Analysis of a Darcy flow model with a dynamic pressure saturation relation*

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Abstract

We consider a new model for groundwater flow. The model differs from previous models in that the saturation-pressure relation is extended with a dynamic term, namely the time derivative of the saturation. The resulting model equation is of nonlinear degenerate pseudo-parabolic type. We give a rigorous analysis of travelling wave solutions and the local behaviour near fronts. Part of this analysis is then used in a formal asymptotic analysis of solutions with initial support in a half line. As an illustrative special case we also consider Heaviside initial data.

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Abbreviated Title. Darcy flow with dynamic pressure.

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1 Introduction

Multiphase flows in porous media occur in various practical situations including oil recovery and unsaturated groundwater flow, see for instance [6]. The mathematical modelling of such flows is therefore of great importance, the aim being to give qualitative and quantitative predictions of the saturation profiles, based on knowledge about initial profiles, in- and outflow boundary conditions, and the governing equations modelling the flow. In the standard approach for two phase flows, such as oil-water or air-water mixtures, one combines the mass conservation equations and Darcy's law for the separate phases with a relation for the pressure difference in the phases. Typically this relation is taken to be of the form

$$p_n - p_w = p_c(S_w), \tag{1.1}$$

where p_n, p_w are respectively the pressure of the non-wetting and the wetting phase, and where S_w is the level of saturation of the wetting phase. The function $p_c(S_w)$ is called the capillary pressure. It is usually assumed to be a bounded decreasing function on the interval $[0, 1]$ with some smoothness properties and satisfying $p_c(1) = 0$. In the case of air (non-wetting) and water (wetting), the air pressure is often assumed to be atmospheric, so that p_n is constant; henceforth we consider this case.

Mass conservation of water is given by

$$\phi \frac{\partial S_w}{\partial t} + \mathrm{div}\ q_w = 0. \tag{1.2}$$

Here ϕ is the porosity of the porous medium and q_w the volumetric water flux. Darcy's law reads

$$q_w = -K(S_w)\ \mathrm{grad}\ p_w, \tag{1.3}$$

where $K(S_w)$ is the hydraulic conductivity, which is a nonnegative increasing function of $S_w \in [0, 1]$. Note that we neglect gravity.

Combining (1.1,1.2,1.3) and restricting to one space dimension, we obtain a single nonlinear degenerate diffusion equation for the saturation S_w [9]:

$$\frac{\partial S_w}{\partial t} = -\frac{\partial}{\partial x}(K(S_w)\frac{\partial}{\partial x}p_c(S_w)) = \frac{\partial}{\partial x}(D(S_w)\frac{\partial S_w}{\partial x}), \tag{1.4}$$

where

$$D(S_w) = -K(S_w)p_c'(S_w). \tag{1.5}$$

Equation (1.4) is of great practical importance because it can actually be used to compute the saturation profile if the the constitutive functions $K(S_w)$ and $p_c(S_w)$ are known. Usually these are determined from experiments and typically the coefficient $D(S_w)$ tends to zero as S_w tends to zero.

Replacing S_w by u and taking as a model for the case of small saturations $D(u) = u^\alpha$ with $\alpha > 0$, we arrive at

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(u^\alpha \frac{\partial u}{\partial x}), \tag{1.6}$$

which is regarded as *the* model equation for degenerate diffusion. Equation (1.6), which also arises in numerous other applications, has been subject of intensive research over the past decades, see for example [2], [17], [1], [12].

A striking property of (1.6) is the occurrence of interfaces: solutions with compactly supported initial data remain compactly supported for all positive times. In terms of the groundwater model this corresponds to flows with a moving front separating the dry and the wet regions. Properties such as waiting-time phenomena, regularity and large time behaviour of the interfaces bounding the (expanding) support have been fully analysed.

A drawback of this model is not so much the explicit choice of $D(u)$ above, but rather the assumption that the capillary pressure is a function of the saturation only. This is generally considered by groundwater engineers to be untrue, measurements for wetting and draining exhibiting different pressure-saturation curves. To capture the dynamic aspects of two-phase flows more accurately, Hassanizadeh and Gray [11] have recently proposed a new approach in which the pressure saturation relation (1.1) is replaced by an expression of the form

$$p_n - p_w = p_c(S_w) - L\frac{\partial S_w}{\partial t}. \quad (L > 0) \tag{1.7}$$

The coefficient L may be viewed as a capillary damping coefficient. A similar relation was also suggested some time ago as an outcome of experiments by Stauffer [16]. This changes equation (1.4) into

$$\frac{\partial S_w}{\partial t} = -\frac{\partial}{\partial x}(K(S_w)\frac{\partial}{\partial x}p_c(S_w)) + \frac{\partial}{\partial x}(K(S_w)L\frac{\partial^2 S_w}{\partial x \partial t}). \tag{1.8}$$

The corresponding model equation is

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(u^\alpha \frac{\partial u}{\partial x}) + \frac{\partial}{\partial x}(u^\beta \frac{\partial^2 u}{\partial x \partial t}), \tag{1.9}$$

where α and β are positive constants. Let us mention here that, as is the case with (1.6), equation (1.9) should be considered as a model for small saturations so that it is not unreasonable to take pure powers in the nonlinearities, which has the advantage of making the analysis more transparent and the calculations more explicit. Models closely related to (1.9) also arise in other contexts; see, for example, [13]. A physically relevant initial boundary value problem consists of studying (1.9) on an x-interval $[0, l]$, with nonnegative initial data $u(x, 0) = u_0(x)$ and zero flux lateral boundary conditions $u^\alpha u_x + u^\beta u_{xt} = 0$ on $x = 0, l$.

Assuming for the moment that $\beta \neq \alpha + 1$, we may transform this problem into two equations which decouple the space and time derivatives, namely

$$u_t + \frac{1}{\alpha - \beta + 1}u^{\alpha-\beta+1} = w \quad \text{for} \quad t \geq 0; \quad u = u_0 \quad \text{at} \quad t = 0; \tag{1.10}$$

and

$$-(u^\beta w_x)_x + w = \frac{1}{\alpha - \beta + 1}u^{\alpha-\beta+1} \quad \text{for} \quad x \in (0, l); \quad u^\beta w_x = 0 \quad \text{on} \quad x = 0, l. \tag{1.11}$$

It is clear from the theory of second order ordinary differential equations that (1.11) is uniquely solvable in w if the function u is positive and smooth (in fact uniform bounds of the form $0 < \delta \leq u \leq M$ and Lebesque measurability suffice if we consider weak solutions). Since the solution w depends smoothly on the choice of u, it follows that, with (1.10), the problem may be written as an ordinary differential equation in an open subset of a Banach space of functions. In particular local well-posedness for piecewise continuous positive initial data is immediate.

However, we are interested here in solutions with moving (or fixed) fronts. In view of the degeneracy near $u = 0$, it is not yet clear what the proper definition of such solutions should be, and what kind of moving fronts one should expect. As in the case of (1.6), one would expect such a solution to be obtained as the limit of smooth solutions u_ϵ with initial data $u_\epsilon(x, 0) = u_0(x) + \epsilon$. Though positivity properties of the approximating smooth solutions may be obtained using maximum principle type arguments, there is, unfortunately, unless $\beta = 0$, no comparison principle which allows us to conclude monotonicity in ϵ. Thus the existence of a (unique) limit as $\epsilon \to 0$ is unknown. (For $\beta = 0$ well-posedness is immediate from (1.10) and (1.11). In this case also a strong maximum principle holds, so that nonnegative solutions instantaneously become strictly positive).

For $\alpha > 0$ and $\beta > 0$ we shall first make (Section 2) a detailed and rigorous study of possible travelling wave solutions and in particular of the behaviour of such travelling waves near $u = 0$. This amounts to a local analysis of the behaviour of general solutions close to a moving front; the global travelling wave profiles are in fact always unbounded. This may seem unphysical at first because we are thinking of saturations which should be bounded by definition; the point, however, is that one expects solutions near moving fronts to behave locally as a travelling wave. In other words, fixing t and zooming in at the front by scaling u and x, the profile selects one of the (local) travelling wave profiles. This is well known for (1.6) where the speed of the selected profile coincides with the speed of the propagating front.

In the case of (1.6), for each $\alpha > 0$, there is, up to translation, only one travelling wave with a front for each (positive) speed. In the travelling wave analysis of (1.9) we encounter what we will call generic and exceptional behaviours near $u = 0$, but there are also values of the parameters for which no travelling fronts exist. In order to give an interpretation of these travelling waves in terms of local front behaviour of general solutions, we shall carry out a formal asymptotic analysis (Section 3) of the limit solution obtained through the regularisation by setting $u_\epsilon(x, 0) = u_0(x) + \epsilon$. In this part of the analysis the limit solution is assumed to exist and behave well under perturbation.

It will turn out that near a moving front the asymptotics select the exceptional behaviour of the solutions of the travelling wave equation near zero. Moreover, for values of the parameters for which there are no travelling wave solutions with a front, the asymptotics indicate that the front is in fact fixed. Numerical experiments carried out by Leijnse [14] at the RIVM in Bilthoven, The Netherlands, also suggest that in some cases the front does not move at all and that the solution develops a rather steep standing profile. To illustrate this behaviour, we also include an asymptotic analysis of solutions with Heaviside initial data (Section 4). In this analysis the global behaviour of the unbounded travelling waves will play a role.

We mention that equation (1.9) differs considerably from another third order extension of the porous medium equation involving mixed derivatives, see [3], where basically in the right-hand side of (1.4) the quantity S_w is replaced by $S_w + \tau \frac{\partial}{\partial t} S_w$. The corresponding model equation would read, in our notation,

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2} \Phi(u + \frac{\partial u}{\partial t}), \tag{1.12}$$

with, for example, $\Phi(s) = s^\alpha$. The main mathematical difference is that (1.12) allows a splitting similar to (1.10,1.11), but with w appearing only on the right hand side of (1.11). Therefore (1.12) is easier to analyse [8].

Equations such as (1.9) and (1.12) are called pseudo-parabolic. They are also, in general, nonlinear and, for appropiate α, β and Φ, degenerate. With, respectively, $\alpha = \beta = 0$ and $\Phi(s) = s$, both (1.9) and (1.12) reduce to the same linear pseudo-parabolic equation. For relevant references the interested reader is referred to the book of Carroll and Showalter [7], and also to [4] and [5], where relaxation terms appear in a different context.

2 Travelling wave solutions

In this section we investigate for which values of α and β (1.9) has advancing front solutions. As observed in the introduction this is a purely local analysis. We note that for a large class of equations including (1.6), such an analysis has been an essential first step towards a complete understanding of the full partial differential equation and the appearance of interfaces, see [10] for an exhaustive discussion. We set

$$u(x,t) = U(\eta), \quad \eta = x + ct \tag{2.1}$$

which gives for $U(\eta)$ that

$$cU' = (U^\alpha U' + cU^\beta U'')', \tag{2.2}$$

where primes denote differentation with respect to η. In this section the profiles we consider are increasing in x and moving to the left. Thus we take $c > 0$.

Since we are interested in moving fronts, we look for solutions with

$$U(\eta) \downarrow 0 \;\text{ and }\; U(\eta)^\alpha U'(\eta) + U(\eta)^\beta U''(\eta) \to 0 \;\text{ as }\; \eta \downarrow \eta_0 > -\infty, \tag{2.3}$$

for some finite η_0 corresponding to the position of the front. The second relation in (2.3) states that at the front the flux should be equal to zero (conservation of mass). In the first part of the analysis we classify the solutions of (2.2) which satisfy (2.3), in terms of both their local and their global behaviour.

We then also identify the speeds for which there are solutions of (2.2) with $U(\eta) \to 1$ as $\eta \downarrow -\infty$ and having an exceptional growth for $\eta \uparrow \infty$. This will be needed for the asymptotic analysis of (1.9) in Section 3. Therefore we choose not to scale c out of the equation, even though this is possible if $\beta \neq 2\alpha$.

Since the equation (2.3) is autonomous we may, without loss of generality, take $\eta_0 = 0$. Using the boundary conditions (2.3) we integrate (2.2) to give

$$U'' + \frac{1}{c}U^{\alpha-\beta}U' = U^{1-\beta}, \tag{2.4}$$

which we rewrite as the system,

$$U' = V; \quad V' = U^{1-\beta} - \frac{1}{c}U^{\alpha-\beta}V = U^{\alpha-\beta}(U^{1-\alpha} - \frac{V}{c}). \tag{2.5}$$

Let $\mathcal{V}^+ = \{(U,V) : \ U > 0, \ V > 0, \ V' < cU^{1-\alpha}\}$ and $\mathcal{V}^- = \{(U,V) : \ U > 0, \ V > 0, \ V' > cU^{1-\alpha}\}$. Note that, assuming $U' = V > 0$,

$$\frac{dV}{dU} = \frac{U^{1-\beta}}{V} - \frac{U^{\alpha-\beta}}{c}. \tag{2.6}$$

We shall also use a number of related equations, namely

$$Z = \frac{1}{U^{\alpha-1}U'};\ s = \log U : \quad \tfrac{dZ}{ds} + (\alpha-1)Z = e^{(2\alpha-\beta)s}Z^2(\tfrac{1}{c} - Z); \tag{2.7}$$

$$W = \frac{2}{\beta}(U^{\frac{\beta}{2}})' : \quad \tfrac{dW}{ds} = \tfrac{1}{W} - \tfrac{2-\beta}{2}W - \tfrac{1}{c}e^{(\alpha-\frac{\beta}{2})s}, \tag{2.8}$$

$$Y = \frac{1}{\beta-\alpha}(U^{\beta-\alpha})' : \quad \tfrac{dY}{ds} = (\beta-\alpha-1)Y - \tfrac{1}{c} + \tfrac{e^{(\beta-2\alpha)s}}{Y}; \tag{2.9}$$

$$A = \frac{1}{Y} = \frac{\beta-\alpha}{(U^{\beta-\alpha})'} : \quad \tfrac{dA}{ds} = A(\tfrac{A}{c} - \beta + \alpha + 1) - e^{(\beta-2\alpha)s}A^3; \tag{2.10}$$

In terms of Z we have that $\mathcal{V}^+ = \{Z > \frac{1}{c}\}$ and $\mathcal{V}^- = \{Z < \frac{1}{c}\}$.

2.1 Advancing fronts for $0 < \alpha < 1$

By straightforward phase plane analysis we see that tracing back orbits in $\mathcal{V}^+$ there are three possibilities: (i) $U \to u_0 > 0$ and $V \to 0$ as $\eta \downarrow 0$, (ii) $U \to 0$ and $V \to 0$ as $\eta \downarrow 0$, or (iii) (U,V) comes out of $\mathcal{V}^-$.

Orbits in $\mathcal{V}^-$ correspond to concave solutions U, so clearly, by shifting η, these have $U \downarrow 0$ as $\eta \downarrow 0$. It is easily seen from integrating (2.6) that for $\beta \geq \alpha + 1$ this happens with unbounded $V = U'$, while $V = U'$ remains bounded for $\beta < \alpha + 1$. We call these solutions generic fronts because nearby solutions of (2.3) also have fronts with the same local behaviour; we denote these 1-parameter families of solutions by U_g.

Since both (i) and (iii) can clearly occur and both are stable with respect to small changes, it follows that there is at least one exceptional solution to (2.5) which satisfies (ii). In fact we shall see that, up to a shift in η, only one such solution exists, and that it satisfies $U \downarrow 0$ and $U' \downarrow 0$ as $\eta \downarrow 0$; we shall call such solutions exceptional fronts and denote them by U_e.

2.1.1 The generic fronts for $0 < \alpha < 1$

As already stated, we have that all solutions in $\mathcal{V}^-$ come from the positive V-axis if $\beta < \alpha + 1$ and in fact they are parametrised by $v_0 = V(0)$. Thus we have

$$U_g(\eta) \sim v_0 \eta \quad \text{as} \quad \eta \downarrow 0 \quad \text{if} \quad \beta < \alpha + 1. \tag{2.11}$$

If $\beta > \alpha+1$ all the generic fronts are concave near $U = 0$ with U' unbounded. For a more detailed analysis of these solutions we employ the Z-equation (2.7). Introducing $P = e^{(\beta-2\alpha)s}$, we rewrite (2.7) as

$$P\frac{dZ}{ds} = (1-\alpha)PZ + Z^2(\frac{1}{c} - Z); \quad P\frac{dP}{ds} = (\beta - 2\alpha)P^2; \tag{2.12}$$

the P-factor in front of the s-derivatives may absorbed into a new independent variable. In the (Z, P)-plane the generic fronts come out of the origin. Using polar coordinates $Z = r\cos\varphi$, $P = r\sin\varphi$, we find critical angles (restricting to $P \geq 0$ and $Z \geq 0$)

$$\varphi = 0; \quad \varphi = \frac{\pi}{2}; \quad \varphi = \arctan\frac{1}{c(\beta-\alpha-1)}. \tag{2.13}$$

Linearising we find that the first two are saddle points but the corresponding unstable manifolds are just the positive Z-and P-axis. The third angle however is an unstable node. It follows that all generic fronts correspond to solutions with

$$Y = U^{\beta-\alpha+1}U' = \frac{P}{Z} \to \frac{1}{c(\beta-\alpha-1)}. \tag{2.14}$$

Returning to the Y-equation (2.9) and writing

$$Y(s) = \frac{1}{c(\beta-\alpha-1)} + X(s), \tag{2.15}$$

we find

$$X(s) = e^{(\beta-\alpha-1)s}\big(K + \int_{-\infty}^{s} \frac{e^{(1-\alpha)a}}{\frac{1}{c(\beta-\alpha-1)} + X(a)} da\big) = \tag{2.16}$$

$$Ke^{(\beta-\alpha-1)s} + \frac{c(\beta-\alpha-1)}{1-\alpha}e^{(\beta-2\alpha)s} + O(e^{(2\beta-3\alpha-1)s}) \quad \text{as} \quad s \to -\infty, \tag{2.17}$$

where K is an arbitrary constant. In particular we thus have

$$U_g(\eta) \sim \big(\frac{\beta-\alpha}{\beta-\alpha-1}\frac{\eta}{c}\big)^{\frac{1}{\beta-\alpha}} \quad \text{as} \quad \eta \downarrow 0 \quad \text{if} \quad \beta > \alpha + 1. \tag{2.18}$$

Finally we consider the case $\beta = \alpha + 1$. For $Y = V = U'$ (which still goes to infinity) we have

$$\frac{dY}{ds} = -\frac{1}{c} + \frac{e^{(1-\alpha)s}}{Y(s)} \to -\frac{1}{c} \quad \text{as} \quad s \downarrow -\infty. \tag{2.19}$$

This gives

$$U_g(\eta) \sim -\frac{1}{c}\eta\log\eta \quad \text{as} \quad \eta \downarrow 0 \quad \text{if} \quad \beta = \alpha + 1. \tag{2.20}$$

2.1.2 The exceptional front for $0 < \alpha < 1$

Here we consider orbits in the open first quadrant of the (U, V)-plane which come out of the origin. These orbits can only lie in the set $\mathcal{V}^+$, so they have $Z > \frac{1}{c}$. We first consider $\beta < 2\alpha$. Writing the Z-equation (2.7) as the system

$$\frac{dZ}{ds} = (1-\alpha)Z + QZ^2(\frac{1}{c} - Z); \quad \frac{dQ}{ds} = (2\alpha - \beta)Q, \tag{2.21}$$

these orbits come out of infinity along the positive Z-axis. Thus in (2.6) the first term on the right-hand side dominates. It follows that $W = U^{\frac{b-2}{2}}V$ goes to a positive constant, and the W-equation (2.8) in fact gives that

$$W = U^{\frac{b-2}{2}}V = \frac{2}{\beta}(U^{\frac{\beta}{2}})' \to \sqrt{\frac{2}{2-\beta}} \quad \text{as} \quad U \downarrow 0. \tag{2.22}$$

Writing (2.8) as the system

$$\frac{dW}{ds} = \frac{1}{W} - \frac{2-\beta}{2}W - \frac{Q}{c}; \quad \frac{dQ}{ds} = (\alpha - \frac{\beta}{2})Q, \tag{2.23}$$

the corresponding orbit comes out of the saddle point $(W, Q) = (\sqrt{\frac{2}{2-\beta}}, 0)$ and is therefore unique. In particular we find that the exceptional front is unique and satisfies

$$U_e(\eta)^{\frac{\beta}{2}} \sim \frac{\beta\eta}{\sqrt{2(2-\beta)}} \quad \text{as} \quad \eta \downarrow 0 \quad \text{if} \quad \beta < 2\alpha. \tag{2.24}$$

Next we consider $\beta > 2\alpha$. In terms of Z, the exceptional front must have the property that $Z > 1/c$ and, by inspection of (2.7) and (2.12), that $Z \to 1/c$ as $s \to -\infty$, so that $U^{\alpha-1}U' \to c$. Introducing as new independent variable,

$$t = \frac{1}{2\alpha - \beta}e^{(2\alpha-\beta)s} = \frac{1}{2\alpha - \beta}U^{2\alpha-\beta}, \tag{2.25}$$

we find

$$\frac{dZ}{dt} = -\frac{1-\alpha}{\beta - 2\alpha}\frac{Z}{t} + Z^2(\frac{1}{c} - Z). \tag{2.26}$$

Writing

$$Z(s) = \frac{1}{c} + R(s), \tag{2.27}$$

we have

$$R(t) = -\frac{1-\alpha}{\beta-2\alpha}\frac{1}{t}\int_{-\infty}^{0}(1+\frac{\xi}{t})^{\frac{\alpha+1-\beta}{\beta-2\alpha}}e^{\frac{\xi}{c^2}}d\xi - \int_{-\infty}^{0}(1+\frac{\xi}{t})^{\frac{1-\alpha}{\beta-2\alpha}}e^{\frac{\xi}{c^2}}F(R(t+\xi))d\xi, \tag{2.28}$$

where $F(R) = R^2(R + \frac{2}{c})$. Using a contraction argument, only one solution of this integral equation with $R(-\infty) = 0$ is seen to exist, and it satisfies

$$R(t) \sim -\frac{1-\alpha}{\beta-2\alpha}\frac{c^2}{t} \quad \text{as} \quad t \to -\infty, \tag{2.29}$$

so that, in terms of U,

$$U_e(\eta)^\alpha \sim c\alpha\eta \quad \text{as} \quad \eta \downarrow 0 \quad \text{if} \quad \beta > 2\alpha. \tag{2.30}$$

Finally for the borderline case $\beta = 2\alpha$ we have that the Z-equation (2.7) reduces to

$$\frac{dZ}{ds} = (1-\alpha)Z + Z^2(1/c - Z), \tag{2.31}$$

and the exceptional front corresponds to the unique (stable) positive zero of the right-hand side of (2.31). Thus we find the exact formula

$$U_e(\eta)^\alpha = \frac{\alpha\eta}{\frac{1}{2c} + \sqrt{\frac{1}{4c^2} + 1 - \alpha}} \quad \text{if} \quad \beta = 2\alpha. \tag{2.32}$$

2.1.3 Behaviour of the fronts for $0 < \alpha < 1$ as $\eta \to \infty$

For the sake of completeness we describe here what happens to the above fronts as η becomes large. Inspection of the (U, V)-phase plane gives that all solutions enter the set $\mathcal{V}^+$, meaning that Z becomes larger than $1/c$. For $\beta = 2\alpha$ it is clear from (2.31) that Z tends to the positive zero of the right-hand side of (2.31), whence all fronts have

$$U(\eta)^\alpha \sim \frac{\alpha\eta}{\frac{1}{2c} + \sqrt{\frac{1}{4c^2} + 1 - \alpha}} \quad \text{as} \quad \eta \to \infty \quad \text{if} \quad \beta = 2\alpha. \tag{2.33}$$

If $\beta > 2$ the linear term in (2.7) dominates the nonlinear term so that $Z(s) \sim K \exp{(1-\alpha)s}$. In terms of V we have that V goes to a positive constant v_∞. This can be seen directly from integrating the V-equation (2.6). Thus we have

$$U(\eta) \sim v_\infty \eta \quad \text{as} \quad \eta \to \infty \quad \text{if} \quad \beta > 2, \tag{2.34}$$

where v_∞ determines the solution completely.

If $\beta = 2$, the W-equation (2.8) for $W = V = U'$ reduces to

$$\frac{dV}{ds} = \frac{1}{V} - \frac{1}{c} e^{(\alpha-1)s}, \tag{2.35}$$

so that

$$U(\eta) \sim \eta\sqrt{2\log\eta} \quad \text{as} \quad \eta \to \infty \quad \text{if} \quad \beta = 2. \tag{2.36}$$

The W-equation (2.8) implies immediately that all solutions have

$$U(\eta)^{\frac{\beta}{2}} \sim \frac{\beta\eta}{\sqrt{2(2-\beta)}} \quad \text{as} \quad \eta \to \infty \quad \text{if} \quad 2\alpha < \beta < 2. \tag{2.37}$$

Finally we see from the Z-equation (2.7) that

$$U(\eta)^\alpha \sim c\alpha\eta \quad \text{as} \quad \eta \to \infty \quad \text{if} \quad \beta < 2\alpha. \tag{2.38}$$

2.2 Advancing fronts for $\alpha = 1$

For $\alpha = 1$ the V-equation (2.6) is separable and can be solved implicitly. We find

$$cU' + c^2 \log|U' - c| + \frac{1}{2-\beta} U^{2-\beta} = C \quad \text{if} \quad \beta \neq 2, \tag{2.39}$$

$$cU' + c^2 \log|U' - c| + \log U = C \quad \text{if} \quad \beta = 2, \tag{2.40}$$

where C is an arbitrary integration constant. In addition $U(\eta) = c\eta$ is always an explicit solution.

2.2.1 The generic fronts for $\alpha = 1$

As before we have

$$U_g(\eta) \sim v_0 \eta \quad \text{as} \quad \eta \downarrow 0 \quad \text{if} \quad \beta < 2, \tag{2.41}$$

where $v_0 \in (0, \infty)$ determines the solution,

$$U_g(\eta) \sim \left(\frac{\beta - 1}{\beta - 2} \frac{\eta}{c}\right)^{\frac{1}{\beta - 1}} \quad \text{as} \quad \eta \downarrow 0 \quad \text{if} \quad \beta > 2, \tag{2.42}$$

and

$$U_g(\eta) \sim -\frac{\eta}{c} \log \eta \quad \text{as} \quad \eta \downarrow 0 \quad \text{if} \quad \beta = 2. \tag{2.43}$$

2.2.2 The exceptional front for $\alpha = 1$

We have

$$U_e(\eta) = c\eta \quad \text{if} \quad \beta \geq 2, \tag{2.44}$$

and

$$U_e(\eta)^{\frac{\beta}{2}} \sim \frac{\beta \eta}{\sqrt{2(2-\beta)}} \quad \text{as} \quad \eta \downarrow 0 \quad \text{if} \quad \beta < 2. \tag{2.45}$$

In both cases the exceptional fronts are unique. Note that for $\beta \geq 2$ exceptional no longer means that the orbit comes out of $(U, V) = (0, 0)$. It is now exceptional in the sense that it is the only solution coming from the positive V-axis.

For $\beta = \alpha = 1$ the exceptional front is given by

$$U_e(\eta) = c\eta + c^2 (e^{-\frac{\eta}{c}} - 1) = \frac{\eta^2}{2} - \frac{\eta^3}{6c} + \frac{\eta^4}{24c^2} - \cdots. \tag{2.46}$$

2.2.3 Behaviour of the fronts for $\alpha = 1$ as $\eta \to \infty$

We have for large η-values that

$$U'(\eta) \to c \quad \text{if} \quad \beta \leq 2, \tag{2.47}$$

and

$$U'(\eta) \to v_\infty \in (0, \infty) \quad \text{if} \quad \beta > 2. \tag{2.48}$$

2.3 Advancing fronts for $\alpha > 1$

In this range, for which $\mathcal{V}^+$ and $\mathcal{V}^-$ are separated by a monotone decreasing curve asymptotic to the two axes, we encounter an unexpected phenomenon:

2.3.1 No advancing fronts for $\alpha > 1$ and $2 \leq \beta < 2\alpha$

We see from (2.21) that for $\beta < 2\alpha$ every orbit comes out of the set where $Z > 1/c$. In terms of (2.5) this means that every orbit comes out of $\mathcal{V}^+$. In particular V is bounded. But then we have, since $\alpha > 1$, that $dV/dU \sim U^{1-\beta}/V$ as $U \to 0$. For $\beta \geq 2$ this implies that every orbit comes from the positive U-axis. Thus we conclude that for $2 \leq \beta < 2\alpha$ every increasing solution of (2.4) begins at a positive minimum, so that no advancing fronts exist.

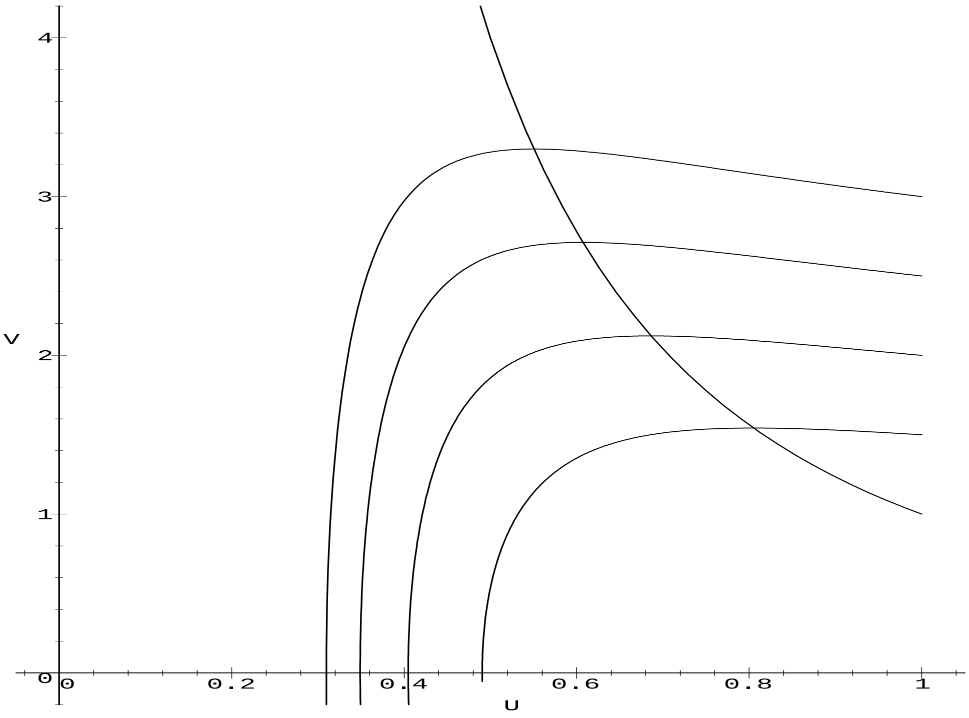

Figure 1. The U,V-phase plane for (2.5) with $\alpha = 3$, $\beta = 5$ and $c = 1$.

2.3.2 Critical wave speed for $\alpha > 1$ and $\beta = 2\alpha$

For $\beta = 2\alpha$ the Z-equation reduces to (2.31), which, since $\alpha > 1$, changes type when c crosses the value $\frac{1}{2\sqrt{\alpha-1}}$. Above this value the analysis is the same as for $2 \leq \beta < 2\alpha$. In terms of U we have

$$c > \frac{1}{2\sqrt{\alpha-1}} \quad \Rightarrow \quad \text{no advancing fronts exist.} \tag{2.49}$$

On the other hand we have

$$c < \frac{1}{2\sqrt{\alpha-1}} \quad \Rightarrow \quad U_g(\eta)^\alpha \sim \frac{\alpha\eta}{\frac{1}{2c} - \sqrt{\frac{1}{4c^2} + 1 - \alpha}} \quad \text{as} \quad \eta \downarrow 0$$

$$\text{and} \quad U_e(\eta)^\alpha = \frac{\alpha\eta}{\frac{1}{2c} + \sqrt{\frac{1}{4c^2} + 1 - \alpha}}. \tag{2.50}$$

Both right-hand sides are exact solutions of (2.4) and correspond, respectively, to the unstable and the stable positive equilibria of (2.31).

Finally we have

$$c = \frac{1}{2\sqrt{\alpha-1}} \quad \Rightarrow \quad U_g(\eta)^\alpha \sim \frac{\alpha\eta}{\sqrt{\alpha-1}} \quad \text{as} \quad \eta \downarrow 0 \quad \text{and} \quad U_e(\eta)^\alpha = \frac{\alpha\eta}{\sqrt{\alpha-1}}, \tag{2.51}$$

so that in the leading order term there is no difference between generic and exceptional solutions. This has consequences for the asymptotic analysis of the solution with Heaviside initial data, see Section 4.4.

2.3.3 Generic and exceptional fronts for $\alpha > 1$ and $\beta < 2$

Since $\beta < 2$ implies $\beta < 2\alpha$, we have again from (2.21) that every orbit comes out of $\mathcal{V}^+$, but now these orbits can also come from the positive V-axis (generic case) or from the origin (exceptional case). Thus we have

$$U_g(\eta) \sim v_0\eta \quad \text{as} \quad \eta \downarrow 0 \quad \text{if} \quad \beta < 2. \tag{2.52}$$

Using the W-equation (2.8) the exceptional front has

$$U_e(\eta)^{\frac{\beta}{2}} \sim \frac{\beta\eta}{\sqrt{2(2-\beta)}} \quad \text{as} \quad \eta \downarrow 0. \tag{2.53}$$

2.3.4 Generic and exceptional fronts for $\alpha > 1$ and $\beta > 2\alpha$

In terms of (2.12), the solution can now come out of the origin $(Z,P) = (0,0)$ (generic fronts), out of the saddle-node $(Z,P) = (1/c, 0)$ (exceptional front), or from $(Z,P) = (\infty, p)$ with $p > 0$ (when $U(\eta)$ has a positive minimum).

The analysis of the generic case is similar to that with $\alpha < 1$ and $\beta > \alpha + 1$ and leads to

$$U_g(\eta) \sim \left(\frac{\beta-\alpha}{\beta-\alpha-1}\frac{\eta}{c}\right)^{\frac{1}{\beta-\alpha}} \quad \text{as} \quad \eta \downarrow 0, \tag{2.54}$$

while the exceptional case is similar to that with $\alpha < 1$ and $\beta > 2\alpha$ and leads to

$$U_e(\eta)^\alpha \sim c\alpha\eta \quad \text{as} \quad \eta \downarrow 0. \tag{2.55}$$

All fronts are contained in the set $\mathcal{V}^-$.

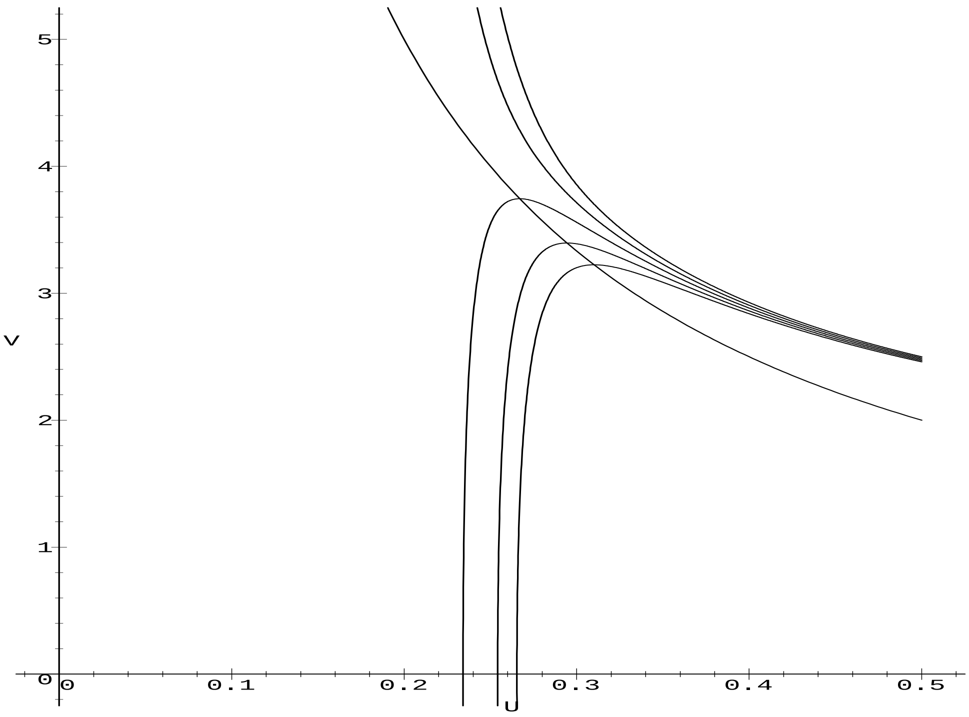

Figure 2. The U, V-phase plane for (2.5) with $\alpha = 2$, $\beta = 6$ and $c = 1$.

2.3.5 Behaviour for $\alpha > 1$ and large η

For $\alpha > 1$ the behaviour of solutions of (2.5) again depends on β but also, in some cases, on the initial data. Let us first observe that all solutions are (eventually) in $\mathcal{V}^-$. Thus they become concave with $U(\eta) \uparrow \infty$ as $\eta \uparrow \infty$.

From the V-equation (2.6) it is clear that, since $\alpha > 1$,

$$\beta > \alpha + 1 \quad \Rightarrow \quad \forall v_\infty \in (0, \infty)\ \exists!\ \text{solution with } U'(\eta) \to v_\infty \text{ as } \eta \to \infty, \tag{2.56}$$

while for $\beta \le \alpha + 1$ all solutions have the property that $V(\eta) = U'(\eta) \to 0$.

We first continue with the case that $\beta > \alpha + 1$. If we take $\beta = 2\alpha$ everything follows again from the reduced Z-equation (2.7). We obtain

$$\beta = 2\alpha \text{ and } c > \frac{1}{2\sqrt{\alpha - 1}} \quad \Rightarrow \quad U'(\eta) \to v_\infty \in (0, \infty) \text{ as } \eta \to \infty. \tag{2.57}$$

For subcritical values of c we obtain as $\eta \to \infty$ two generic cases and one exceptional case:

$$\beta = 2\alpha \text{ and } c < \frac{1}{2\sqrt{\alpha-1}} \quad \Rightarrow \quad U'(\eta) \to v_\infty \in (0,\infty) \text{ (generic)},$$
$$\text{or } U(\eta)^\alpha = \frac{\alpha\eta}{\frac{1}{2c} - \sqrt{\frac{1}{4c^2}+1-\alpha}} \text{ (exceptional)},$$
$$\text{or } U(\eta)^\alpha \sim \frac{\alpha\eta}{\frac{1}{2c} + \sqrt{\frac{1}{4c^2}+1-\alpha}} \text{ (generic)}. \tag{2.58}$$

For the critical wave speed we have as $\eta \to \infty$

$$\beta = 2\alpha \text{ and } c = \frac{1}{2\sqrt{\alpha-1}} \quad \Rightarrow \quad U'(\eta) \to v_\infty \in (0,\infty) \text{ (generic)},$$
$$U(\eta)^\alpha = \frac{\alpha\eta}{\sqrt{\alpha-1}} \text{ (exceptional)}, \quad \text{or } U(\eta)^\alpha \sim \frac{\alpha\eta}{\sqrt{\alpha-1}} \text{ (generic)}, \tag{2.59}$$

while for $\beta > 2\alpha$ we find the same picture as for $\beta = 2\alpha$ with c supercritical:

$$\beta > 2\alpha \quad \Rightarrow \quad U'(\eta) \to v_\infty \in (0,\infty) \text{ as } \eta \to \infty. \tag{2.60}$$

This is easily seen from the Z-equation (2.7) in which the linear term dominates.

Next we consider $\alpha + 1 < \beta < 2\alpha$. Here we find a similar picture as for $\beta = 2\alpha$ with c subcritical. We use (2.10), which we may also write as a system,

$$\frac{dA}{ds} = A(\frac{A}{c} - \beta + \alpha + 1) - PA^3; \quad \frac{dP}{ds} = (\beta - 2\alpha)P. \tag{2.61}$$

From (2.61), combined with (2.12), we obtain that as $s \to \infty$, positive solutions of (2.10) either go to zero (generic), to $c(\beta - \alpha - 1)$ (exceptional) or to infinity (generic, corresponding to $Z(s) \to \frac{1}{c}$). Respectively, in terms of U, as $\eta \to \infty$ we have

$$\alpha + 1 < \beta < 2\alpha \quad \Rightarrow \quad U'(\eta) \to v_\infty \in (0,\infty) \text{ (generic)},$$
$$\text{or } U(\eta)^{\beta-\alpha} \sim \frac{\beta-\alpha}{\beta-\alpha-1}\frac{\eta}{c} \text{ (exceptional)}, \quad \text{or } U(\eta)^\alpha \sim c\alpha\eta \text{ (generic)}. \tag{2.62}$$

Finally we consider $\beta \leq \alpha + 1$. It now follows that every solution has $A(s) \to \infty$ as $s \to \infty$. In terms of Z, $Z(s) \to \frac{1}{c}$. Thus we find that the behaviour for large η is independent of the initial conditions:

$$\beta \leq \alpha + 1 \quad \Rightarrow \quad U(\eta)^\alpha \sim c\alpha\eta \text{ as } \eta \to \infty. \tag{2.63}$$

2.4 Advancing fronts with a positive limit

In this section we consider solutions of the full travelling wave equation (2.2) with the property that

$$U(\eta) \to 1 \quad \text{as} \quad \eta \to -\infty. \tag{2.64}$$

These solutions will play an important role in Section 3. Integration of (2.2) now yields

$$U'' + \frac{1}{c}U^{\alpha-\beta}U' = U^{1-\beta} - U^{-\beta}, \tag{2.65}$$

or, written as a system,

$$U' = V; \quad V' = U^{-\beta}(U - 1 - \frac{U^{\alpha}V}{c}), \tag{2.66}$$

or

$$\frac{dV}{dU} = \frac{U-1}{VU^{\beta}} - \frac{U^{\alpha-\beta}}{c}. \tag{2.67}$$

Clearly we have that for positive c the solution $U = U_1$ of (2.2) with (2.64) is given by the unique orbit coming out of the saddle point $(U,V) = (1,0)$ into the first quadrant of the (U,V)-plane. This orbit corresponds to the positive eigenvalue

$$\lambda_+ = \lambda_+(c) = -\frac{1}{2c} + \sqrt{\frac{1}{4c^2} + 1}, \tag{2.68}$$

and satisfies

$$V \sim \lambda_+(c)(U-1) \quad \text{as} \quad U \to 1. \tag{2.69}$$

For fiven c, U_1 is therefore unique up to translations in η.

The leading order behaviour for large η of increasing solutions of (2.66) is exactly the same as for (2.5), because in all cases $U(\eta) \to \infty$ so that the extra term is of lower order. In all cases where there is only one large η behaviour, this evidently also applies to U_1. For those values of α and β with more then just one kind of generic large η behaviour, we can expect the existence of an exceptional speed $c = c^*$ for which $U_1(\eta)$ is exceptional.

2.4.1 Exceptional speeds for $\alpha > 1$ and $\alpha + 1 < \beta < 2\alpha$

We recall that for this range there were no advancing fronts. The exceptional solution we construct plays a key role in the asymptotic analysis given below.

The solution $U_1 = U_{1,c}$ corresponds to a solution $V(U) = V_{1,c}(U)$ of (2.67). It is clear that this function increases as c increases. To show that an exceptional speed exists, we first prove that for large and small c the large η behaviours are generic and are different. We then conclude that in between there is a value $c = c^*$ for which the large η behaviour is exceptional.

We first oberve that the Z-equation (2.7) and the A-equation (2.10) now read

$$\frac{dZ}{ds} + (\alpha - 1)Z = e^{(2\alpha-\beta)s}Z^2(\frac{1}{c} - (1 - e^{-s})Z), \tag{2.70}$$

and

$$\frac{dA}{ds} = A(\frac{A}{c} - \beta + \alpha + 1) - e^{(\beta-2\alpha)s}(1-e^{-s})A^3. \tag{2.71}$$

For $c \to \infty$ equation (2.67) reduces to the limit equation

$$V\frac{dV}{dU} = \frac{U-1}{U^\beta}, \tag{2.72}$$

so the limit solution is

$$\frac{1}{2}V_{1,\infty}(U)^2 = \frac{1}{(\beta-1)U^{\beta-1}} - \frac{1}{(\beta-2)U^{\beta-2}} + \frac{1}{(\beta-1)(\beta-2)}, \tag{2.73}$$

and in particular $V_{1,\infty}(\infty) > 0$. Using the smooth dependence on the parameter $1/c$ of the unstable manifold of $(U,V) = (1,0)$ of the system (2.66), it follows that $V_{1,c}(U) \to V_{1,\infty}(U)$ uniformly on bounded intervals of the form $\{1 \le U \le M\}$. Since the second term on the right-hand side of (2.67) is integrable with respect to U, the convergence is also uniform on $\{1 \le U \le \infty\}$. This yields for c sufficiently large a positive limit $v_{\infty,c}$, so that $V_{1,c}(U) \to v_{\infty,c} > 0$ as $U \to \infty$, and thus

$$U_{1,c}'(\eta) \to v_{\infty,c} > 0 \quad \text{as} \quad \eta \to \infty, \tag{2.74}$$

which is one the two generic behaviours.

Next we show that by making c small we obtain the other generic behaviour for the orbit coming out of $(U,V) = (1,0)$.

In view of the fact that the orbit is bounded by $V \le c(\alpha-1)^{\alpha-1}\alpha^{-\alpha}$ (which is the maximum V-value in $\{U > 0\}$ in the (U,V)-plane along the curve where $V' = 0$), we employ the scaling $V = c\phi$, $\eta = c\tau$, which changes the system (2.66) into

$$\dot{U} = c^2\phi; \quad \dot{\phi} = U^{-\beta}(U - 1 - U^\alpha\phi). \tag{2.75}$$

Thus we can consider the limit system with $c = 0$ (in the fast variable τ only), which has a one-dimensional manifold of critical points given by

$$M_0 \equiv \{\phi = \frac{U-1}{U^\alpha}\}, \tag{2.76}$$

which is easily seen to be normally hyperbolic in the sense of singular geometric perturbation theory, see for instance [15]. This implies that in a compact neighbourhood of $(U,\phi) = (1,0)$ there exists for every sufficiently small $c > 0$ a one-dimensional invariant manifold M_c of (2.75) which converges to M_0 as $c \to 0$. More precisely, M_c is of the form $M_c = \{\phi = \phi_c(U)\}$ and $\phi_c(U) \to (U-1)U^{-\alpha}$ as $c \to 0$. Clearly, M_c can only be the unstable manifold of $(U,\phi) = (1,0)$. Also there is no restriction on the right-hand side boundary of the U-interval on which ϕ_c is defined, so we may take it as large as we want. It is not hard to see that, in terms $Z(s)$, this means that we can keep $Z(s)$ near $1/c$ for as large an s-value as we like, implying that from there on the right hand side of (2.70) will force $Z(s)$ to converge to $1/c$. This corresponds to the other type of generic behaviour.

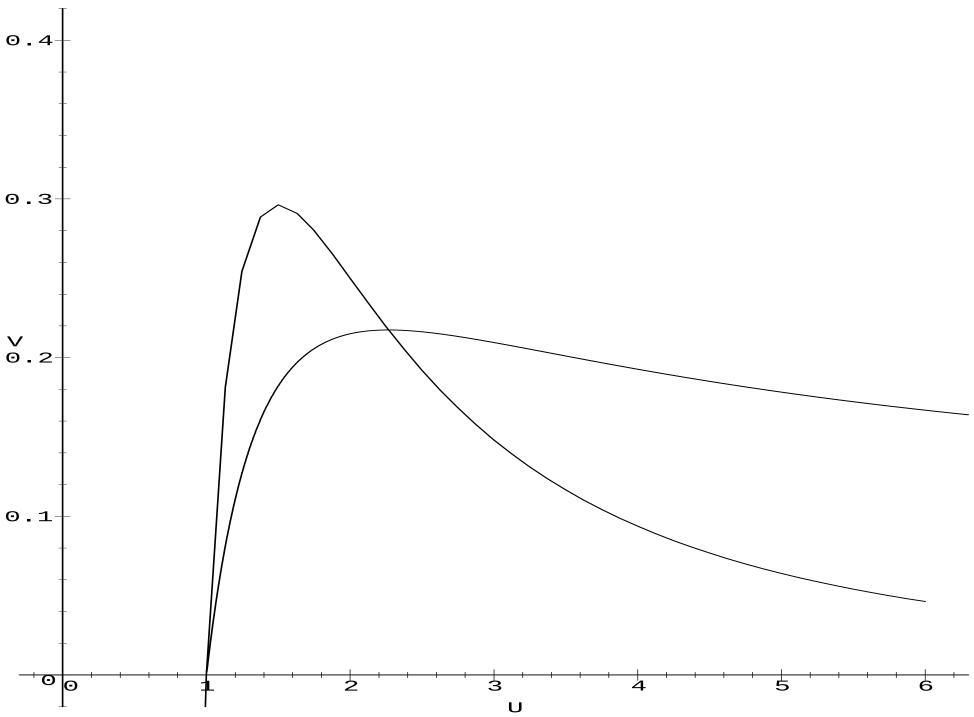

Figure 3. The isocline $V' = 0$ and the orbit coming out of $(1, 0)$ in the U, V-phase plane for (2.66) with $\alpha = 3$, $\beta = 5$ and $c = 2$ (i.e. c large).

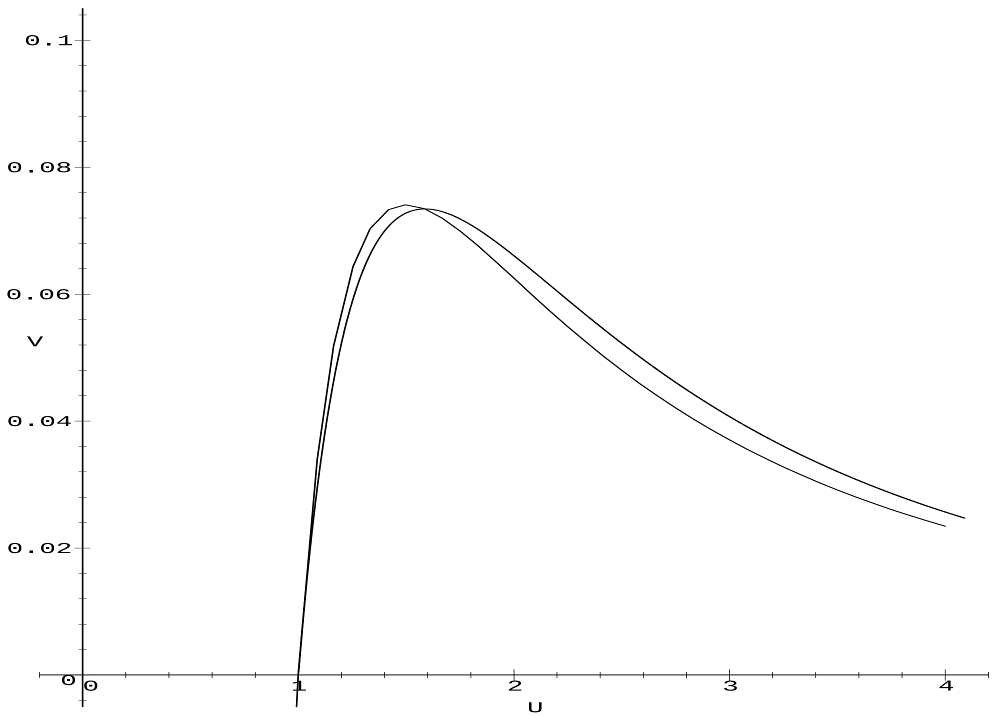

Figure 4. The isocline $V' = 0$ and the orbit coming out of $(1, 0)$ in the U, V-phase plane for (2.66) with $\alpha = 3$, $\beta = 5$ and $c = \frac{1}{2}$ (i.e. c small).

In view of the different generic behaviours for large and small c, it follows that there must exists a speed $c = c^*$ for which $U_{1,c}$ has exceptional behaviour for large η, meaning that

$$U_{1,c^*}(\eta)^{\beta-\alpha} \sim \frac{\beta - \alpha}{\beta - \alpha - 1} \frac{\eta}{c^*} \quad \text{as} \quad \eta \to \infty. \tag{2.77}$$

The monotonicity with respect to c in (2.67) implies that $V_{1,c}(U)$ is increasing in c and hence the corresponding $A_{1,c}(s)$ is decreasing in c. Since exceptional large η behaviour corresponds to the unique orbit with $A(s) \to c(\beta - \alpha - 1)$, the limit being increasing with respect to c, the exceptional speed c^* for which $A_{1,c^*}(s) \to c^*(\beta - \alpha - 1)$, is unique.

2.4.2 Exceptional speeds for $\alpha > 1$ and $\beta = 2\alpha$

Here the Z-equation reduces to

$$\frac{dZ}{ds} = -(\alpha - 1)Z + Z^2(\frac{1}{c} - (1 - e^{-s})Z) \geq -(\alpha - 1)Z + Z^2(\frac{1}{c} - Z), \quad (2.78)$$

where $Z(s) \to \infty$ as $s \downarrow 0$. The exceptional speed coincides with the critical speed:

$$c^* = \frac{1}{2\sqrt{\alpha - 1}}. \quad (2.79)$$

This is the only value of c for which $U_{1,c}$ selects the exceptional large η behaviour. Indeed, the third order term prevents $Z(s)$ from going back to infinity, and the inequality (2.78) keeps $Z(s)$ above the zero's of $-(\alpha - 1)Z + Z^2(\frac{1}{c} - Z)$. Consequently it follows for $c > c^*$ (when there is no exceptional behaviour) that $Z(s) \to 0$, while for $c < c^*$ we have $Z(s) \to \frac{1}{2c} + \sqrt{\frac{1}{4c^2} + 1 - \alpha}$, which is one of the generic behaviours. For $c = c^*$ it follows that $Z(s) \to \sqrt{\alpha - 1}$, but only with an algebraic convergence rate.

2.5 Advancing fronts, summary

In the table below we list the local behaviours near zero of solutions of (2.4), and the large η behaviours of increasing solutions of (2.4) and (2.65). In all cases where there are three large η behaviours, the second one listed is exceptional. For $\alpha + 1 < \beta \leq 2\alpha$ there is an exceptional wave speed c^* for which there is a unique solution of (2.65) with $U(\eta) \to 1$ as $\eta \to -\infty$ having exceptional large η behaviour. For $\beta = 2\alpha$ this exceptional speed coincides with the critical speed.

$\alpha < 1$	exceptional	generic	large η
$\beta < 2\alpha$	$U_e^{\frac{\beta}{2}} \sim \frac{\beta\eta}{\sqrt{2(2-\beta)}}$	$U_g \sim v_0\eta$	$U^\alpha \sim c\alpha\eta$
$\beta = 2\alpha$	$U_e^\alpha \sim \frac{\alpha\eta}{\frac{1}{2c}+\sqrt{\frac{1}{4c^2}+1-\alpha}}$	$U_g \sim v_0\eta$	$U^\alpha \sim \frac{\alpha\eta}{\frac{1}{2c}+\sqrt{\frac{1}{4c^2}+1-\alpha}}$
$2\alpha < \beta < 1+\alpha$	$U_e^\alpha \sim c\alpha\eta$	$U_g \sim v_0\eta$	$U^{\frac{\beta}{2}} \sim \frac{\beta\eta}{\sqrt{2(2-\beta)}}$
$\beta = 1+\alpha$	$U_e^\alpha \sim c\alpha\eta$	$U_g \sim \frac{\eta}{c}\log(\frac{1}{\eta})$	$U^{\frac{\beta}{2}} \sim \frac{\beta\eta}{\sqrt{2(2-\beta)}}$
$1+\alpha < \beta < 2$	$U_e^\alpha \sim c\alpha\eta$	$U_g^{\beta-\alpha} \sim \frac{\beta-\alpha}{\beta-\alpha-1}\frac{\eta}{c}$	$U^{\frac{\beta}{2}} \sim \frac{\beta\eta}{\sqrt{2(2-\beta)}}$
$\beta = 2$	$U_e^\alpha \sim c\alpha\eta$	$U_g^{\beta-\alpha} \sim \frac{\beta-\alpha}{\beta-\alpha-1}\frac{\eta}{c}$	$U \sim \eta\sqrt{2\log\eta}$
$\beta > 2$	$U_e^\alpha \sim c\alpha\eta$	$U_g^{\beta-\alpha} \sim \frac{\beta-\alpha}{\beta-\alpha-1}\frac{\eta}{c}$	$U \sim v_\infty\eta$
$\alpha = 1$	exceptional	generic	large η
$\beta < 2$	$U_e^{\frac{\beta}{2}} \sim \frac{\beta\eta}{\sqrt{2(2-\beta)}}$	$U_g \sim v_0\eta$	$U \sim c\eta$
$\beta = 2$	$U_e = c\eta$	$U_g \sim \frac{\eta}{c}\log(\frac{1}{\eta})$	$U \sim c\eta$
$\beta > 2$	$U_e = c\eta$	$U_g^{\beta-1} \sim \frac{\beta-1}{\beta-2}\frac{\eta}{c}$	$U \sim v_\infty\eta$
$\alpha > 1$	exceptional	generic	large η
$\beta < 2$	$U_e^{\frac{\beta}{2}} \sim \frac{\beta\eta}{\sqrt{2(2-\beta)}}$	$U_g \sim v_0\eta$	$U^\alpha \sim c\alpha\eta$
$2 \leq \beta \leq 1+\alpha$	none	none	
$1+\alpha < \beta < 2\alpha$	none	none	$U^\alpha \sim c\alpha\eta$ $U^{\beta-\alpha} \sim \frac{\beta-\alpha}{\beta-\alpha-1}\frac{\eta}{c}$ $U \sim v_\infty\eta$
$\beta = 2\alpha$ $c > \frac{1}{2\sqrt{\alpha-1}}$	none	none	$U \sim v_\infty\eta$
$\beta = 2\alpha$ $c = \frac{1}{2\sqrt{\alpha-1}}$	$U_e^\alpha = \frac{\alpha\eta}{\sqrt{\alpha-1}}$	$U_g^\alpha \sim \frac{\alpha\eta}{\sqrt{\alpha-1}}$	$U \sim v_\infty\eta$ $U^\alpha \sim \frac{\alpha\eta}{\sqrt{\alpha-1}}$ $U^\alpha \sim \frac{\alpha\eta}{\sqrt{\alpha-1}}$
$\beta = 2\alpha$ $c < \frac{1}{2\sqrt{\alpha-1}}$	$U_e^\alpha = \frac{\alpha\eta}{\frac{1}{2c}+\sqrt{\frac{1}{4c^2}+1-\alpha}}$	$U_g^\alpha \sim \frac{\alpha\eta}{\frac{1}{2c}-\sqrt{\frac{1}{4c^2}+1-\alpha}}$	$U \sim v_\infty\eta$ $U^\alpha \sim \frac{\alpha\eta}{\frac{1}{2c}-\sqrt{\frac{1}{4c^2}+1-\alpha}}$ $U^\alpha \sim \frac{\alpha\eta}{\frac{1}{2c}+\sqrt{\frac{1}{4c^2}+1-\alpha}}$
$\beta > 2\alpha$	$U_e^\alpha \sim c\alpha\eta$	$U_g^{\beta-\alpha} \sim \frac{\beta-\alpha}{\beta-\alpha-1}\frac{\eta}{c}$	$U \sim v_\infty\eta$

3 Asymptotic methods

The preceding analysis has been rigorous but was limited to a very special class of solutions. We now commence with the study of more general solutions of (1.9) by means of formal asymptotic methods. Nevertheless, this analysis will rely quite heavily on the travelling wave solutions discussed above.

3.1 Moving fronts

In order to determine the appropriate moving boundary conditions for (1.9), we consider the physically motivated regularisation whereby the initial data takes the form

$$u(x,0) = G(x) + \epsilon, \quad G(x) = 0 \text{ for } x > 0, \quad 0 < \epsilon << 1. \tag{3.1}$$

Note that we now typically will have a movement to the right, so that any travelling wave appearing here will be reversed with respect to the travelling waves in the previous analysis. We assume that the solution u_ϵ of (1.9) with these initial data is sufficiently smooth and examine its behaviour as $\epsilon \to 0$. Here we expect a two-layer structure: an outer region on the left where $u_\epsilon \sim u_0$ as $\epsilon \to 0$, with $u_0 > 0$ satisfying (1.9), and an inner region situated at the boundary of this outer region. We will search for a scaling of u and x which allows the leading order behaviour of the interior layer to match with the outer behaviour. Locating the inner region at

$$x = s(t;\epsilon) \sim s_0(t) \quad \text{as} \quad \epsilon \to 0, \tag{3.2}$$

we will thus have

$$u_0 = 0 \text{ for } x \geq s_0(t). \tag{3.3}$$

We introduce inner scalings, with $\gamma > 0$ to be chosen later,

$$u = \epsilon v, \quad x = s(t;\epsilon) + \epsilon^\gamma z, \quad t = \tau \tag{3.4}$$

and assume that

$$v \sim v_0 \quad \text{as} \quad \epsilon \to 0; \tag{3.5}$$

(v_0 in this section differs in meaning from that of Section 2), (1.9) then transforms into

$$\frac{\partial v}{\partial \tau} - \epsilon^{-\gamma}\dot{s}\frac{\partial v}{\partial z} = \epsilon^{\alpha-2\gamma}\frac{\partial}{\partial z}(v^\alpha \frac{\partial v}{\partial z}) + \epsilon^{\beta-2\gamma}\frac{\partial}{\partial z}(v^\beta \frac{\partial^2 v}{\partial z \partial \tau}) - \epsilon^{\beta-3\gamma}\dot{s}\frac{\partial}{\partial z}(v^\beta \frac{\partial^2 v}{\partial z^2}). \tag{3.6}$$

Since $\gamma > 0$ the terms containing τ-derivatives are negligible as $\epsilon \to 0$, so that

$$\dot{s}\frac{\partial v}{\partial z} + \epsilon^{\alpha-\gamma}\frac{\partial}{\partial z}(v^\alpha \frac{\partial v}{\partial z}) \sim \epsilon^{\beta-2\gamma}\dot{s}\frac{\partial}{\partial z}(v^\beta \frac{\partial^2 v}{\partial z^2}). \tag{3.7}$$

We must choose γ to give a limit equation for v_0 which is solvable under the condition that

$$v_0 \to 1 \quad \text{as} \quad z \to +\infty, \tag{3.8}$$

and which is such that the growth of the solution as $z \to -\infty$ matches the scaling (3.4), which requires that

$$v_0 = O((-z)^{\frac{1}{\gamma}}) \quad \text{as} \quad z \to -\infty. \tag{3.9}$$

Three cases must be discussed separately.

3.1.1 $\beta > 2\alpha$

Here we need $\gamma = \alpha$, so the first two terms in (3.7) balance and dominate the third. Hence we have

$$\dot{s}_0 \frac{\partial v_0}{\partial z} + \frac{\partial}{\partial z}(v_0^\alpha \frac{\partial v_0}{\partial z}) = 0, \tag{3.10}$$

so that

$$\dot{s}_0(v_0 - 1) + v_0^\alpha \frac{\partial v_0}{\partial z} = 0, \tag{3.11}$$

giving

$$v_0^\alpha \sim -\alpha \dot{s}_0 z \quad \text{as} \quad z \to -\infty, \tag{3.12}$$

as required to match. In terms of u_0 and x this leads to the porous medium equation balance

$$u_0(x,t)^\alpha \sim \alpha \dot{s}_0(t)(s_0(t) - x) \quad \text{as} \quad x \uparrow s_0(t). \tag{3.13}$$

The regularisation thus selects the exceptional front, leading to a uniquely specified solution; this will be true of the other cases also.

3.1.2 $\beta < 2\alpha$

Here we cannot take $\gamma = \alpha$ because then we would obtain in the limit for v_0 the equation $(v_0^\beta v_{0zz})_z = 0$, which with $v_0 \to 1$ as $z \to +\infty$ gives only $v_0 \equiv 1$. Thus we have to balance the terms in another way. The only possible choice turns out to be

$$\gamma = \frac{\beta}{2}, \tag{3.14}$$

and the limit equation becomes

$$\dot{s}_0 \frac{\partial v_0}{\partial z} = \dot{s}_0 \frac{\partial}{\partial z}(v_0^\beta \frac{\partial^2 v_0}{\partial z^2}). \tag{3.15}$$

Assuming $\dot{s}_0 \neq 0$ we obtain

$$\frac{\partial^2 v_0}{\partial z^2} = v_0^{1-\beta} - v_0^{-\beta}, \tag{3.16}$$

and for $\beta < 2$ we find

$$\frac{\partial v_0}{\partial z} = -\sqrt{\frac{2}{2-\beta} v_0^{2-\beta} - \frac{2}{1-\beta} v_0^{1-\beta} + \frac{2}{(1-\beta)(2-\beta)}}, \tag{3.17}$$

(in the special case $\beta = 1$ there are obvious changes) so that

$$v_0^\beta \sim \frac{\beta^2}{2(2-\beta)} z^2 \quad \text{as} \quad z \to -\infty, \tag{3.18}$$

which, in terms of u_0 and x leads to the moving boundary condition

$$u_0(x,t)^\beta \sim \frac{\beta^2}{2(2-\beta)} (s_0(t) - x)^2 \quad \text{as} \quad x \uparrow s_0(t). \tag{3.19}$$

This does not involve $\dot{s}_0(t)$ or even the sign of $\dot{s}_0(t)$.

For $\beta = 2$, however, we find

$$\frac{\partial v_0}{\partial z} = -\sqrt{2}\sqrt{\log(v_0) + \frac{1}{v_0} - 1}, \tag{3.20}$$

incompatible with the scaling, which requires linear growth in this case. For $\beta > 2$ we find that v_0 has linear growth, again incompatible with the scaling and therefore unmatchable. We thus deduce that there are no solutions with travelling fronts for $2 \leq \beta < 2\alpha$, which is consistent with our travelling wave analysis in Section 2.3.1.

3.1.3 $\beta = 2\alpha$

Now, with $\gamma = \alpha$, all three terms in (3.7) balance as $\epsilon \to 0$ and we obtain upon integration

$$\dot{s}_0(v_0 - 1) + v_0^\alpha \frac{\partial v_0}{\partial z} = \dot{s}_0 v_0^\beta \frac{\partial^2 v_0}{\partial z^2}, \quad \beta = 2\alpha. \tag{3.21}$$

This is the traveling wave equation (2.65) for fronts with a positive limit (with the change of notation $U \leftrightarrow v_0,\ c \leftrightarrow \dot{s}_0,\ \eta \leftrightarrow -z$). The analysis depends on α and was performed in Section 2.4.

For $0 < \alpha < 1$ we only have to transcribe (2.33) and obtain

$$v_0(z)^\alpha \sim \frac{-\alpha z}{\frac{1}{2\dot{s}_0} + \sqrt{\frac{1}{4\dot{s}_0^2} + 1 - \alpha}} \quad \text{as} \quad z \to -\infty, \tag{3.22}$$

or, in terms of u_0,

$$u_0(x,t)^\alpha \sim \frac{\alpha}{\frac{1}{2\dot{s}_0} + \sqrt{\frac{1}{4\dot{s}_0^2} + 1 - \alpha}} (s_0(t) - x) \quad \text{as} \quad x \uparrow s_0(t). \tag{3.23}$$

Thus the exponents are the same as in the case of porous medium equation balance (3.13), but the coefficient is different.

The case $\alpha > 1$ corresponds to Section 2.4.2. We find the same balance as in (3.23), but with the restriction that the speed of the front is subcritical:

$$\dot{s}_0 \leq \frac{1}{2\sqrt{\alpha - 1}}. \tag{3.24}$$

In case of equality the generic solutions in (2.51) are also allowed in the matching procedure, see also Section 4.4.

Finally $\alpha = 1$ leads again to the porous medium equation balance

$$u_0(x,t) \sim \dot{s}_0(t)(s_0(t) - x) \quad \text{as} \quad x \uparrow s_0(t). \tag{3.25}$$

3.2 Fixed fronts for $2 \leq \beta < 2\alpha$

Here we consider the remaining range of parameters $2 \leq \beta < 2\alpha$, for which the methods in Section 3.1 failed to give any moving fronts. It will turn out that the front is in fact a fixed boundary, at which the no-flux condition

$$u_0^\alpha \frac{\partial u_0}{\partial x} + u_0^\beta \frac{\partial^2 u_0}{\partial x \partial t} = 0 \quad \text{at} \quad x = 0 \tag{3.26}$$

has to be imposed on the outer solution $u_0(x,t)$, which satisfies (1.9) for $x < 0$ with initial data $u_0(x,0) = G(x)$. We assume here that this solution exists and that

$$U(t) = \lim_{x \to 0^-} u_0(x,t) > 0, \tag{3.27}$$

(in fact this relies on the local behaviour of $G(x)$ as $x \to 0^-$, but it is certainly true if $G(0) > 0$). We also define

$$W(t) = \dot{U}(t) + \frac{1}{\alpha - \beta + 1} U(t)^{\alpha - \beta + 1}, \tag{3.28}$$

Now let u_ϵ be the solution of (1.9) with initial data (3.1). We start by scaling the space variable and set $x = \delta X$, where $\delta = \delta(\epsilon) \to 0$ as $\epsilon \to 0$. Assuming that $u_\epsilon \sim U_0(X,t)$ as $\epsilon \to 0$, we obtain for U_0 that

$$\frac{\partial}{\partial X}\left(U_0^\alpha \frac{\partial U_0}{\partial X} + U_0^\beta \frac{\partial^2 U_0}{\partial X \partial t}\right) = 0. \tag{3.29}$$

Integrating once we find that

$$U_0^\alpha \frac{\partial U_0}{\partial X} + U_0^\beta \frac{\partial^2 U_0}{\partial X \partial t} = 0, \tag{3.30}$$

(the constant of integration must be zero as a consequence of the rescaling) and, integrating and matching it follows that

$$\frac{\partial U_0}{\partial t} + \frac{1}{\alpha - \beta + 1} U_0^{\alpha - \beta + 1} = W(t), \tag{3.31}$$

again with the obvious change if $\beta = \alpha + 1$.

We now let $X = s(t,\epsilon)$, with $s(t,\epsilon) \sim s_0(t)$ as $\epsilon \to 0$, denote the position of the edge of the support of U_0 so that (3.31) is subject to the initial condition $U_0 = 0$ at $X = s_0(t)$. To the left of this front we assume the leading order behaviour of u_ϵ to be given by (3.31) whereas to the right we assume that $u_\epsilon = o(1)$ as $\epsilon \to 0$.

3.2.1 $2 < \beta < \alpha + 1$

When $\beta < \alpha + 1$, the second term on the left-hand side of (3.31) is small near $X = s_0(t)$, so that

$$\dot{s}_0 \frac{\partial U_0}{\partial X} \sim -W(t), \quad \text{as} \quad X \to s_0(t)^-, \tag{3.32}$$

which calls for a linear scaling of u and X near $X = s(t, \epsilon)$, i.e.

$$u = \epsilon v, \quad X = s(t, \epsilon) + \epsilon z, \tag{3.33}$$

with $v \sim v_0$ as $\epsilon \to 0$. The third term in (1.9) then dominates the second but balances the first if we choose

$$\delta = \epsilon^{\frac{\beta-2}{2}} \quad \text{to give} \quad \frac{\partial v_0}{\partial z} = \frac{\partial}{\partial z}(v_0^\beta \frac{\partial^2 v_0}{\partial z^2}). \tag{3.34}$$

Imposing $v_0 \to 1$ as $z \to -\infty$ then yields

$$\frac{\partial v_0}{\partial z} = -(\frac{2}{(\beta-2)(\beta-1)} - \frac{1}{\beta-2} v_0^{2-\beta} + \frac{1}{\beta-1} v_0^{1-\beta})^{\frac{1}{2}}, \tag{3.35}$$

and in particular

$$v_0 \sim -\sqrt{\frac{2}{(\beta-2)(\beta-1)}} z \quad \text{as} \quad z \to -\infty. \tag{3.36}$$

We have from (3.32) that

$$U_0 \sim \frac{W(t)}{\dot{s}_0(t)}(s_0(t) - X) \quad \text{as} \quad X \to s_0(t)^-, \tag{3.37}$$

so matching with (3.36) implies that

$$s_0(t) = \sqrt{\frac{(\beta-2)(\beta-1)}{2}} \int_0^t W(\tau) d\tau. \tag{3.38}$$

In the original (x, t) coordinates the 'front' is given by

$$x \sim \epsilon^{\frac{\beta-2}{2}} s_0(t), \tag{3.39}$$

and in the limit $\epsilon \to 0$ we thus obtain a fixed front.

3.2.2 $\beta = 2$

The main difference from the previous section is that the scalings read

$$x = \delta X = \delta s(t, \epsilon) + \epsilon z, \quad \delta = (\log(1/\epsilon))^{-\frac{1}{2}}. \tag{3.40}$$

The formulae for v_0 and s_0 change accordingly. We have

$$\frac{\partial v_0}{\partial z} = -\sqrt{2}\big(\log v_0 + \frac{1}{v_0} - 1\big)^{\frac{1}{2}}, \quad v_0 \sim -\sqrt{2}z(\log(-z))^{\frac{1}{2}} \text{ as } z \to -\infty, \tag{3.41}$$

so that, again matching with (3.32), the above choice of δ follows. The 'front' is given by

$$x \sim (\log(1/\epsilon))^{-\frac{1}{2}} s_0(t) = (\log(1/\epsilon))^{-\frac{1}{2}} \frac{1}{\sqrt{2}} \int_0^t W(\tau)d\tau. \tag{3.42}$$

3.2.3 $\alpha + 1 < \beta < 2\alpha$

When $\beta > \alpha + 1$, the second term on the left-hand side of (3.31) is large near $X = s_0(t)$, so that

$$\dot{s}_0 \frac{\partial U_0}{\partial X} \sim \frac{1}{\alpha - \beta + 1} U_0^{\alpha-\beta+1} \text{ as } X \to s_0(t)^-, \tag{3.43}$$

which calls for the rescaling

$$u = \epsilon v, \quad X = s(t, \epsilon) + \epsilon^{\beta-\alpha} z. \tag{3.44}$$

To obtain a balance in (1.9) we must now choose

$$\delta = \epsilon^{\alpha - \frac{\beta}{2}}, \tag{3.45}$$

to give the full travelling wave balance

$$\frac{\partial v_0}{\partial z} = -\frac{1}{\dot{s}_0} \frac{\partial}{\partial z}(v_0^\alpha \frac{\partial v_0}{\partial z}) + \frac{\partial}{\partial z}(v_0^\beta \frac{\partial^2 v_0}{\partial z^2}). \tag{3.46}$$

Thus

$$v_0^\beta \frac{\partial^2 v_0}{\partial z^2} - \frac{1}{\dot{s}_0} v_0^\alpha \frac{\partial v_0}{\partial z} - v_0 + 1 = 0, \tag{3.47}$$

where we require a solution which has a behaviour for $z \to -\infty$ consistent with the scaling (3.44). This is (apart from a change of notation $v_0 \leftrightarrow U$, $c \leftrightarrow \dot{s}_0$, $\eta \leftrightarrow -z$) the equation (2.65), for which the analysis in Section 2.4.1 gave the existence of a unique exceptional speed $\dot{s}_0(t) = c^*$ for which (in terms of v_0) we have

$$v_0^{\beta-\alpha} \sim -\frac{1}{c^*} \frac{\beta - \alpha}{\beta - \alpha - 1} z. \tag{3.48}$$

In terms of U_0 it follows from (3.43) that

$$U_0^{\beta-\alpha} \sim \frac{1}{c^*} \frac{\beta - \alpha}{\beta - \alpha - 1} (s_0(t) - X) \text{ as } X \to s_0(t)^-, \tag{3.49}$$

and matching is thus accomplished if

$$s_0(t) = c^* t. \tag{3.50}$$

In the original (x, t) coordinates the 'front' is thus given by

$$x \sim \epsilon^{\alpha - \frac{\beta}{2}} c^* t. \tag{3.51}$$

3.2.4 $\beta = \alpha + 1$

Here equation (3.31) changes into

$$\frac{\partial U_0}{\partial t} + \log U_0 = W(t), \tag{3.52}$$

so that (3.43) is replaced by

$$\dot{s}_0 \frac{\partial U_0}{\partial X} \sim \log U_0 \quad \text{as} \quad X \to s_0(t)^-, \quad \text{giving} \quad U_0 \sim \frac{1}{\dot{s}_0}(s_0 - X)\log(1/(s_0 - X)). \tag{3.53}$$

We employ almost the same scaling as in the case $2 < \beta < \alpha + 1$,

$$u = \epsilon v, \quad x = \delta s(t, \epsilon) + \epsilon^{\frac{\beta}{2}} z, \tag{3.54}$$

which again leads to and (3.35) and (3.36). Matching with (3.52) now gives that

$$\delta = \epsilon^{\frac{\beta-2}{2}} \log(1/\epsilon), \quad \dot{s}_0(t) = \sqrt{\frac{(\beta-1)(\beta-2)}{2}}, \tag{3.55}$$

so that the 'front' is given by

$$x \sim \epsilon^{\frac{\beta-2}{2}} \log(1/\epsilon) s_0(t) = \epsilon^{\frac{\beta-2}{2}} \log(1/\epsilon) \sqrt{\frac{(\beta-1)(\beta-2)}{2}} t. \tag{3.56}$$

4 Heaviside initial data

We now consider initial data of the form

$$u(x, 0) = H(-x), \tag{4.1}$$

where H is the Heaviside function and we assume that the solution is that obtained as a formal limit of solutions u_ϵ with initial data $u_\epsilon(x, 0) = H(-x) + \epsilon$. The analysis in Section 3.2 has the surprising corollary that in the range $2 \leq \beta < 2\alpha$ this limit solution does not move at all, in other words, $u(x, t) = H(-x)$.

For the remaining cases we shall look for a self-similar structure of the limit solution with Heaviside initial data as $t \to 0^+$ which has a front behaviour consistent with the analysis in Section 3.1. At this point we note that equation (1.9) allows self-similar solutions of the form

$$u(x, t) = t^{\frac{1}{\beta-\alpha}} f(\eta), \quad \eta = x t^{-\frac{\beta}{2(\beta-\alpha)}} \quad \text{for} \quad \beta \neq \alpha, \tag{4.2}$$

and

$$u(x, t) = \exp(\gamma t) f(\eta), \quad \eta = x \exp(-\alpha\gamma t/2) \quad \text{for} \quad \beta = \alpha. \tag{4.3}$$

Such self-similar solutions of the full equation will play a role in the asymptotic analysis of small time behaviour in the range $\beta > \max(\alpha + 1, 2\alpha)$. To determine the local self-similarity of the solution with Heaviside initial data, we first examine the behaviour of $u(0^+, t)$ for $0 < t << 1$. We use the decoupling of (1.9), see (1.10, 1.11) with $w = \frac{\partial u}{\partial t} + \log u$ if $\beta = \alpha + 1$. Assuming the continuity of w we have

$$\frac{\partial}{\partial t}[u]_{0^-}^{0^+} + \frac{1}{\alpha - \beta + 1}[u^{\alpha-\beta+1}]_{0^-}^{0^+} = 0 \quad \text{if} \quad \beta \neq \alpha + 1, \tag{4.4}$$

and

$$\frac{\partial}{\partial t}[u]_{0^-}^{0^+} + [\log u]_{0^-}^{0^+} = 0 \quad \text{if} \quad \beta = \alpha + 1. \tag{4.5}$$

We hence obtain that $[u]_{0^-}^{0^+} = -\exp(-t)$ if $\beta = \alpha$.

At this stage we claim that for general α and β, the rate of change of $u(0^+, t)$ is much larger than that of $u(0^-, t)$ because $\beta > 0$. Consequently,

$$u(0^+, t) \sim \frac{1}{\alpha - \beta + 1} t \quad \text{as} \quad t \to 0 \quad \text{for} \quad \beta < \alpha + 1; \tag{4.6}$$

$$u(0^+, t) \sim t \log(1/t) \quad \text{as} \quad t \to 0 \quad \text{for} \quad \beta = \alpha + 1; \tag{4.7}$$

$$u(0^+, t) \sim \left(\frac{\beta - \alpha}{\beta - \alpha - 1} t\right)^{\frac{1}{\beta - \alpha}} \quad \text{as} \quad t \to 0 \quad \text{for} \quad \beta > \alpha + 1. \tag{4.8}$$

In the latter two cases our claim above follows directly from (4.4) and (4.5). In the first it can readily be verified a posteriori; see Section 4.1 below. We now discuss these cases separately.

4.1 $\beta < \min(2, \alpha + 1)$

We look for a self-similar form consistent with (4.6). The only possibility is

$$u \sim t f(\eta), \quad \eta = x t^{-\beta/2}, \quad s \sim \eta_0 t^{\beta/2}, \tag{4.9}$$

which satisfies the balance

$$\frac{\partial u}{\partial t} \sim \frac{\partial}{\partial x}\left(u^\beta \frac{\partial^2 u}{\partial x \partial t}\right). \tag{4.10}$$

In terms of f this leads to

$$f(\eta)^\beta = \frac{\beta^2}{2(2-\beta)}(\eta_0 - \eta)^2, \quad \eta_0 = \frac{(2(2-\beta))^{\frac{1}{2}}}{(\alpha - \beta + 1)^{\beta/2}\beta}. \tag{4.11}$$

For $\beta > 2\alpha$ however this is not consistent with the local behaviour found in Section 3.1 and an additional region is then needed with with scalings

$$x = s(t) + t^{\frac{\beta(2-\beta)}{2(\beta - 2\alpha)}}\zeta, \quad u = t^{\frac{2-\beta}{\beta - 2\alpha}}\phi, \quad (2\alpha < \beta < \alpha + 1), \tag{4.12}$$

with leading order balance again being of travelling wave type:

$$-\frac{\beta}{2}\eta_0\phi_0 = \phi_0^\alpha \frac{\partial \phi_0}{\partial \zeta} - \frac{\beta}{2}\eta_0\phi_0^\beta \frac{\partial^2 \phi_0}{\partial \zeta^2}. \tag{4.13}$$

It then follows from Section 2.1.3 that the exceptional travelling wave solution matches the outer and the inner behaviours.

Notice that the accumulative amount of mass that flows through $x = 0$ from left to right is of order $t^{1+\beta/2}$, so that in particular $u(0^-, t) = 1 + o(t)$ as $t \to 0$. This justifies our claim that $u(0^-, t)$ changes much more slowly than $u(0^+, t)$.

4.2 $\beta = \alpha + 1$, $\alpha < 1$

For $\alpha < 1$ the differences from the analysis in Section 4.1 are that

$$u \sim t \log(1/t) f(\eta), \quad \eta = x(t\log(1/t))^{-\beta/2}, \quad s \sim \eta_0 (t\log(1/t))^{\beta/2}, \tag{4.14}$$

with

$$\eta_0 = (2(2-\beta))^{1/2}/\beta,$$

and

$$x = s(t) + t^{\frac{\beta}{2}} (\log(1/t))^{-\frac{\beta^2}{2(2-\beta)}} \zeta, \quad u = t(\log(1/t))^{-\frac{\beta}{2-\beta}} \phi \tag{4.15}$$

now give the scalings in the travelling wave region.

4.3 $\beta > \max(2\alpha, \alpha + 1)$

The local behaviour (4.8) leads to (4.2), with boundary conditions

$$f(0) = (\frac{\beta - \alpha}{\beta - \alpha - 1})^{\frac{1}{\beta - \alpha}}, \quad f(\eta)^\alpha \sim \frac{\alpha\beta}{2(\beta-\alpha)} \eta_0 (\eta_0 - \eta) \text{ as } \eta \to \eta_0^-, \tag{4.16}$$

where η_0 is one of the unknowns. We conjecture that there is a unique solution $f(\eta)$.

4.4 $\beta = 2\alpha$, $\alpha > 1$

As in Section 4.3 we are led to (4.2), i.e. to

$$u(x,t) \sim t^{\frac{1}{\alpha}} f(\eta), \quad \eta = x/t, \tag{4.17}$$

which has to satisfy

$$f(0) = (\frac{\alpha}{\alpha-1})^{\frac{1}{\alpha}} \text{ and } f(\eta)^\alpha \sim \frac{\alpha}{2(\alpha-1)\eta_0}(1 - \sqrt{1 - 4(\alpha-1)\eta_0^2})(\eta_0 - \eta) \tag{4.18}$$

as $\eta \to \eta_0^-$. Condition (3.24) implies that $2\sqrt{\alpha-1}\eta_0 \le 1$. Since the right hand side above is in fact a solution of the corresponding full similarity equation, but satisfies the wrong boundary condition at $\eta = 0$, it follows that

$$\eta_0 = 1/(2\sqrt{\alpha - 1}). \tag{4.19}$$

Consequently the local front behaviour is non-generic in this case, but see however the final remarks of Section 2.3.2. Here too we conjecture the existence of a unique solution $f(\eta)$.

4.5 $\alpha = 1$, $\beta = 2$

Here we find

$$u(x,t) \sim t \log(1/t) f(\eta), \quad \eta = x/(t\sqrt{\log(1/t)}), \tag{4.20}$$

leading to the third order term being dominating in (1.9), giving $f(\eta) = 1 - \eta$ and $\eta_0 = 1$.

5 Discussion

Our analysis has shown that basically there are three possible fronts for (1.9). One is of porous medium type ($\beta > 2\alpha$), and another (new) one has local behaviour in which the second order term in (1.9) is irrelevant ($\beta < \min(2, 2\alpha)$). One of the key results of the foregoing analysis is that there is a third range of (α, β) for which (1.9) exhibits fixed fronts and u develops a discontinuity. This represents a new phenomenon in the current context and is consistent with certain experimental observations which formed part of the motivation for this work [14]. Here we conclude with a short discussion of two issues related to this aspect.

5.1 Other models

An alternative model closely related to (1.12) is (cf. [3], [4] [5])

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(u^\alpha \frac{\partial u}{\partial x} + \frac{1}{\beta + 1}\frac{\partial^2}{\partial x \partial t}u^{\beta+1}). \tag{5.1}$$

We briefly discuss the more general case

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(u^\alpha \frac{\partial u}{\partial x} + u^\beta \frac{\partial^2 u}{\partial x \partial t} + \gamma u^{\beta-1}\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}), \tag{5.2}$$

with $\alpha > 0$, $\beta > 0$, which has as special cases both (1.9) ($\gamma = 0$) and (5.1) ($\gamma = \beta$). For $\beta > 2\alpha$ solutions to (5.2) have travelling wave fronts of porous medium type and there is little qualitative difference between solutions for different γ. For $\beta < 2\alpha$, we investigate the possibility of fixed fronts by first seeking a local balance of the form

$$u \sim A(s(t) - x)^{\frac{2}{\beta}} \quad \text{as} \quad x \to s(t)^-, \tag{5.3}$$

from which we find that $A^\beta = \beta^2/(2(2\gamma + 2 - \beta))$. For a moving front we require that $\beta < 2\gamma + 2$, while for

$$2\gamma + 2 < \beta < 2\alpha, \tag{5.4}$$

we conclude that fixed fronts occur. For $\gamma = 0$, (5.4) implies $2 < \beta < 2\alpha$ (we omit discussion of borderline cases here) whereas (5.4) cannot be satisfied for $\gamma = \beta$. We thus in particular conclude that (5.1) cannot exhibit fronts which are fixed for all time and this is a very important qualitative difference between (1.9) and (5.1).

5.2 The porous medium equation limit

Here we consider

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(u^\alpha \frac{\partial u}{\partial x} + \mu\delta u^\beta \frac{\partial^2 u}{\partial x \partial t}), \tag{5.5}$$

where $\delta << 1$ will be chosen shortly and $\mu = O(1)$, subject to (3.1) and with $2 < \beta < 2\alpha$. If we take the limit $\delta \to 0$ followed by $\epsilon \to 0$ we obtain, as the leading order outer problem for u_0, the porous medium equation (1.6) with the usual condition (3.13) at the moving front. However, it follows from the analysis in Section 3.2 that if we take the limits in the opposite order ($\epsilon \to 0$ followed by $\delta \to 0$) we obtain (1.6) subject not to (3.13) but to

$$\frac{\partial u_0}{\partial x} = 0 \quad \text{at} \quad x = 0, \tag{5.6}$$

taking $x = 0$ to be the initial location of the right-hand front. Thus (1.6) with (3.13) is structurally unstable to perturbation by the δ term and here we briefly discuss (omitting almost all of the details) the transition between (3.13) and (5.6).

5.2.1 $2 < \beta < \alpha + 1$

Here the appropriate choice to describe the transition is $\delta = \epsilon^{\beta-2}$. There are inner regions with scalings $x - s(t;\epsilon) = O(\epsilon^{\beta-2})$, $u = O(1)$ and $x - s(t;\epsilon) = O(\epsilon^{\beta-1})$, $u = O(\epsilon)$. Matching to these (cf. Section 3.2.1) shows that (1.6) is to be solved subject to

$$u_0^{\alpha-\beta+1} = \frac{\sqrt{2\mu}(\alpha-\beta+1)}{\sqrt{(\beta-1)(\beta-2)}}\dot{s}_0, \quad u_0^{\alpha-1}\frac{\partial u_0}{\partial x} = -\dot{s}_0 \quad \text{at} \quad x = s_0. \tag{5.7}$$

Hence in the limit $\mu \to 0$ we recover (3.13) while for $\mu \to \infty$ we obtain $\dot{s}_0 \to 0$ and thus (5.6).

5.2.2 $\alpha + 1 < \beta < 2\alpha$

Here we must take $\delta = \epsilon^{2\alpha-\beta}$ and there are two cases to consider. With two inner layers with scalings $x - s(t;\epsilon) = O(\epsilon^{2\alpha-\beta})$, $u = O(1)$ and $x - s(t;\epsilon) = O(\epsilon^{\alpha})$, $u = O(\epsilon)$, we obtain (in a somewhat similar fashion to Section 3.2.3) the matching condition

$$u_0^{\alpha-1}\frac{\partial u_0}{\partial x} = -\dot{s}_0 = -c^*/\sqrt{\mu} \quad \text{at} \quad x = c^* t/\sqrt{\mu}, \tag{5.8}$$

while with a single inner layer of scaling $x - s(t;\epsilon) = O(\epsilon^{\alpha})$, $u = O(\epsilon)$, we arrive at (3.13). It follows from Section 2.4.1 that successful matching into the travelling wave layer in the latter case requires that $\dot{s}_0 < c^*/\sqrt{\mu}$, giving the required generic behaviour. We conclude that (1.6) is now to be solved subject to

$$u_0^{\alpha-1}\frac{\partial u_0}{\partial x} = -\dot{s}_0, \quad \dot{s}_0 = c^*/\sqrt{\mu}, \quad u_0 > 0 \quad \text{at} \quad x = s_0 \quad \text{if} \quad t < t_c, \tag{5.9}$$

$$u_0 \to 0, \quad u_0^{\alpha-1}\frac{\partial u_0}{\partial x} \to -\dot{s}_0, \quad \dot{s}_0 < c^*/\sqrt{\mu} \quad \text{as} \quad x \to s_0^- \quad \text{if} \quad t > t_c. \tag{5.10}$$

Thus at some finite time t_c, $u_0(s_0(t), t)$ drops to zero and the standard porous medium balance (3.13) is then recovered (in contrast to (5.7) where it is obtained only as $t \to \infty$). For $t < t_c$, the inner region travelling front is exceptional, and hence determines $\dot{s}_0$ (but $u_0(s_0(t), t)$ is then free) whereas for $t > t_c$ it is generic, $\dot{s}_0$ being determined instead by the outer problem with $u_0(s_0(t), t) = 0$. The behaviour at the front may, however, switch back; this will depend on the initial data. We note that (5.9)-(5.10) represent novel moving boundary conditions for the porous medium equation.

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