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Travelling wave solutions of a fourth order semi-linear diffusion equation

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1. Introduction. In this paper we are interested in travelling wave solutions $u = u(x,t)$ of the fourth order equation

$$\frac{\partial u}{\partial t} = -\gamma \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + f(u), \quad f(u) = (u-a)(1-u^2), \tag{1.1}$$

where $-1 < a \leq 0$ and $\gamma > 0$, connecting the two stable states $u = \pm 1$ of the ordinary differential equation $u' = (u-a)(1-u^2)$. This equation has many applications in e.g. population genetics and pattern formation, for references see [PT1].

When $\gamma = 0$ a travelling wave solution is given by

$$u(x,t) = \tanh\Big(\frac{x - a\sqrt{2}t}{\sqrt{2}}\Big), \tag{1.2}$$

with wave speed $c_0 = a\sqrt{2}$, which is negative if $a < 0$. The wave profile is independent of a and for $a = 0$ this travelling wave solution is a stationary solution of (1.1) with $\gamma = 0$.

Equation (1.1) with $\gamma = 0$ with a slightly more general non-linearity has been studied extensively in [AW1], [AW2], [FM] amongst many others. It has been shown that for this type of non-linearity there exists a unique (except for symmetry and translation) travelling wave solution, i.e. a solution of the form $u(x,t) = u(\xi)$ where $\xi = x - ct$ for some c, connecting the stable states $u = \pm 1$.

If $\gamma > 0$ and $a = 0$ equation (1.1) is called the Extended Fisher-Kolmogorov equation (EFK), which is a fourth order extension of the classical Fisher-Kolmogorov equation (FK). In a series of papers [PT1-4], [PTV] existence results on stationary solutions and their properties have been proved. In these papers one studies, in view of the symmetry of the non-linearity, odd solutions and hence the conditions $u(0) = 0$, $u''(0) = 0$ and $u(\infty) = 1$ are imposed. One distinguishes two different cases $\gamma \leq \frac{1}{8}$ and $\gamma > \frac{1}{8}$ where the behaviour of solutions is different.

In both cases an energy identity can be used to reduce the order of the equation. If $\gamma \leq \frac{1}{8}$ the order can be reduced further by assuming monotonicity of the solutions and the remaining problem is of second order. In [PT1] this is used to show that there is a unique solution for $\gamma \leq \frac{1}{8}$. If $\gamma > \frac{1}{8}$ monotonicity is lost. This can be seen by linearising the equation at $u = \pm 1$. The eigenvalues are now complex so that any solution converging to $u = \pm 1$ must be oscillatory. In [PT4] a shooting method is used to prove the existence of families of different kinks. In [PTV] the variational structure of the stationary equation is used to prove existence of odd equilibrium solutions connecting $u = \pm 1$.

In this paper we look for travelling wave solutions of equation (1.1). The resulting travelling wave equation neither has a conserved energy nor a variational structure and also the symmetry is lost. Thus the methods of [PT1] and [PTV] cannot be applied directly here. For small γ however, equation (1.1) can be seen as a perturbation of (1.1) with $\gamma = 0$ and it is this view that is taken in this paper. With the methods of geometric singular perturbation theory as developed in [F] and [J] we prove the following

Theorem 1. *For $\gamma > 0$ sufficiently small there exists a $c = c(\gamma)$ for which there is a travelling wave solution of (1.1) connecting the steady states $u = \pm 1$. The rate of change of the wave speed with respect to γ is given by*

$$\left.\frac{dc}{d\gamma}\right|_{\gamma=0} = -\tfrac{1}{5}\sqrt{2}a(2a^2-3).$$

The paper is divided as follows. In section 2 we describe how geometric perturbation theory is used to construct a locally invariant manifold M_γ for the travelling wave equation when γ is small and positive. In section 3 we use this manifold to obtain a travelling wave solution. In the last section we compute the rate at which the wave speed changes when the fourth order term is added.

2. Geometric Singular Perturbation Theory. Our approach in this section is similar to that in [GJ] where existence of a travelling wave solution is proved for a sixth order equation, but our calculations are more explicit. After substituting $u = u(\xi)$, where $\xi = x - ct$, and setting $\gamma = \varepsilon^2$ where $\varepsilon > 0$ in (1.1) we obtain the following boundary value problem:

$$(P_\varepsilon)\begin{cases} -\varepsilon^2 u'''' + u'' + cu' + (u-a)(1-u^2) = 0 \quad \text{on} \quad \mathbf{R}, \\ \lim\limits_{\xi\to-\infty} u(\xi) = -1, \quad \lim\limits_{\xi\to\infty} u(\xi) = 1, \end{cases}$$

where primes mean differentiation with respect to ξ.

We can write the differential equation in problem (P_ε) as a first order system

$$(S_\varepsilon)\begin{cases} u' = v, \quad v' = w, \quad \varepsilon w' = z, \\ \varepsilon z' = w + cv + (u-a)(1-u^2), \end{cases}$$

and setting $\xi = \varepsilon\eta$ we obtain

$$(F_\varepsilon)\begin{cases} \dot{u} = \varepsilon v, \quad \dot{v} = \varepsilon w, \quad \dot{w} = z, \\ \dot{z} = w + cv + (u-a)(1-u^2), \end{cases}$$

where dots denote differentiation with respect to η. Note that (S_ε) is singular at $\varepsilon = 0$ because (S_0) is not a well-defined dynamical system in $\mathbf{R^4}$. Having set $\xi = \varepsilon\eta$ we overcome this problem. The time scale given by ξ is said to be slow whereas that for η is fast, hence the corresponding systems are called the slow system (S_ε) and the fast system (F_ε). The latter is well-defined for all ε including $\varepsilon = 0$. For $\varepsilon \neq 0$, (F_ε) and (S_ε) are equivalent and the critical points are $(-1,0,0,0)$, $(a,0,0,0)$ and $(1,0,0,0)$.

If $\varepsilon = 0$ we define M_0 to be the two dimensional manifold of critical points of (F_0):

$$M_0 := \{(u,v,w,z) \in \mathbf{R^4} | \quad z = 0, w = -cv - (u-a)(1-u^2)\}.$$

Geometric perturbation theory uses both the above systems: (F_ε) provides us with an invariant manifold M_ε close to M_0 and we study the flow of (S_ε) restricted to this manifold. The main theorem that we use is the invariant manifold theorem due to Fenichel and we use the version formulated by Jones [J]. In our context this theorem yields the following:

Theorem 2. *If M_0 is a normally hyperbolic manifold, then for all $R > 0$, for all open intervals I with $c_0 \in I$ and for all $k \in \mathbf{N}$ there exists an $\varepsilon_0 > 0$ depending on R, I and k such that for all $\varepsilon \in (0, \varepsilon_0)$ there exists a manifold M_ε, given by*

$$M_\varepsilon = \{(u,v,w,z) \in \mathbf{R^4} | \quad w = \phi(u,v,c,\varepsilon), z = \psi(u,v,c,\varepsilon), (u,v) \in B_R(0), c \in I\}$$

with ϕ and ψ in $C^k(\overline{B_R(0)} \times \overline{I} \times [0, \varepsilon_0])$, which is locally invariant under the flow of (F_ε).

In order to apply this theorem we must ensure that the hypothesis on M_0 is satisfied. The radius R that we choose must be so large that $M_0 \cap B_R(0)$ contains the connection from -1 to 1 at $\varepsilon = 0$. We also fix $k \geq 2$.

For M_0 to be normally hyperbolic we must check that the eigenvalues μ associated to the eigenvectors of the linearised problem for any point in M_0 at $\varepsilon = 0$, which are transversal to the tangent space, have non-zero real part. Note that c can either be seen as a parameter in which case M_0 is a two-dimensional manifold, parametrised by u and v, or as an extra variable in which case we need to add the equation $c' = 0$ and M_0 becomes a three-dimensional manifold.

The matrix of the linearisation of (F_0) at the point $(u,v,w,z) \in M_0$ is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ f'(u) & c & 1 & 0 \end{pmatrix}.$$

Its set of eigenvalues is always $\{0, 0, -1, 1\}$. Only the two zero eigenvalues have eigenvectors tangent to M_0. Thus M_0 is normally hyperbolic. If we view c as a variable instead of as a parameter, M_0 remains normally hyperbolic.

Since ϕ and ψ are C^k functions in u, v, c and ε we can write down their Taylor series in ε, i.e.

$$\phi(u,v,c,\varepsilon) = \sum_{i=0}^{k} \phi_i(u,v,c)\varepsilon^i + \Phi(u,v,c,\varepsilon)\varepsilon^k, \quad \psi(u,v,c,\varepsilon) = \sum_{i=0}^{k} \psi_i(u,v,c)\varepsilon^i + \Psi(u,v,c,\varepsilon)\varepsilon^k, \tag{2.1}$$

where Φ and Ψ are continuous in $\varepsilon = 0$ with $\Phi(u,v,c,0) = 0$ and $\Psi(u,v,c,0) = 0$. In the remainder of this section we compute the coefficients of ϕ and ψ explicitly. Clearly we have

$$\phi_0(u,v,c) = -cv - (u-a)(1-u^2) \quad \text{and} \quad \psi_0(u,v,c) = 0$$

Since M_ε is locally invariant, the fast vector field $(\varepsilon v, \varepsilon w, z, w + cv + f(u))$ is perpendicular to the two normals $(\frac{\partial\phi}{\partial u}, \frac{\partial\phi}{\partial v}, -1, 0)$ and $(\frac{\partial\psi}{\partial u}, \frac{\partial\psi}{\partial v}, 0, -1)$ of M_ε. Taking the inner product of the fast vector field with each of these normals we obtain the following two coupled non-linear partial differential equations

$$\psi(u,v,c,\varepsilon) = \varepsilon\Big(v\frac{\partial\phi(u,v,c,\varepsilon)}{\partial u} + \phi(u,v,c,\varepsilon)\frac{\partial\phi(u,v,c,\varepsilon)}{\partial v}\Big), \tag{2.2}$$

$$\phi(u,v,c,\varepsilon) + cv + f(u) = \varepsilon\Big(v\frac{\partial\psi(u,v,c,\varepsilon)}{\partial u} + \phi(u,v,c,\varepsilon)\frac{\partial\psi(u,v,c,\varepsilon)}{\partial v}\Big), \tag{2.3}$$

for ϕ and ψ. We now successively compute the coefficients in (2.1) from (2.2) and (2.3). We already know the zero-th order coefficients for ϕ and ψ and substituting the zero-th order term of ϕ into

(2.2) gives the first order term for ψ. Substituting the zero-th order term of ψ, which equals zero, into (2.3) we see that the first order term of ϕ vanishes. Thus

$$\phi_1(u,v,c) = 0 \quad \text{and} \quad \psi_1(u,v,c) = v(1+c^2-3u^2+2au) + c(u-a)(1-u^2).$$

Similarly we find second order terms

$$\begin{aligned} \phi_2(u,v,c) = {} & [-v^2(-6u+2a) + 2v\{-2u(u-a)+1-u^2\}c - c^3 v \\ & + (1-u^2)(u-a)\{-2u(u-a)+1-u^2\} - (1-u^2)(u-a)c^2], \\ \psi_2(u,v,c) = {} & 0. \end{aligned} \tag{2.4}$$

Continuing in this way we can solve for all the coefficients whereby we remark that all the odd coefficients of ϕ and all the even ones of ψ are zero.

3. The flow on M_ε : construction of the travelling wave. In this section we prove the existence of a travelling wave solution for (1.1) for sufficiently small γ by showing that the heteroclinic orbit corresponding to (1.2) as a solution of the second order problem (P_0), is a transversal intersection of the unstable and stable manifolds of respectively $u=-1$ and $u=1$. We consider the slow equations restricted to the invariant manifold M_ε in Theorem 2. The resulting reduced slow system is well defined for $\varepsilon = 0$:

$$(S'_\varepsilon) \begin{cases} u' = v \\ v' = w = \phi(u,v,c,\varepsilon), \end{cases} \qquad \text{and} \quad (S'_0) \begin{cases} u' = v \\ v' = -cv - f(u). \end{cases}$$

The latter are the phase plane equations for the second order travelling wave equation in (P_0). This system has three non-degenerate critical points $(-1,0)$, $(a,0)$ and $(1,0)$ and thus it follows from the implicit function theorem that for ε small there are still three critical points, which depend in a C^k fashion on c and ε. Since these three points must correspond to the three critical points $(-1,0,0,0), (a,0,0,0)$ and $(1,0,0,0)$ of the full system (S_ε), they are independent of ε and c.

For $\varepsilon = 0$ and $c = c_0 = a\sqrt{2}$ the travelling wave solution (1.2) corresponds to a saddle connection in the phase plane of (S'_0) connecting the saddle points $(-1,0)$ and $(1,0)$. This connection is given by

$$v = \tfrac{1}{\sqrt{2}}(1-u^2).$$

By the stable manifold theorem we can for small ε still parametrize the unstable manifold of $(-1,0)$ and the stable manifold of $(1,0)$ in the phase plane of (S'_ε) locally as C^k functions of u. Denoting these functions by $h_0(u,c,\varepsilon)$ and $h_1(u,c,\varepsilon)$, we have $h_0(-1,c,\varepsilon)=0$ and $h_1(1,c,\varepsilon)=0$. Using smooth dependence on initial data we can continue h_0 and h_1 to $u=0$ if ε is small. We want to show that, for possibly even smaller ε, there exists a unique $c=c(\varepsilon)$ such that $h_0(0,c(\varepsilon),\varepsilon) = h_1(0,c(\varepsilon),\varepsilon)$. Thus we introduce

$$G(c,\varepsilon) := h_0(0,c,\varepsilon) - h_1(0,c,\varepsilon).$$

The existence of $c(\varepsilon)$ will follow from the implicit function theorem if we prove that $\frac{\partial G}{\partial c}(c_0,0) \neq 0$.

For $v \neq 0$ we can rewrite (S'_ε) as

$$\frac{dv}{du} = \frac{\phi(u,v,c,\varepsilon)}{v} \tag{3.1}$$

When we first differentiate (3.1) with respect to c and set $c = c_0$ and $\varepsilon = 0$, we get

$$\frac{dw}{du}(u) = -1 + \frac{f(u)}{h_0^2(u, c_0, 0)} w(u) \quad \text{for} \quad w(u) = \frac{\partial h_0}{\partial c}(u, c_0, 0). \tag{3.2}$$

Since $h_0(-1, c, \varepsilon) = 0$ we have $w(-1) = 0$. Note that all the higher order terms of ϕ have disappeared because we have set $\varepsilon = 0$. Similarly, we get for $\tilde{w}(u) = \frac{\partial h_1}{\partial c}(u, c_0, 0)$, that

$$\frac{d\tilde{w}}{du}(u) = -1 + \frac{f(u)}{h_1^2(u, c_0, 0)} \tilde{w}(u) \tag{3.3}$$

and $\tilde{w}(1) = 0$. Since $h_0(u, c_0, 0) = h_1(u, c_0, 0) = \frac{1}{\sqrt{2}}(1 - u^2)$, (3.2) and (3.3) both read

$$\frac{dw}{du} = -1 + \frac{2(u - a)}{1 - u^2} w, \tag{3.4}$$

whence, in view of $w(-1) = \tilde{w}(1) = 0$,

$$w(0) = -\int_{-1}^{0} (1 - s)^{1-a}(1 + s)^{1+a} ds \quad \text{and} \quad \tilde{w}(0) = \int_{0}^{1} (1 - s)^{1-a}(1 + s)^{1+a} ds.$$

Thus

$$\frac{\partial G}{\partial c}(c_0, 0) = w(0) - \tilde{w}(0) = -\int_{-1}^{1} (1 - s)^{1-a}(1 + s)^{1+a} ds =: I_1 < 0. \tag{3.5}$$

Hence by the implicit function theorem there exists a neighbourhood U of 0 such that we can find a C^k map $c : U \to \mathbf{R}$ such that $G(c(\varepsilon), \varepsilon) = 0$ for all $\varepsilon \in U$. So we have found a connecting orbit of (S'_ε) from $(-1, 0)$ to $(1, 0)$ for ε sufficiently close to 0 and thus we have proved existence of a travelling wave equation of equation (1.1) for sufficiently small $\gamma > 0$.

4. Rate of change of the wave speed. The implicit function theorem also gives us the dependence of the wave speed on ε or rather on $\gamma = \varepsilon^2$, because as we saw earlier, the only non-vanishing terms in ϕ, see (2.1) and thereafter, are the even powers of ε. We next compute $\frac{\partial G}{\partial \gamma}$.

As before we start with (3.1) and differentiating with respect to γ followed by setting $\gamma = 0$ and $c = c_0$ we get, in view of (2.4),

$$\frac{dz}{du}(u) = -2\sqrt{2}u(3u^2 - 2) + \frac{2(u - a)}{1 - u^2} z(u) \quad \text{for} \quad z(u) = \frac{\partial h_0}{\partial \gamma}(u, c_0, 0). \tag{4.1}$$

Going through similar calculations as before we get for z and $\tilde{z}(u) = \frac{\partial h_1}{\partial \gamma}(u, c_0, 0)$ that

$$\frac{\partial G}{\partial \gamma}(c_0, 0) = z(0) - \tilde{z}(0) = -2\sqrt{2} \int_{-1}^{1} s(3s^2 - 2)(1 - s)^{1-a}(1 + s)^{1+a} ds =: I_2, \tag{4.2}$$

whence, with (3.5), using either the calculus of residues or gamma functions,

$$\left.\frac{dc}{d\gamma}\right|_{\gamma=0} = -\frac{\partial G}{\partial \gamma} \left(\frac{\partial G}{\partial c} \right)^{-1} = -\frac{I_2}{I_1} = -\tfrac{1}{5}\sqrt{2}a(2a^2 - 3).$$

Note that $\frac{dc}{d\gamma}$ is negative, so by adding a fourth order perturbation to the second order travelling wave equation the wave speed, which is negative for our choice of a, decreases. In other words the absolute value of the wave speed increases under the perturbation of (1.1) with $\gamma = 0$ with the fourth order term in (1.1).

In the special case when $a = 0$ the wave speed $c_0 = 0$ and the travelling wave is an odd stationary solution. It is shown in [PT1] that this 'kink' solution satisfies

$$\frac{1}{\sqrt{2}(1+4\gamma)^{\frac{1}{4}}} < u'(0,\gamma) < \frac{1}{\sqrt{2}} \quad \text{for} \quad 0 < \gamma < \tfrac{1}{8}. \tag{4.3}$$

Here $u'(0,\gamma)$ is the derivative of u with respect to ξ. From our calculations we have

$$\frac{\partial u'}{\partial \gamma}(0,0) = \frac{\partial h_0}{\partial \gamma}(0,0,0) = z(0) = -\int_{-1}^{0} 2\sqrt{2}s(3s^2-2)(1-s)(1+s)ds = -\frac{1}{\sqrt{2}},$$

which is consistent with (4.3) and shows that the lower bound is sharp.

Acknowledgement. We would like to thank L.A. Peletier, D. Terman and W.C. Troy for [PTT] and for pointing out the paper by R. Gardner and C.R.K.T. Jones [GJ].

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