1/2}\eta) \quad \text{in } B_0 = \{0 \le \eta < \psi(\tau_0)\}.$

Problem (4.5a-b) admits as stationary self-similar solution the profile f_0.

4.2. Interface equation. Let us review here some facts about the interface for further reference. In view of the assumed regularity, by differentiation of the free-boundary condition $u(\zeta(t), t) = 0$ we have on the interface $r = \zeta(t)$ the equation $u_r(\zeta, t)\zeta' + u_t(\zeta, t) = 0$, which gives

$$\zeta'(t) = u_t(\zeta(t), t) = u_{rr}(\zeta(t), t) - \frac{N-1}{\zeta(t)}, \tag{4.7}$$

where we have also used the second free-boundary condition $u_r = -1$. In terms of the rescaled variables it becomes

$$\psi'(\tau) = w_\tau(\psi(\tau), \tau) = \frac{1}{2}\psi(\tau) + w_{\eta\eta} - \frac{N-1}{\psi(\tau)}. \tag{4.8}$$

4.3. First bounds. In view of the Strong Maximum Principle for uniformly parabolic equations, one can see that under the above assumptions the solution $u(r, t)$ is *eventually monotone*

$$u_r(r, t) < 0 \quad \text{for} \quad 0 < r < \zeta(t) \text{ and } t \approx T.$$

This means that we may suppose from the beginning that u_0 (also w_0) is strictly monotone and then

$$w_\eta < 0 \quad \text{for } 0 < \eta < \psi(\tau).$$

This will quite useful in a number of the arguments below.

On the other hand, it follows from a general estimate of Bernstein type [CV] that

$$|w_\eta| \le M \quad \text{in} \quad \Omega \tag{4.9}$$

(here and later on M, $M_1, \ldots$ denote different positive constants). By means of a so-called *Monotonicity Formula*, Caffarelli [C] has recently shown that an interior gradient bound holds for much more general two-phase problems. In our radially symmetric situation a bound is obtained from the Maximum Principle for u_r once the data are smooth.

¿From the above gradient bound we have

Proposition 4.1. *There holds*

$$w \le M_1 \quad in \;\; \Omega. \tag{4.10}$$

Proof. We argue by contradiction. Assume that for a given $K \gg 1$ there exists $\tau_K > \tau_0$ such that $w(0,\tau_K) \ge K$. Then (4.9) implies that

$$w(\eta,\tau_K) \ge (K - M\eta)_+ \quad \text{for } \eta \ge 0. \tag{4.11}$$

Therefore, we conclude that

$$w(\eta,\tau_K) \gg f_0(\eta) \quad \text{in } [0,\eta_0]. \tag{4.12}$$

[In this context we use the sign $\gg$ to mean strict separation in the sense that not only the first function is strictly larger than the second where it is positive but also that the boundaries of the supports do not touch]. (4.12) contradicts the fact that by construction both solutions, $u(r,t)$ and $u_s(r,t) = (T-t)^{1/2} f_0(\eta)$, have been chosen to have the same extinction time T. Indeed, rewriting (4.12) in the original variables, we have that at $t = t_K$

$$u(r,t_K) \gg u_s(r,t_K) \equiv (T-t_K)^{1/2} f_0(|x|(T-t_K)^{-1/2}) \tag{4.13}$$

in $[0, \eta_0(T-t_K)^{1/2}]$. By continuity of the solution u and the interface we deduce that there exists small $\Delta t > 0$ such that $u(r, t_K + \Delta t) \ge u_s(r, t_K)$. Then by comparison with the explicit radial solutions (Theorem 1.8 or [CV]) it follows that $u(r, t + \Delta t) \ge u_s(r,t)$ for all $t \in (t_K, T)$. Setting here $t = T - \Delta t$ we obtain the inequality $u(r,T) \equiv 0 \ge u_s(r, T - \Delta t)$, which yields the contradiction to the fact that by construction both solutions, $u(r,t)$ and $u_s(r,t)$, have the same extinction time T. $\square$

5. Set of similarity profiles. The annular solution

We expand here on the theory of the stationary rescaled equation

$$\mathbf{A}(f) \equiv \frac{1}{\eta^{N-1}}(\eta^{N-1} f')' - \frac{1}{2}\eta f' + \frac{1}{2} f = 0. \tag{5.1}$$

In this study it is convenient to consider solutions with any sign defined for $0 \le \eta < \infty$. The reader will bear in mind that when considered as a solution to our free-boundary problem every such solution must be restricted to its positivity domain. Alternatively, we may consider the solution as continued by zero outside the positivity domain (which corresponds to the physical situation in the flame problem).

The self-similar regular profile f_0 uniquely determined by problem (4.2) can be expressed by means of the Kummer series

$$f_0(\eta) = c_0 \left[1 - \frac{\eta^2}{4N} - \ldots - \frac{(2k-1)!!}{4^{k+1}(k+1)!} \frac{\eta^{2k+2}}{N(2+N)(4+N)\ldots(2k+N)} - \ldots \right], \tag{5.2}$$

which converges everywhere. Here $c_0 = c_0(N) > 0$ is a constant. It follows from (5.2) that $f_0(\eta)$ is decreasing, concave for $\eta > 0$ and has a unique positive root $\eta_0(N) < 2\sqrt{N}$. The profile has an exponential behaviour at infinity

$$f_0(\eta) \approx -a_0 \eta^{-(N+1)} e^{\eta^2/4} \quad \text{for } \eta \gg 1,\ a_0 = a_0(N) > 0. \tag{5.3}$$

Another linear independent solution $f_1(\eta)$ is almost linear at infinity

$$f_1(\eta) \approx a_1 \eta \quad \text{as } \eta \to \infty,\ a_1 = a_1(N) > 0, \tag{5.4}$$

and singular at the origin: as $\eta \to 0$

$$f_1(\eta) \approx \{c_1 \log \eta \text{ if } N = 2, \text{ and } -c_1 \eta^{2-N} \text{ if } N \geq 3\}, \tag{5.5}$$

where $c_1(N) > 0$ is a constant. The function $f_1(\eta)$ is strictly increasing and vanishes at a unique point $\eta_1(N) < \eta_0$ such that $f_1'(\eta_1) = 1$ (one can see that if these properties are not true then the function $Bf_1(\eta)$ with a certain constant $B \in \mathbf{R}$ will touch $f_0(\eta)$ contradicting the uniqueness for the ordinary differential equation (5.1)). Thus, the general solution of equation (5.1) is

$$f(\eta) = Af_0(\eta) + Bf_1(\eta), \quad A,\ B \in \mathbf{R}. \tag{5.6}$$

5.1. Self-similar profiles with distant interfaces. We now consider a class of stationary profiles which will play an important role in the use of the intersection comparison argument. Namely, we fix $l \geq \eta_0$ and consider the solution $f(\eta; l)$ of equation (5.1) for $\eta < l$ with the boundary conditions

$$f(l; l) = 0, \quad f_\eta'(l; l) = -1. \tag{5.7}$$

It follows from (5.6) that this solution has the form

$$f(\eta; l) = A(l) f_0(\eta) + B(l) f_1(\eta), \tag{5.8}$$

with

$$A(l) = \frac{f_1(l)}{d(l)}, \quad B(l) = -\frac{f_0(l)}{d(l)}, \tag{5.9.a}$$

Here $d(l)$ is the Wronskian, hence $d \neq 0$. Moreover, $d(l) > 0$ for all $l > 0$. It is precisely given by the formula

$$d(l) = f_0(l) f_1'(l) - f_0'(l) f_1(l) = \frac{1}{2} a_0 a_1 l^{1-N} e^{l^2/4}. \tag{5.9.b}$$

If $l = \eta_0$ then $B(l) = 0$ and (5.8) gives the self-similar solution f_0. In the case $l > \eta_0$ we have $A(l) > 0$ and $B(l) > 0$, and therefore (5.8) implies that $f(\eta; l)$ vanishes at some other point $\eta = \eta_*(l) > 0$ so that $f(\eta; l) > 0$ on (η_*, l).

Let us consider the limits of this construction. One can see from (5.8) and the asymptotics of f_0 and f_1 given above that

$$\eta_*(l) \to 0, \;\; f'_\eta(\eta_*; l) \to \infty \quad \text{as} \;\; l \to \eta_0^+. \tag{5.10}$$

Consider now the other limit, $l \to +\infty$. By using (5.3) and (5.4) we conclude that uniformly on any compact subset $[\delta, 1/\delta]$ with $0 < \delta \ll 1$ as $l \to \infty$

$$f(\eta; l) = \left[\frac{2}{a_0}\, l^N e^{-l^2/4} f_0(\eta) + \frac{2}{a_1} l^{-2} f_1(\eta) \right] (1 + o(1)) \to 0, \tag{5.11}$$

and also $f'(\eta; l) \to 0$. In particular, we also deduce that

$$\eta_*(l) \to \eta_1^- \quad \text{and} \;\; f'(\eta_*; l) \to 0 \quad \text{as} \; l \to \infty. \tag{5.12}$$

5.3. Self-similar profiles with small interfaces. Consider the case $l < \eta_0$. Then from (5.8) and (5.9) we deduce that as $l \to 0$ there holds $d(l) \approx f_0(0) f'_1(l) \to -\infty$ and hence

$$f(\eta; l) \approx \frac{f_1(l)}{f'_1(l)} - \frac{f_1(\eta)}{f'_1(l)} > 0, \quad f'(\eta; l) \approx -\frac{f'_1(\eta)}{f'_1(l)} < 0 \;\; \text{on} \;\, (0, l) \;\, \text{as} \;\, l \to 0, \tag{5.13}$$

so that $f(\eta; l) \gg 1$ becomes steeper for small $\eta > 0$ provided that $l \to 0$.

5.4. Single point extinction on an annulus, a self-similar solution.

Proposition 5.1. *There exists $l_* > \eta_0$ such that the solution $f_*(\eta) = f(\eta; l_*)$ satisfies the correct free-boundary condition*

$$f'_*(\eta_*(l_*)) = 1. \tag{5.14}$$

Proof. Existence follows from (5.10) and (5.12) and from the continuous dependence of the solution $f(\eta; l)$ to (5.1) upon the parameter l. □

The corresponding self-similar solution

$$u_*(x, t) = (T - t)^{1/2} f_*(\eta), \quad \eta = \frac{|x|}{(T - t)^{1/2}} \tag{5.15}$$

describes a new type of single-point extinction. Both interfaces, the outer one, $|x| = (T - t)^{1/2} l_*$, and the inner one, $|x| = (T - t)^{1/2} \eta_*$ $(\eta_* < l_*)$, tend to the origin as $t \to T$ where the solution vanishes so that the extinction set is $E_* = \{0\}$. However, in rescaled space the support of the solution for every τ and its extinction set are an annulus. Such an unusual extinction behaviour of radial solutions must be unstable since it implies the exact coincidence of the times where both interfaces reach the origin. This case plays the role of a border-line case between the other more structurally stable asymptotics described in Sections 6 and 8 respectively. We will return to this question in Section 8.

6. Asymptotic behaviour of single point extinction

6.1. BOUNDS ON THE INTERFACE. Consider Problem (P) of Section 4. In order to attack the asymptotic behaviour we need bounds on the interface. We begin with the upper bound.

Proposition 6.1. *There holds*

$$\psi(\tau) \le M_2 \quad \textit{for all } \tau > \tau_0. \tag{6.1}$$

Proof. The argument is based on nonstandard comparison with the self-similar solutions $f(\eta; l)$ constructed in Section 5. Without loss of generality we assume that $w_0(\eta)$ is regular enough. It follows from property (5.11) that there exists $l_2 \gg 1$ such that the function $f(\eta; l_2)$ intersects $w_0(\eta)$ exactly once, so that the number of 'intersections' $I(\tau)$ of the solutions $w(\eta, \tau)$ and $f(\eta; l_2)$ in the interval $(\eta_*(l_2), \psi(\tau))$ (more precisely, we mean the number of the sign changes of the difference $z(\eta, \tau) = w(\eta, \tau) - f(\eta; l_2)$) satisfies

$$I(\tau_0) = 1. \tag{6.2}$$

We now adapt the well-known Intersection Comparison argument originated with Sturm [St] (for the heat equation) and extended by several authors, cf. [N], [S],.... We apply it to the linear parabolic equation satisfied by the difference $z = w - f$

$$z_\tau = \frac{1}{\eta^{N-1}}(\eta^{N-1} z_\eta)_\eta - \frac{1}{2} z_\eta \eta + \frac{1}{2} z$$

in $Q_s = (\eta_*, h(\tau)) \times (\tau_0, s)$, $s > \tau_0$, with $h(\tau) = \min\{\psi(\tau), l_2\}$. By the classical results $I(s)$ is not larger than the number of sign changes of $z(\eta, \tau)$ on the parabolic boundary ∂Q_s. Therefore, we have to check the behaviour of sign changes of the difference on both lateral boundaries.

Consider first the left boundary and let us prove that new intersections cannot appear at the left lateral boundary $\eta = \eta_*$, $\tau > \tau_0$. For that we look at the situation when the interface of w lies just on the left boundary, i.e., $\psi(\tau_1) = \eta_*$ for some $\tau_1 \in (\tau_0, s)$, so that $Q_s \cap \{\tau = \tau_1\} = \emptyset$. Assume also that $\psi(\tau) > \eta_*$ for $\tau - \tau_1 > 0$ small. It is clear then by the regularity and transversality of the profiles at the interfaces, $w_\eta(\eta_*, \tau) = -1$, $f'(\eta_*) > 0$, that for any $\tau \approx \tau_1$, such that $\psi(\tau) > \eta_*$, there holds $I(\tau) = 1$. In the other possible case when ψ does not cross the boundary $\eta = \eta_*$ the sign of z is locally constant in time. Hence, the result is true.

Consider now the difference on the right lateral boundary:

$$g(\tau) = z(h(\tau), \tau) = w(h(\tau), \tau) - f(h(\tau); l_2).$$

By construction $g(\tau) < 0$ for $\tau - \tau_0 > 0$ small. Assume g changes sign and that $\tau = \tau_1$ is the first zero, i.e., $g(\tau) < 0$ for $\tau < \tau_1$ and $g(\tau_1) = 0$. Then necessarily $\psi(\tau_1) = l_2$. Let us prove that the intersection (there is only one) disappears at that moment at $\eta = l_2$. Indeed, if the intersection line stays away from $\eta = l_2$ as $\tau \to \tau_1$ from below, then by Hopf's boundary lemma we have different spatial slopes of w and f at $\eta = l_2$ when $\tau = \tau_1$, contradicting the free boundary condition. On the other hand, if along a sequence $\{\tau_n\} \to \tau_1$ the intersection line converges to l_2 then in the limit we will have

$$w(\eta, \tau_1) \geq f(\eta; l_2), \tag{6.3}$$

so that $I(\tau_1) = 0$. This means that for $\tau > \tau_1$ we have $\psi(\tau) \geq l_2$. Now, the interfaces are analytic functions so that the equality $\psi = l_2$ is excluded even locally, hence $\psi(\tau) > l_2$ for $\tau > \tau_1$. But the situation

$$\psi(\tau) > l_2, \quad z(\eta, \tau) \geq 0 \quad \text{for } \eta > 0, \tag{6.4}$$

contradicts the assumption that the solutions of the heat equation corresponding to the rescaled functions $w(\eta, \tau)$ and $f(\eta; l_2)$ have the same extinction time T (see the proof of Proposition 4.1). We conclude that necessarily $\psi(\tau) < l_2$, whence (6.1). □

Remark. The reader will have noticed that Propositions 4.1 and 6.1 are based in the end on a simple common idea: it is impossible for w to be above a certain self-similar profile with the same extinction time.

Proposition 6.2. *There exists a constant $m_2 > 0$ such that*

$$\psi(\tau) \geq m_2 \quad \textit{for all} \ \ \tau > \tau_0. \tag{6.5}$$

Proof. Since we have a gradient bound, (4.9), we do not need to use the Intersection Comparison argument. Indeed, if $\psi(\tau)$ is very small at a certain τ then the profile $w(\cdot, \tau)$ is uniformly small, hence by the usual Maximum Principle (as in Proposition 4.1) the extinction time is less that T. □

Corollary 6.3. *There exist constants m_1 and $m_3 > 0$ such that*

$$\|w(\cdot, \tau)\|_\infty \equiv w(0, \tau) \geq m_1 \quad \textit{for} \ \ \tau > \tau_0, \tag{6.6}$$

$$\int_0^{\psi(\tau)} \eta^{N-1} w(\eta, \tau) d\eta \geq m_3. \tag{6.7}$$

Proof. Multiplying equation (4.6.a) by η^{N-1} and integrating over $(0, \psi(\tau))$ we obtain the following *mass equation*

$$\frac{d}{d\tau} E(\tau) = \frac{N+1}{2} E(\tau) - \psi^{N-1}(\tau), \qquad E(\tau) = \int_0^{\psi(\tau)} \eta^{N-1} w(\eta, \tau) d\eta. \tag{6.8}$$

This holds for $\tau > \tau_0$. From this using (6.5) we have

$$E(\tau) \geq c m_2^{N-1} \quad \text{for all } \tau > \tau_0,$$

otherwise by (6.8) w would vanish in a finite τ. Now, by the interface estimate (6.1)

$$E(\tau) \leq w(0, \tau) \psi^N(\tau)/N \leq w(0, \tau) M_2^N/N,$$

hence (6.8) implies (6.6) with $m_1 = N m_2^{N-1}/M_2^N$. □

Remark. By estimates (4.10) and (6.1) the mass is also bounded from above. On the other hand, we will see later that equation (6.8) stabilizes as $\tau \to \infty$ in the sense that

$$\frac{N+1}{2}E(\tau) - \psi^{N-1}(\tau) \to 0.$$

6.2. ASYMPTOTIC ANALYSIS AND MAIN RESULT. In view of bounds (4.9), (4.10) and (6.1) we introduce the ω-limit set

$$\begin{aligned} \omega(w_0) = \{g \in C : \ \exists \text{ a sequence } \{\tau_k\} \to \infty \text{ such that} \\ w(\cdot, \tau_k) \to g(\cdot) \text{ uniformly}\}. \end{aligned} \tag{6.9}$$

First, we prove that $\omega(w_0)$ consists of stationary profiles. We begin with the following preliminary result.

Lemma 6.4. *There holds*

$$\omega(w_0) \subseteq \{\alpha f_0(\eta), \ 0 < m_3 \le \alpha \le M_3 < \infty\}. \tag{6.10}$$

Proof. Rewriting equation (4.6.a) in divergence form

$$\rho w_\tau = (\rho w_\eta)_\eta + \frac{1}{2}\rho w, \quad \text{with} \ \ \rho(\eta) = \eta^{N-1}e^{-\eta^2/4}, \tag{6.11}$$

and using the regularity results of Section 3 and the interface equation

$$\psi'_\tau(\tau) = w_\tau(\psi(\tau), \tau), \quad \tau > \tau_0, \tag{6.12}$$

see (4.8), one can check that the problem admits a standard **Lyapunov function** of the form

$$L[w](\tau) = \frac{1}{2}\int_0^{\psi(\tau)} \rho[(w_\eta)^2 - \frac{1}{2}w^2]d\eta + \frac{1}{2}\int_{\psi(\tau_0)}^{\psi(\tau)} \rho(s)ds. \tag{6.13}$$

Namely, it is decreasing in time:

$$\frac{d}{d\tau}\, L[w](\tau) = -\int_0^{\psi(\tau)} \rho\,(w_\tau)^2 d\eta \le 0. \tag{6.14}$$

In particular, using the above bounds we have that

$$\int_{\tau_0}^{\infty} \left(\int_0^{\psi(\tau)} \rho(w_\tau)^2(\tau)d\tau \right) d\eta < \infty. \tag{6.15}$$

Therefore, taking a monotone sequence $\{\tau_k\} \to \infty$ we have that uniformly in $s \in [0, C]$

$$\begin{aligned} &\int \rho|w(\cdot, \tau_k + s) - w(\cdot, \tau_k)|^2 \le C\int \rho| \int_{\tau_k}^{\tau_k+s} \frac{\partial}{\partial\tau}\, w(\cdot, \tau)|^2 \\ &\le C\int_{\tau_k}^{\tau_k+s} \int \rho(w_\tau)^2 \le C\int_{\tau_k}^{\infty} d\tau \int \rho(w_\tau)^2 d\eta \to 0 \end{aligned} \tag{6.16}$$

as $k \to \infty$. Thus, passing to the limit in equation (6.11) with $\tau = \tau_k + s \to \infty$ and using the bounds (4.9), (4.10) and (6.15) we conclude that the limit function does not depend on the new time s. Since by the regularity in the positivity domain, $g'(0) = 0$ for any $g \in \omega(w_0)$, we obtain (6.10), where the estimates for the parameter α follow from (4.10) and (6.5). $\square$

Lemma 6.5. *There exists a finite limit*

$$\psi_\infty = \lim_{\tau\to\infty} \psi(\tau). \tag{6.17}$$

Proof. We argue by contradiction. Assume that the limit in (6.17) does not exist. Since the interface is uniformly bounded there exists then a finite $l > 0$ such that the function $\psi(\tau)$ has an infinite number of oscillations around the point $\eta = l$ as $\tau \to \infty$. This means that there exists a monotone sequence $\{\tau_k\} \to \infty$ such that, say,

$$\psi(\tau_{2m}) < l \le \psi(\tau_{2m+1}), \quad m = 1, 2, \dots . \tag{6.18}$$

Let $I(\tau; l)$ be the number of intersections of the solution $w(\eta, \tau)$ and the profile $f(\eta; l)$ in $(\eta_*(l), l)$. Since as in the proof of Proposition 6.1, new intersections cannot appear on the lateral boundary, it follows from (6.18) that such oscillations are possible if and only if

$$I(\tau; l) = \infty \quad \text{for all } \tau > \tau_0. \tag{6.19}$$

Indeed, it is shown in the proof of Proposition 6.1 that each oscillation around the point $\eta = l$ means that at least one intersection disappears on the interface, whence (6.19). See a similar analysis for a quasilinear degenerate parabolic equation in [GV2, Section 8]. Now, since both profiles, $w(\eta, \tau)$ and $f(\eta; l)$, are analytic functions in η in the positivity domains, see Section 3, if $\psi(\tau) \not\equiv l$ the intersection number must be finite, thus contradicting (6.9).

Actually, in the above argument we have assumed that $l > \eta_0$. However, the case $l \le \eta_0$ is similar. $\square$

Theorem 6.6. *Under the above assumptions we have as* $\tau \to \infty$

$$w(\cdot, \tau) \to f_0(\cdot) \quad \textit{uniformly}, \tag{6.20}$$

with f_0 *given by problem (4.2). Also* $\psi(\tau) \to \eta_0$.

Proof. Let $g \in \omega(w_0)$. We have proved that $g = \alpha f_0$ for some positive constant α. Therefore the support of g is the interval $[-\eta_0, \eta_0]$. It follows from an elementary topological argument that $\psi_\infty \ge \eta_0$. By standard parabolic regularity we have $g \in C^\infty([0, \psi_\infty - \varepsilon])$ for any $\varepsilon > 0$ and it is positive there (since it cannot vanish identically). Therefore, Therefore, $\psi_\infty = \eta_0$.

On the other hand, passing to the limit $\tau \to \infty$ in the mass equation (6.7) we obtain that the limit equality $(N+1)E(\infty)/2 = \psi_\infty^{N-1}$. This means that $\alpha = 1$ in (6.10), which implies (6.20).

Convergence of the interfaces is proved via a semiconvexity argument in Section 7. $\square$

7. The one-dimensional problem

In this section we consider the one-dimensional problem. Here we can prove convergence to a symmetric profile at extinction with the same rates as before even for non-radially symmetric data. Assuming up to space translations that $0 \in E$ (E is the extinction set, defined in (0.6)) and making the same scaling transformation we arrive at the problem

$$w_\tau = \mathbf{A}(w) \equiv w_{\eta\eta} - \frac{1}{2}\eta\, w_\eta + \frac{1}{2}w \quad \text{in } \Omega = S(\tau) \times (\tau_0, \infty), \tag{7.1}$$

where $S(\tau) = \{w(\eta,\tau) > 0\} = (\phi(\tau), \psi(\tau))$. We assume that u_0 has a connected support, a quite general assumption for asymptotic purposes), with boundary conditions on the interfaces

$$w = 0,\ w_\eta = -1 \ \text{ for } \eta = \psi(\tau); \quad w = 0,\ w_\eta = 1 \ \text{ for } \eta = \phi(\tau),\ \ \tau > \tau_0, \tag{7.2}$$

and smooth initial data

$$w(\eta, \tau_0) = w_0(\eta) \quad \text{in} \ \ S_0 = (\phi(0), \psi(0)). \tag{7.3}$$

7.1. Self-similar solution is asymptotic profile. Equation (7.1) admits the symmetric stationary solution

$$f_0(\eta) = c_0 \left(e^{\eta^2/4} - \frac{\eta}{2} \int_0^\eta e^{\zeta^2/4} d\zeta \right) \quad \text{on } (-\eta_0, \eta_0), \tag{7.4}$$

where $\eta_0 > 0$ is the unique positive root of the equation

$$\eta \int_0^\eta e^{\zeta^2/4} d\zeta = 2e^{\eta^2/4}, \tag{7.5}$$

and $c_0 > 0$ is chosen so that $f_0'(\eta_0) = -1$. This yields

$$c_0 = 2 \left(\int_0^{\eta_0} e^{\zeta^2/4} d\zeta \right)^{-1}.$$

One can see from (7.5) that $\eta_0 > \sqrt{2}$. Our main result about the one-dimensional mode of extinction is expressed in rescaled variables as follows:

Theorem 7.1. *Let $0 \in E(u_0)$. Under the given assumptions, the asymptotic convergence (6.20) still holds with f_0 given by (7.4). We have also convergence of the interfaces, $\phi(\tau) \to -\eta_0$ and $\psi(\tau) \to \eta_0$ as $\tau \to \infty$.*

The proof of this result follows the general outline of Section 6 with some difficulties due to the lack of symmetry.

7.2. First bounds. Integrating the heat equation, $u_t = u_{xx}$, we have that $(d/dt)\int u = -2$, i.e., $\int u(t)dx = 2(T-t)$. In terms of the rescaled function this means a mass estimate

$$\int w(\tau)\, d\eta = 2 \quad \text{for all } \tau \geq \tau_0, \tag{7.6}$$

which repeats and improves (6.8). A gradient bound follows from [CV] (or simply use the Maximum Principle for w_η)

$$|w_\eta| \leq M \quad \text{in } \Omega. \tag{7.7}$$

¿From both estimates it immediately follows that

$$w \leq M_1 \quad \text{in } \Omega. \tag{7.8}$$

We now prove a semiconvexity estimate.

Lemma 7.2. *There holds*

$$w_{\eta\eta} \leq M_2 e^{-\tau/2} \quad \text{in } \Omega. \tag{7.9}$$

Proof. Let us revert to standard variables. The second derivative $z = u_{xx}$ satisfies in Ω the heat equation $z_t = z_{xx}$. Arguing as in (4.7) we obtain on the right interface $x = \zeta(t)$ the relation

$$u_x(\zeta, t)\zeta' + u_t(\zeta, t) = 0. \tag{7.10}$$

Since $u_t = u_{xx}$ and $u_x = -1$ on the interface $x = \zeta(t)$, there holds $\zeta' = u_{xx}$. Moreover, from $u_x(\zeta(t), t) = -1$ we get

$$u_{xx}(\zeta, t)\zeta' + u_{xt}(\zeta, t) = 0,$$

so that

$$z_x = u_{xt} = -u_{xx}\zeta' = -(u_{xx})^2 \leq 0. \tag{7.11}$$

Similarly, $z_x = (u_{xx})^2 \geq 0$ on the left interface. Therefore, by the Maximum Principle $z = u_{xx} \leq M_2 = \sup u_0''$ in Ω which implies (7.9) for the rescaled function. □

¿From (7.9) and (7.6) we have the following property by simple calculus.

Lemma 7.3. *If for large τ the solution support, $|S(\tau)| \equiv \psi(\tau) - \phi(\tau)$, is large then $w(\eta, \tau)$ is small. More precisely, we have an estimate of the form*

$$w(x, \tau) \leq \frac{C}{|S(\tau)|}. \tag{7.12}$$

7.3. Set of stationary profiles. The general solution of the stationary equation

$$\mathbf{A}(f) \equiv f'' - \frac{1}{2}\eta f' + \frac{1}{2} f = 0, \quad \text{in } \mathbf{R} \tag{7.13}$$

has the form

$$f(\eta) = A f_0(\eta) + B\eta. \tag{7.14}$$

Therefore, the solution $f(\eta; l)$ of problem (7.12) with data $f = 0$, $f' = -1$ at $\eta = l$ (cf. (5.7)) is given by

$$f(\eta; l) = \frac{l}{d(l)} f_0(\eta) - \frac{f_0(l)}{d(l)} \eta, \quad d(l) = c_0 e^{l^2/4} > 0. \tag{7.15}$$

Hence, for any $l > \eta_0$ the function $f(\eta; l)$ vanishes at a point $\eta_*(l) \in (-\eta_0, 0)$ (notice that for $N = 1$ $\eta^*(l)$ is negative). One can see from (7.15) that as $l \to \infty$ uniformly on any compact subset (cf. (5.11))

$$f(\eta; l) \approx \frac{l}{c_0}\, e^{-l^2/4} f_0(\eta) + \frac{a_0}{c_0} l^{-2}\, \eta \to 0. \tag{7.16}$$

Since $f_0(\eta) \approx c_0(1 - \eta^2/4)$ for $|\eta| \ll 1$, we conclude that

$$\eta_*(l) \approx -\frac{c_0}{a_0}\, l^3\, e^{-l^2/4} \to 0^- \quad \text{as } l \to \infty, \tag{7.17}$$

and also

$$f'(\eta_*; l) \to 0 \quad \text{as } l \to \infty. \tag{7.18}$$

7.4. Bounds on the interfaces. We begin with the proof of the following

Lemma 7.4. *$\psi(\tau)$ is uniformly bounded from above.*

Proof. (i) We argue by contradiction. Assume that

$$\lim\ \sup_{\tau \to \infty} \psi(\tau) = \infty. \tag{7.19}$$

As in (4.8) we have the following interface equation in rescaled variables

$$\psi'(\tau) = \frac{1}{2}\psi(\tau) + w_{\eta\eta}(\psi, \tau), \tag{7.20}$$

that follows immediately from (7.10). In the same way we have for the left interface

$$\phi'(\tau) = \frac{1}{2}\phi(\tau) - w_{\eta\eta}(\phi, \tau), \tag{7.21}$$

In view of (7.9) these estimates show that there is a certain uniformity in the growth of the interfaces once τ is big enough. This is reflected in the inequality

$$\psi'(\tau) \le \psi(\tau) \quad \text{for } \tau \text{ and } \psi \gg 1 \tag{7.22}$$

that uses (7.9) [here $\gg$ means as usual much larger than]. Under our assumption (7.19) there exists $\tau_2 \gg 1$ such that $\psi(\tau_2) = 2L$, where L is a fixed but very large number. It follows from (7.22) and the continuity of the interface that if we choose the moment $\tau_1 < \tau_2$ such that

$$\psi(\tau_1) = L \quad \text{and} \quad \psi(\tau) \ge L \ \text{ for } \ \tau \in [\tau_1, \tau_2], \tag{7.23}$$

then

$$\tau_2 - \tau_1 \ge \frac{1}{3}. \tag{7.24}$$

Now we have to consider separately the cases when ϕ is also very large and when ϕ is not.

(ii) Let us check first the case when

$$\phi(\tau_2) \gg 1, \tag{7.25}$$

The support of the solutions is then moved a long distance to the right away from 0 at this time and then a contradiction immediately follows with the assumption that $0 \in E$ by a comparison argument with the standard self-similar solutions shifted a bit to the right. Let us see the details. At $t = t_2$ the profile $u(x, t_2)$ has support

$$\text{supp } u(x, t_2) = (\xi(t_2), \zeta(t_2)), \quad \text{and } \ \xi(t_2) \gg 2\eta_0 (T - t_2)^{1/2}.$$

We now consider the self-similar solution $u_s(x - x_0, t_2)$ centered at $x_0 = \xi(t_2)/2$ and having the same extinction time $T - t_2$. Since $\text{supp}\, u_s(x - x_0, t_2) \cap \text{supp}\, u(x, t_2) = \emptyset$ the number of intersections of the solutions in $\{|x - x_0| < \eta_0 (T - t_2)^{1/2}\}$ is

$$I(t_2) = 0. \tag{7.26}$$

A main fact of the Intersection Comparison theory already explained in the proof of Proposition 6.1 says that new intersections can appear only on the parabolic boundary. In this construction a new intersection can appear only on the right interface $x = x_0 + \eta_0 (T - t)^{1/2}$ and

$$I(t) \le 1 \quad \text{for all } \ t \in (t_2, T). \tag{7.27}$$

Then one can see that the left interfaces of the solutions, $x = \xi(t)$ and $x = x_0 - \eta_0 (T - t)^{1/2}$, must be ordered

$$\xi(t) \ge x_0 - \eta_0 (T - t)^{1/2} \quad \text{in } (t_2, T). \tag{7.28}$$

Otherwise, if (7.28) does not hold for some $t_3 \in (t_2, T)$, then we arrive at the case of zero intersections which has been studied in the proof of Proposition 4.1 (i.e., the extinction time of u_s is less than that of u). Now, inequality (7.28) implies that $0 \notin E$, against the first assumption of this section. The proof shows in fact that condition (7.25) can be precised in the form

$$\phi(\tau_2) \geq 2\eta_0. \tag{7.29}$$

(iii) We now consider the second case where

$$\phi(\tau) = O(1) \quad \text{for all } \tau \in [\tau_1, \tau_2]. \tag{7.30}$$

In fact, we may assume that $\phi(\tau) \ll l \ll L$ in $[\tau_1, \tau_2]$. Since by assumption both interfaces ϕ and ψ stay uniformly bounded away from $\eta = l$ for all $\tau \in (\tau_1 + \varepsilon, \tau_2 - \varepsilon)$, $\varepsilon > 0$ small, by the standard interior regularity for uniformly parabolic equations the derivatives w_η, $w_{\eta\eta}$ are uniformly bounded in $Q_\varepsilon = [l - \varepsilon, l + \varepsilon] \times (\tau_1 + \varepsilon, \tau_2 - \varepsilon)$. Since by Lemma 7.3 the solution is uniformly small in $S(\tau)$ for $\tau \in [\tau_1, \tau_2]$ we may assume that $w_\eta \approx 0$ in Q_ε. Therefore, integrating equation (7.1) over $(\psi(\tau), l)$ yields

$$\frac{\partial}{\partial \tau} \int_{\phi(\tau)}^{l} w(\tau) d\eta = w_\eta(l, \tau) - 1 - \frac{1}{2} w(l, \tau) l + \int_{\phi(\tau)}^{l} w(\tau) d\eta \approx -1 \tag{7.31}$$

for all $\tau \in (\tau_1 + \varepsilon, \tau_2 - \varepsilon)$. Observe that the last term is of order $(l - \phi) \max w$ and by Lemma 7.3 this can be made very small when $l \ll L$. This means that $w(\eta, \tau)$ must vanish on the set $\eta \leq l/2$ in a time which is less than

$$2 \int_{\phi(\tau_1 + \varepsilon)}^{l} w(\tau_1 + \varepsilon) d\eta \ll 1. \tag{7.32}$$

This contradicts assumption (7.30).

(iv) It remains to consider the case where ϕ is large and negative, but in that case the proof of Proposition 6.1 applies. This completes the proof of the boundedness of ψ. $\square$

Similarly, $\phi(\tau)$ is uniformly bounded from below. From these bounds

$$\psi(\tau) \leq M_2 \quad \text{and} \quad \phi(\tau) \geq -m_2 \quad \text{for all } \tau > \tau_0 \tag{7.33}$$

and the mass estimate (7.6) it follows that

$$\|w(\cdot, \tau)\|_\infty \geq M_3 > 0 \quad \text{for } \tau > \tau_0. \tag{7.34}$$

7.5. Asymptotic behaviour. We introduce the ω-limit set (6.9). Since equation (7.1) with the boundary conditions (7.2) admits a Lyapunov function of the form (cf. (6.13))

$$\begin{aligned} L[w](\tau) &= \frac{1}{2}\int_{S(\tau)} \rho[(w_\eta)^2 - \frac{1}{2}w^2]\,ds \\ &+ \frac{1}{2}\left(\int_{\psi(\tau_0)}^{\psi(\tau)} \rho(s)ds + \int_{\phi(\tau_0)}^{\phi(\tau)} \rho(s)ds\right), \quad \rho(s) = e^{-s^2/4}, \end{aligned} \tag{7.35}$$

we conclude that $\omega(w_0)$ consists of bounded stationary profiles of the form $\alpha f(\eta; l)$ with $\alpha > 0$ and $l \in \mathbf{R}$. Observe that by (7.34)

$$\{0\} \notin \omega(w_0). \tag{7.36}$$

Exactly as in the proof of Lemma 6.5 we establish that there exist finite limits

$$\psi(\tau) \to \psi_\infty \quad \text{and} \quad \phi(\tau) \to \phi_\infty \quad \text{as } \tau \to \infty\ . \tag{7.37}$$

Passing to the limit $\tau \to \infty$ in (7.9) we conclude that any $g \in \omega(w_0)$ satisfies

$$g_{\eta\eta} \le 0, \tag{7.38}$$

and hence by (7.37)

$$g(\eta) > 0 \quad \text{in} \quad (\phi_\infty, \psi_\infty), \;\; g(\phi_\infty) = g(\psi_\infty) = 0. \tag{7.39}$$

In the limit $\tau \to \infty$ of (7.6) we get

$$\int_{\phi_\infty}^{\psi_\infty} g d\eta = 2. \tag{7.40}$$

Integrating now equation (7.13) over $(\phi_\infty, \psi_\infty)$ yields

$$g'(\psi_\infty) - g'(\phi_\infty) = -2. \tag{7.41}$$

We now show that

$$g'(\psi_\infty) = -1 \quad (\text{then by (7.41) } g'(\phi_\infty) = 1). \tag{7.42}$$

Indeed, $g'(\psi_\infty)$ cannot be less than -1 because this contradicts the fact that g is a limit of profiles which are almost concave in the sense of (7.9) and satisfy the boundary condition -1 on the slope. If $g'(\psi_\infty) > -1$, then by (7.41) we arrive at the same contradiction at the left-end point $\eta = \phi_\infty$.

In order to complete the proof of Theorem 7.1 we need the following *uniqueness* result for the stationary problem.

Lemma 7.5. *Let g satisfying (7.39) be a solution of $\mathbf{A}(g) = 0$ in $(\phi_\infty, \psi_\infty)$ such that*

$$g'(\phi_\infty) = 1, \quad g'(\psi_\infty) = -1.$$

Then $g(\eta)$ is the symmetric function (7.4) and in particular $\phi_\infty = -\eta_0$ and $\psi_\infty = \eta_0$.

One can see that in view of (7.42) Lemma 7.5 implies that $\omega(w_0) = \{f_0\}$ and Theorem 7.1 follows.

Proof of Lemma 7.5. It is based on an evolution property of problem (7.1)-(7.2). Another possible proof based on an ODE analysis seems to be more complicated.

Assume that there exists $l > \eta_0$ such that the function $f(\eta; l)$ given by (7.15) is also a solution, i.e., $f'_\eta(\eta_*(l); l) = 1$. Let us check the number of intersections of the profiles $f_0(\eta)$ and $f(\eta; l)$. See see from (7.15) that all the intersections are determined from the equation

$$f_0(\eta) = -\frac{f_0(l)}{d(l) - l}\,\eta. \tag{7.43}$$

Since $l > \eta_0$, we have $f_0(l) < 0$. Set $G(l) = d(l) - l$. Then $G(\eta_0) = 0$, and

$$\begin{aligned} G'(l) &= \frac{c_0}{2} l e^{l^2/4} - 1 \equiv \frac{c_0}{2}\left(l e^{l^2/4} - \int_0^{\eta_0} e^{\zeta^2/4} d\zeta\right) \\ &> \frac{c_0}{2}\left(l e^{l^2/4} - \int_0^{l} e^{\zeta^2/4} d\zeta\right) > 0. \end{aligned}$$

Therefore, the coefficient of η in the right-hand side of (7.43) is strictly positive for $l > \eta_0$. Hence, there exists a unique intersection point of $f_0(\eta)$ and $f(\eta; l)$ in $(-\eta_0, \eta_0)$. By uniqueness for the ODE (7.13) this intersection is transversal, i.e., $f_0' \neq f'$ at the point where $f_0 = f$.

Consider now two solutions of the free boundary problem,

$$u(x,t) = (T-t)^{1/2} f_0\left(\frac{x-\delta}{(T-t)^{-1/2}}\right), \quad \tilde{u}(x,t) = (T-t)^{1/2} f\left(\frac{x}{(T-t)^{-1/2}}; l\right), \tag{7.44}$$

having the same extinction time. Here $\delta > 0$ is a small constant such that by continuity the number of intersections between solutions (7.44) satisfies

$$I(0; \delta) = 1. \tag{7.45}$$

Then (7.27) holds, and as in the proof of Proposition 6.1 we conclude that for all $t \in (0, T)$ the left interface $x = \tilde{\xi}(t)$ of $\tilde{u}$ is not less than the left interface $x = \xi(t)$ of u. This leads to the contradiction since after δ-shifting in x of u the solutions (7.44) have different extinction points, $x = \delta > 0$ and $x = 0$. Indeed, we have for $t \approx T$

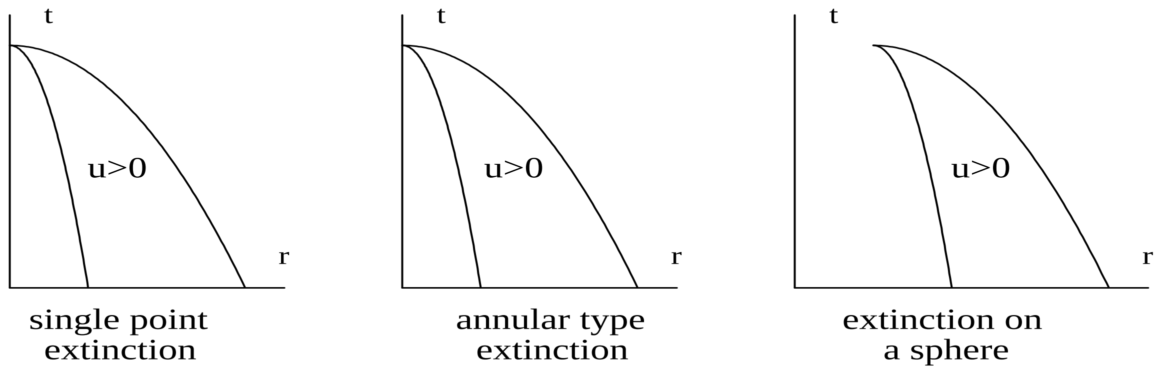
$$\tilde{\xi}(t) \geq \xi(t) \equiv \delta - \eta_0 (T-t)^{1/2} > \delta/2,$$

i.e., $\tilde{\xi}(t) \not\to 0$ as $t \to T$.

The case $l \in (0, \eta_0)$ is studied similarly. $\square$

8. Extinction on a sphere

This section is devoted to study the phenomenon of extinction concentrated in a sphere $S = \{|x| = r_* > 0\}$. Such a possiblity may arise for solutions with initial data concentrated in an annulus, when the outer interface moves inwards to meet the inner one at a finite radius $r_* > 0$. This will happen for instance in the case of a thin annulus with a big hole inside. On the contrary, when the anunulus is thick and the hole is small the inner interface will reach $x = 0$ very soon while the outer interface is still far away, so that the hole disappears and we get the situation of Sections 4-6. Both situations are robust. As we have said, the annular solutions of Section 5 represent a borderline case, where the two interfaces meet at $x = 0$, precisely at the extinction time.



Scheme of the three extinction types for data with annular support

We state conditions under which the extinction set contains a sphere S and not the origin (Theorem 8.1), more in particular, we assume that the support of $u_0(r)$ is an annulus, e.g.,

$$\operatorname{supp} u_0 = \{1 < r < 2\}, \tag{8.1}$$

and then prove that the extinction profile and extinction rates are the same as in the one-dimensional problem, as stated in Theorem 8.2 below. The following result which is proved by comparison with stationary solutions of problem (0.1)-(0.2).

Lemma 8.1. *Assume that for all $r \in (1,2)$*

$$\begin{aligned} &u_0(r) \le \min\left\{\log r, \quad 2\log\frac{2}{r}\right\} \quad \text{for } N = 2,\\ &u_0(r) \le \frac{1}{N-2}\min\left\{1 - r^{-(N-2)}, \quad 2^{N-1}r^{-(N-2)} - 2\right\} \quad \text{for } N \ge 3. \end{aligned} \tag{8.2}$$

Then for all $t \in (0,T)$

$$\operatorname{supp} u(r,t) \equiv (\xi(t), \zeta(t)) \subset [1,2], \qquad \text{and hence } 0 \notin E. \tag{8.3}$$

8.1. Rescaled equation. Main result. Thus, assuming that there exists a sphere $\{r = r_* \in [1,2]\}$ in the extinction set E, we introduce the rescaled variables (cf. (4.4))

$$w(\eta,\tau) = (T-t)^{-1/2} u(r,t), \quad \eta = \frac{r-r_*}{(T-t)^{1/2}}, \quad \tau = -\log(T-t). \tag{8.4}$$

The rescaled function w solves the following equation

$$w_\tau = \mathbf{A}(w) + e^{-\tau/2}\mathbf{B}(w), \tag{8.5}$$

where $\mathbf{A}$ is the one-dimensional operator (as in (8.1))

$$\mathbf{A}(w) = w_{\eta\eta} - \frac{1}{2} w_\eta \eta + \frac{1}{2} w \tag{8.6}$$

and

$$\mathbf{B}(w) = \frac{N-1}{r_* + e^{-\tau/2}\eta}\, w_\eta. \tag{8.7}$$

The initial function for $\tau = \tau_0$ we denote by $w_0(\eta)$. Equation (8.5) looks like an exponentially small perturbation of the one-dimensional equation (7.1). We have

Theorem 8.2. *Assume that (8.1) and (8.2) hold. Then there exists a unique $r_* > 0$ such that (6.20) is valid with f_0 given by (7.4). The interfaces also converge.*

8.1. First bounds. A gradient bound has been proved in [CV],

$$|w_\eta| \le M \quad \text{in} \quad \Omega = S(\tau) \times (\tau_0, \infty), \tag{8.8}$$

where $S(\tau) \equiv \operatorname{supp} w(\eta,\tau) = (\phi(\tau), \psi(\tau))$. In this radial case it also follows from the Maximum Principle for $r^{N-1}u_r$. As in the previous problems, (8.8) implies uniform boundedness of the solution:

Lemma 8.3. *$w \le M_1$ in Ω.*

Proof. Assuming that for some $\tau_K > \tau_0$ $K = \|w(\cdot,\tau_K)\|_\infty \equiv w(\eta_K,\tau_K) \gg 1$, we derive a contradiction for the extinction time of the corresponding solution of (0.1). Indeed, from (8.8) we conclude that

$$w(\eta,\tau_K) \ge [K - M|\eta - \eta_K|]_+ \,. \tag{8.9}$$

Consider the solution $u(r,t)$ for $t = t_K \equiv T - e^{-\tau_K}$. It follows from (8.9) and (8.4) that

$$u(r,t_K) \ge (T-t_K)^{1/2}\Big[K - M|(r-r_*)(T-t_K)^{-1/2} - \eta_K|\Big]_+ \quad \text{for } r \ge 0. \tag{8.10}$$

Therefore, we can fix the explicit radial symmetric solution of the form

$$u_s(x,t_K) \equiv (T-t_K)^{1/2} f_0(|x-x_K|(T-t_K)^{-1/2}),$$

with an arbitrary $x_K \in \{|x| = r_K\}$, $r_K = r_* + \eta_K(T-t_K)^{-1/2}$, and with f_0 given by (5.2), such that by (8.10)

$$u(r,t_K) \gg u_s(x,t_K) \quad \text{in } \operatorname{supp} u_s(x,t_K). \tag{8.11}$$

This leads to contradiction since by comparison (8.11) means that the solutions u and u_s cannot have the same extinction time, see the proof of Proposition 4.1. $\square$

We now show that the limit profile is concave.

Lemma 8.4. *We have*

$$w_{\eta\eta} \le M_2 e^{-\tau/2} \quad \textit{in } \Omega. \tag{8.12}$$

Proof. The function $z = \Delta u \equiv u_{rr} + ((N-1)/r)u_r$ satisfies in Ω the heat equation $z_t = \Delta z$. Using the identities similar to (7.10) we have that on the right interface $r = \zeta(t)$ of the solution $u(r,t)$ instead of (7.11) there holds

$$z_r = -z u_{rr} \equiv -z^2 - \frac{N-1}{\zeta}\, z \le -\frac{N-1}{\zeta}\, z. \tag{8.13}$$

Similarly, on the left interface $r = \xi(t)$

$$z_r = z\left(z - \frac{N-1}{\xi}\right) \ge 0 \quad \text{if } z \ge (N-1)/\xi. \tag{8.14}$$

Since by Lemma 8.1 $\xi \in [1,2]$, we have that $z_r \ge 0$ at $r = \xi$ provided that $z \ge N-1$. Finally, from (8.13), (8.14) by the Maximum Principle we conclude that $z \le M' = \max\{\sup \Delta u_0,\ N-1\}$ in Ω. Since $|u_r| \le M$, we have for all $r \in (\xi(t), \zeta(t))$

$$u_{rr} \le M' - \frac{N-1}{r}\, u_r \le M_3.$$

Hence, (8.12) follows by the scaling transformation (8.4). $\square$

8.3. Mass equation. We now estimate the mass function

$$E(\tau) = \int_{\phi(\tau)}^{\psi(\tau)} w(\eta,\tau)\, d\eta. \tag{8.15}$$

¿From the support bounds (8.3) and the w estimate (Lemma 8.3) we get a first estimate

$$E(\tau) = O(e^{\tau/2}). \tag{8.16}$$

Integrating equation (8.5) over $S(\tau) = (\phi(\tau), \psi(\tau))$ we arrive at the *mass equation*

$$\frac{dE}{d\tau} = E(\tau) - 2 + e^{-\tau/2} \int \mathbf{B}(w)(\tau) d\eta. \tag{8.17}$$

Now, we have

$$\int \mathbf{B}(w) d\eta = (N-1)\, e^{-\tau/2} \int \frac{w}{(r_* + \eta e^{-\tau/2})^2} d\eta = O(e^{-\tau/2}) E(\tau).$$

and hence (8.17) becomes

$$\frac{dE}{d\tau} = E(\tau)[1 + O(e^{-\tau})] - 2 \quad \text{for } \tau \gg 1, \tag{8.18}$$

Lemma 8.5. *As $\tau \to \infty$ we have*

$$E(\tau) = 2 + O(e^{-\tau}). \tag{8.19}$$

Proof. If E is larger than the estimate (8.19) the trajectory will go up to ∞ with a growth of the form $E(\tau) \geq O(e^{\tau})$, against (8.16). On the other hand, if the bound is violated from below the function E will reach the value 0 for a finite τ. against the definition of the rescaled time τ. $\square$

With these preliminaries the estimate of Lemma 7.3 is also valid.

8.4. Bounds on interfaces.

Lemma 8.6. *There holds*

$$\psi(\tau) \leq M \quad \textit{for } \tau > \tau_0. \tag{8.20}$$

Proof. We argue as in the proof of Lemma 7.4. Assume that there exists $\tau_2 \gg 1$ such that

$$\psi(\tau_2) \gg 1.$$

Consider first the case when $\phi(\tau) > c > 0$ for arbitrarily larger times. From the interface equation we obtain for the left interface

$$\phi' = -w_\tau \equiv -w_{\eta\eta} + \frac{1}{2}\phi - e^{-\tau/2}\frac{N-1}{r_* + e^{-\tau/2}\phi},$$

and hence by (8.12)

$$\phi' \geq \frac{1}{2}\phi - M' e^{-\tau/2} \quad \text{for } \tau > \tau_2 \gg 1. \tag{8.21}$$

Integrating over (τ_2, τ) we deduce that

$$\phi(\tau) \geq \phi(\tau_2) e^{(\tau-\tau_2)/2}(1 + o(1)) \quad \text{as } \tau \to \infty. \tag{8.22}$$

It then follows from (8.4) that the left interface in the original spatial variable r satisfies

$$\xi(t) \geq r_* + \phi(\tau_2) e^{-\tau_2/2}(1 + o(1)) \quad \text{for } t \approx T, \tag{8.23}$$

which indeed contradicts the definition of $r = r_*$ as the extinction point, $r_* \in E$.

In the second case when ϕ is bounded and negative we apply a mass analysis. Namely, under the same hypotheses as in the proof of Lemma 7.4 by integrating equation (8.5) over $(\phi(\tau), l)$ we arrive at estimate (7.31) with an extra term in the right-hand side:

$$\frac{\partial}{\partial \tau} \int_{\phi(\tau)}^{l} w(\tau) d\eta \approx -1 + e^{-\tau/2} \int_{\phi(\tau)}^{l} \mathbf{B}(w(\tau)) d\eta. \tag{8.24}$$

Integrating by parts in the last term we obtain

$$e^{-\tau/2}\int_\phi^l \mathbf{B}(w) = (N-1)e^{-\tau/2}\left\{\frac{w(l,\tau)}{r_*+e^{-\tau/2}l}+e^{-\tau/2}\int_\phi^l \frac{w}{(r_*+e^{-\tau/2}\eta)^2}\right\}\to 0.$$

The rest of the proof is the same as in (7.30)-(7.32). The remaining case is similar. □

¿From the above results, (7.33) and (7.34) follow.

8.5. Approximate Lyapunov function. Since equation (8.5) is not autonomous in time, it does not admit an explicit Lyapunov function like (7.35). Nevertheless, it is an approximate Lyapunov function and the following estimate holds.

Lemma 8.7. *We have*

$$\int_{\tau_0}^{\infty}\left(\int_{\phi(\tau)}^{\psi(\tau)}\rho(w_\tau)^2\right)d\tau<\infty. \tag{8.25}$$

Proof. Multiplying equation (8.5) by ρw_τ and integrating over $S\times(\tau_1,\tau_2)$ with fixed $\tau_1\gg 1$ and an arbitrary $\tau_2>\tau_1$ yield

$$\int_{\tau_1}^{\tau_2}\left(\int_\phi^\psi \rho(w_\tau)^2\right) = L[w](\tau_1)-L[w](\tau_2)+\int_{\tau_1}^{\tau_2}e^{-\tau/2}\int_\phi^\psi \rho\mathbf{B}(w)w_\tau.$$

Using an obvious estimate in the last term

$$\int_\phi^\psi \rho\mathbf{B}(w)w_\tau \le M\left|\int_\phi^\psi \rho w_\eta w_\tau\right| \le M\int_\phi^\psi \sqrt{\rho}|w_\tau|$$
$$\le M\left(\int_\phi^\psi \rho(w_\tau)^2\right)^{1/2}\le M\left(1+\int_\phi^\psi \rho(w_\tau)^2\right),$$

we obtain that

$$\int_{\tau_1}^{\tau_2}\left(\int_\phi^\psi \rho(w_\tau)^2\right)\le M,$$

where the constant M does not depend on τ_2 that yields the result. □

As in the proof of Lemma 6.4, by passing to the limit in equation (8.5) we establish that the ω-limit set, $\omega(w_0)$, consists of nontrivial bounded profiles $\{f\}$ satisfying the limit stationary equation

$$\mathbf{A}(f)=0, \tag{8.26}$$

with $\mathbf{A}$ given in (8.6). It follows from the mass estimate (8.19) that for any $f\in\omega(w_0)$

$$\int f\,d\eta = 2, \tag{8.27}$$

and (8.12) implies

$$f''\le 0. \tag{8.28}$$

The rest of the proof of Theorem 8.2 is very similar to the end of proof of Theorem 7.1. □

9. Focusing at the origin

We end our present investigation with a different singular phenomenon, namely the behaviour of a solution focusing at the origin in a finite time. By this we mean that the solution support (corresponding to the fresh zone) contains a hole where $u = 0$ and the support expands to fill this hole in finite time, the question being to describe the profile of the solution when it approaches the focusing and the rate of convergence to such a profile. The problem has no difficulty in one dimension because both components of the support, to the left and to the right of the hole, behave independently until they meet, and the interface behaves linearly near $t = T$. Therefore, we take $N \geq 2$. As in previous chapters there is a self-similar model for the focusing phenomenon, given here by formula (9.2). In particular, the interface behaviour as $t \to T$ will be quadratic, cf. formula (9.5).

Let $N \geq 2$ and let $u_0(r)$ be a smooth, bounded function satisfying

$$u_0 > 0,\ \ u_0' > 0 \ \text{ on } (1,\infty);\ u_0(1) = 0; \quad |u_0'| \leq M,\ \ |u_0''| \leq M. \tag{9.1}$$

Then, there exists a unique solution $u(r,t)$ of the radially symmetric problem with initial data u_0. Such a proof offers no difficulties with respect to what has been proved in Sections 1-3 and we will not give more details at this point. Under suitable growth conditions on u_0 the solution will exist globally, moreover it exhibits *finite time focusing:* there exists $T = T(u_0) > 0$ such that the unique interface $r = \zeta(t)$ reaches the origin

$$\lim_{t\to T} \inf\, \zeta(t) = 0 \quad \text{and} \quad \zeta(t) > 0 \text{ for } t < T.$$

The model example for such behaviour is given by the self-similar solution

$$u_1(r,t;T) = (T-t)^{1/2}[f_1(r/(T-t)^{1/2})]_+. \tag{9.2}$$

with profile $f_1(\eta)$ constructed in Section 5 and parameter $T > 0$. If we add the condition

$$u_0(x) \geq u_1(x,0,T) \quad \text{for some } T > 0. \tag{9.3}$$

then a simple comparison shows that the solution will fill its hole in a finite time and then it becomes positive everywhere (thus, a standard caloric function) for $t > T$, so that it exists globally in time. We can establish the following focusing description.

Theorem 9.1. *Let (9.1) and (9.3) be valid and T be the finite focusing time. Then as $t \to T^-$*

$$w(\eta,t) \equiv (T-t)^{-1/2}u(\eta(T-t)^{1/2},t) \to [f_1(\eta)]_+ \tag{9.4}$$

uniformly on compact subsets in η. The interface also converges,

$$\zeta(t) = \eta_1(T-t)^{1/2}(1+o(1)) \quad \text{as } t \to T^-, \tag{9.5}$$

where $\eta_1 = \eta_1(N) > 0$ is the unique vanishing point of the function $f_1(\eta)$.

Let us remark that the selection of a unique pattern is a typical nonlinear effect. In terms of the original variables Theorem 9.1 says that the solution u is given in first approximation near the focusing point, $r = 0$, $t = T$, by the self-similar formula (9.2). Such approximation is established in certain partial neighbourhoods of the form

$$D_{c,\varepsilon} = \{(r,t) : T - \varepsilon < t < T, 0 \le r \le c(T-t)^{1/2}\}, \tag{9.6}$$

i.e., near the free boundary. This is called the inner behaviour. It leaves out the behaviour of u on the line $t = T$ away from the singularity $r = 0$ (the outer behaviour). We want to prove that it agrees with the solution u_1 as $r \to 0$, i.e., that both behaviours match as we approach the singular point.

Theorem 9.2. *Under the above assumptions*

$$u(r,T) = a_1 r(1 + o(1)) \quad \text{as } r \to 0, \tag{9.7}$$

where $a_1(N) > 0$ is the constant in the asymptotic expansion (5.4).

The universal profile $f_1(\eta)$ and the corresponding constants η_1 and a_1 can be calculated explicitly for $N = 3$, when

$$f_1(\eta) = \frac{1}{2}\left(\eta - \frac{2}{\eta}\right) \quad \text{and} \quad \eta_1 = \sqrt{2}, \ a_1 = \frac{1}{2}. \tag{9.8}$$

On the other hand, for $N = 5$

$$f_1(\eta) = a_1\left(\eta - \frac{4}{\eta} - \frac{4}{\eta^3}\right) \quad \text{with} \quad \eta_1 = [2(1+\sqrt{2})]^{1/2}, \quad a_1 = \frac{3+2\sqrt{2}}{4(2+\sqrt{2})}. \tag{9.9}$$

We have explicitly used the notation $[f_1]_+$ to recall the reader that, when considered as a solution of our problem, the positive part has to be taken in the above formulas, cf. the opening comment in Section 5. This will be implicit in the sequel. Observe that u_1 takes at $t = T$ the exact form $u_1(r,T) = a_1 r$.

9.1. General comment on the problem and method. Our focusing problem has the typical features of blow-up processes for nonlinear heat equations, namely: there arises in a *finite time* ($t \to T^- < \infty$) a singular behaviour around a *finite point* (such point is here the origin $x = 0$). These two *finite* spatio-temporal conditions make the problem both interesting and difficult, since it is impossible to use in the analysis the usual techniques of shifting the solution in time and space in order to construct suitable barriers and to apply other standard techniques from the theory of parabolic equations. This difficulty is a typical one in blow-up problems. To avoid this, in the present paper as well as in other blow-up studies we apply the Intersection Comparison argument to construct nonstandard "barriers". Nevertheless, let us observe that in the present problem the asymptotic behaviour reduces to the study of infinite-time stabilization for an autonomous in time parabolic equation (with a free boundary), a problem which is well known in the general theory. Actually, this is a common property of focusing stated for pure parabolic operators with power-like nonlinearities. (Extra difficulties concerning the Lyapunov instability of the a priori bounds via Intersection Comparison). A very useful tool that applies in this case is the existence of a standard explicit or implicit Lyapunov function (formally, this can be done for *any quasilinear* autonomous parabolic equation [Z], and it is necessary to check its good properties in order to pass to the limit).

Therefore, in comparison with known blow-up problems where more complicated, essentially nonlinear nonautonomous equations appear in the dynamical systems are needed (see e.g. [GV1], [GV2], [GV3] and references therein), focusing problems can be fortunately solved in a rather standard way using some of the ideas developed in blow-up research.

We propose here a somewhat general scheme to study focusing free-boundary problems which is expected to be useful for different equations with other quasilinear diffusion operators and nonlinear lower order terms. In general, the present analysis consists of five main steps.

(i) Control of the interface (free boundary) of the rescaled function w from above and below by using suitable geometrical properties of the stationary rescaled profiles (via the Intersection Comparison technique).

(ii) A semiconvexity argument.

(iii) A mass equation.

(iv) A standard Lyapunov analysis in passing to the limit in the rescaled equation to prove Theorem 9.1.

(v) A matching procedure to continue the expansion on compact subsets in the rescaled spatial variable η to small subsets in the original variable $r = |x|$, Theorem 9.2.

9.2. Existence of focusing. This can be easily proved by comparison with the self-similar solutions with single-point extinction on an annulus of Section 5. On the other hand, it is easy to see that this can be done by comparison with subsolutions. namely, we fix an arbitrary $l > \eta_0$ and set

$$\overline{u}(r,t) = (T_0 - t)^{1/2} c_l f(\eta; l), \quad \eta = \frac{r}{(T_0 - t)^{1/2}}, \; T_0 > 0, \tag{9.10}$$

where $c_l > 0$ is chosen so that $c_l f'(\eta_*(l); l) \geq 1$. By comparison we have

Lemma 9.3. *Let in addition to (9.1)*

$$u_0(r) \geq \overline{u}(r,0) \quad \textit{for } r \geq 1. \tag{9.11}$$

Then there exists focusing at the origin and $T(u_0) \leq T_0$.

9.3. Bounds on the interface and eventual monotonicity. We now consider the rescaled equation (4.6.a) in $\Omega = \{\eta > \psi(\tau)\} \times (\tau_0, \infty)$, where $\eta = \psi(\tau)$ denotes the interface, with the conditions

$$w = 0, \;\; w_\eta = 1 \quad \text{for} \;\; \eta = \psi(\tau), \; \tau \geq \tau_0, \tag{9.12}$$

and smooth, bounded and increasing initial data $w_0(\eta)$, see (9.1).

Lemma 9.4. *There holds*

$$0 < m_2 \leq \psi(\tau) \leq M_2 \quad \textit{for } \tau > \tau_0. \tag{9.13}$$

Proof. It is similar to the proof of Proposition 6.1.

Lower bound. We fix $l_2 \approx \eta_0,\ l_2 > \eta_0$, and consider the function

$$\tilde{f}(\eta; l_2) = cf(\eta; l_2), \tag{9.14}$$

where the function f is as in (5.7) - (5.9) and the constant c is such that $\tilde{f}'(\eta_*; l_2) = 1$. It follows from (5.10) that in the positivity domain

$$\tilde{f}(\eta; l_2) \to 0 \quad \text{as} \ \ l_2 \to \eta_0^+ \text{ uniformly}. \tag{9.15}$$

Therefore, we may suppose that the number of intersections of the functions $\tilde{f}(\eta; l_2)$ and $w_0(\eta)$ satisfies $I(\tau_0; l_2) \le 1$ (cf. (6.2)). As in (6.3) we get the lower bound in (9.13) with $m_2 = \eta_*(l_2) > 0$ since both solutions are assumed to have the same focusing extinction time at the origin.

Upper bound. Denote by $\bar{f}(\eta; l)$ the solution of (5.1) for $\eta > l$ with boundary conditions (cf. (5.7))

$$\bar{f}(l; l) = 0,\ \ \bar{f}'(l; l) = 1. \tag{9.16}$$

It follows from (5.8) and from asymptotics (5.3) and (5.4) that for $l \gg 1$

$$\bar{f}(\eta; l) \approx \frac{2}{l}\left[\left(\frac{l}{\eta}\right)^{N+1} e^{(\eta^2 - l^2)/4} - \frac{\eta}{l}\right] \quad \text{on} \ \ [l, l+1], \tag{9.17}$$

and therefore it is easy to check that

$$\bar{f}'_\eta(\eta; l) \to \infty \quad \text{as } l \to \infty \tag{9.18}$$

uniformly on any set $\{\eta > l:\ 0 \le m \le \bar{f} \le M\}$. Hence, in view of the regularity of w_0 we have that there exists $l_2 \gg 1$ such that (6.2) holds, i.e., one intersection. Therefore, by comparison $\psi(\tau) \le l_2$ for all $\tau > \tau_0$ whence the upper bound in (9.13) with $M_2 = l_2$. $\square$

9.4. Semiconvexity. It follows from [CV] that

$$|w_\eta| \le M \quad \text{in } \Omega, \tag{9.19}$$

and hence

$$w(\eta, \tau) \le M(1 + \eta). \tag{9.20}$$

Using exactly the same argument as in the proof of Proposition 8.4, from (9.1) we have $u_r \ge 0$ and therefore we arrive at the inequalities

$$u_{rr} \le M' - \frac{N-1}{r} u_r \le M,$$

whence

$$w_{\eta\eta} \le M e^{-\tau/2} \quad \text{in } \Omega. \tag{9.21}$$

9.5. The ω-limit set. We now introduce the ω-limit set

$$\begin{aligned} \omega(w_0) = \{g \in C :\ \exists \text{ a sequence } \{\tau_k\} \to \infty \text{ such that} \\ w(\cdot, \tau_k) \to g(\cdot) \text{ uniformly on compact subsets}\}. \end{aligned} \tag{9.22}$$

Then we have from (9.19)-(9.21) that any $g \in \omega(w_0)$ satisfies

$$0 \le g(\eta) \le M(1+\eta), \quad g_{\eta\eta} \le 0. \tag{9.23}$$

9.6. Weighted mass equation. Rewriting the equation in divergence form, (6.11), and integrating it over $S(\tau) = \{\eta > \psi(\tau)\}$ we arrive at the equation

$$\frac{d}{d\tau} E(\tau) = \frac{1}{2} E(\tau) - \rho|_{\eta=\psi(\tau)}, \tag{9.24}$$

for the weighted mass

$$E(\tau) = \int \rho w \, d\eta = \int_{\psi(\tau)}^{\infty} w(\eta,\tau) e^{-\eta^2/4} \eta^{N-1} d\eta. \tag{9.25}$$

Lemma 9.5. *There holds*

$$0 < m_3 \le E(\tau) \le M_3 \quad \textit{for } \tau > \tau_0. \tag{9.26}$$

Proof. The upper bound follows from (9.20). To prove the lower one we assume for a moment $E(\tau_1) \ll 1$ for some $\tau_1 > \tau_0$. Since ψ is bounded below, (9.13), we get from (9.24) that for $\tau > \tau_1$

$$E'(\tau) \le \frac{1}{2} E(\tau_1) - c < 0, \quad c = \min\{\rho(m_2),\ \rho(M_2)\},$$

i.e., $E(\tau)$ vanishes in a finite time contradicting the Strong Maximum Principle. $\square$

Corollary 9.6. *We have separation from the trivial limit:*

$$0 \notin \omega(w_0). \tag{9.27}$$

9.7. Lyapunov analysis. Proof of Theorem 9.1. One can check that under the above hypotheses the problem admits an explicit Lyapunov function (cf. (6.13)) of the form

$$L[w](\tau) = \frac{1}{2} \int_{\psi(\tau)}^{\infty} \rho \left[(w_\eta)^2 - \frac{1}{2} w^2 \right] + \frac{1}{2} \int_{\psi(\tau)}^{\infty} \rho(s) ds, \tag{9.28}$$

which is nonincreasing on evolution orbits. Therefore, $\omega(w_0)$ consists of stationary profiles $\{f\}$ satisfying equation (5.1) and conditions (9.23). It then follows from the properties of stationary profiles in Section 5 that

$$\omega(w_0) \subseteq \{\alpha f_1(\eta),\ \ 0 < \alpha \le M\}, \tag{9.29}$$

and the concavity result (9.21) implies convergence of the interface

$$\psi(\tau) \to \eta_1 \quad \text{as } \tau \to \infty, \tag{9.30}$$

which proves (9.5). It remains to prove that $\alpha = 1$ in (9.29). By passing to the limit in the mass equation (9.29) we obtain that there exists the limit $E(\infty) = 2\rho(\eta_1)$. This means that if $g = \alpha f_1 \in \omega(w_0)$, then

$$\alpha = \frac{2\rho(\eta_1)}{\int \rho f_1} = 1. \tag{9.31}$$

Therefore, $\omega(w_0) = \{f_1\}$, which completes the proof of Theorem 9.1. $\square$

9.8. Proof of Theorem 9.2. Matching. Since we have a precise information about the free boundary $r = \zeta(t)$ for $t \approx T$ and $u(x,t)$ is the solution of the heat equation in $\{(x,t) : 0 < t \le T, |x| \ge \zeta(t)\}$, from the linear growth rate at $t = 0$ we conclude that the solution exists and is locally bounded up to $t = T$. Indeed, estimates (9.19)-(9.21) can be rephrased as

$$u(r,t) \le M(T - t + r), \quad |u_r| \le M, \qquad u_{rr} \le M, \tag{9.32}$$

uniformly in $r \ge \zeta(t)$, $0 < t < T$. In order to compare this with the 'inner' behaviour of Theorem 9.1 we introduce for all fixed $\alpha \in [0,T)$ the rescaled functions

$$v_\alpha(r,t) = \varepsilon^{-1/2} u(r\varepsilon^{1/2}, \alpha + \varepsilon t), \tag{9.33}$$

where $\varepsilon = \varepsilon(\alpha) \equiv (T-\alpha)/T > 0$. It is immediate that v_α is a solution of the heat equation (written in radial coordinates) in the domain $\{0 \le t \le T,\ r \ge \zeta_\alpha(t) = \varepsilon^{-1/2}\zeta(\alpha + \varepsilon t)\}$. Rephrasing the results of previous subsections we can say that for $0 \le t \le T - \delta < T$ the family converges,

$$\lim_{\alpha \to T} v_\alpha(r,t) = u_1(r,t) \tag{9.34}$$

uniformly on compact subsets $0 \le r \le K$. Moreover, the interfaces also converge uniformly for $0 \le t \le T$. Now, in every domain of the form $D_K = \{(r,t) : 0 \le t \le T, K \le r\}$ with K large the v_α's are a family of positive solutions of the heat equation satisfying uniformly the estimates listed in (9.32). It follows that on compact subsets of D_K it converges to a limit $\tilde{v}$ along a subsequence α_n. The point here is that we allow $t \to T$ for fixed $r > 0$. Now, we know that on any subset away from $t = T$ the family v_α converges to $\tilde{u} = u_1$. We conclude that the whole family v_α converges to u_1 and formula (9.34) is true on compact subsets of D_K. We may now take a fixed point (r_0, T) to get

$$\lim_{\alpha \to T} v_\alpha(r_0, T) = u_1(r_0, T),$$

which is equivalent to (9.7).

Remark. After the focusing time the hole and the free boundary disappear and the solution can be naturally continued as a solution of the heat equation in the whole space. The behaviour at $r \approx 0$ for $t > T$, $t \approx T$, is easily shown to be self-similar with

$$u(0,t) \approx c_2 (t-T)^{1/2}. \tag{9.35}$$

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