Asymptotic analysis of an elliptic-parabolic moving boundary problem*

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Abstract

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1 Introduction

In this letter we are primarily interested in the behaviour of radial solutions of an elliptic-parabolic problem with nonzero constant inward flux on a fixed spherical boundary. That is, for $u = u(r,t)$ we consider the equation

$$\frac{\partial}{\partial t}c(u(r,t)) = \frac{\partial^2 u}{\partial r^2} + \frac{N-1}{r}\frac{\partial u}{\partial r}, \quad 0 \le r < 1, \quad 0 < t < T, \tag{1.1}$$

where the nonlinear function c is defined by

$$c(s) = \min(s, 0). \tag{1.2}$$

At the fixed boundaries we prescribe

$$\frac{\partial u}{\partial r} = 0 \ \text{ on } \ r = 0 \ \text{ and } \ \frac{\partial u}{\partial r} = 1 \ \text{ on } \ r = 1, \quad (0 < t < T), \tag{1.3}$$

and as initial conditions we take

$$c(u(r,0)) = c(u_0(r)). \tag{1.4}$$

This problem arises in the mathematical modelling of saturated-unsaturated flows in porous media, see [9] for references. The part of the domain where u, which is a measure for the hydrostatic pressure, is negative, is the unsaturated zone: here equation (1.1) is of parabolic type. The region where u is positive, is the saturated region: here the equation reduces to Laplace's equation with t as a parameter. We note that with $N = 2$ this model also appears in the context of superconductors, see [5] for references.

Under mild conditions on the initial data, see e.g. [8], there exists a unique weak solution $u(r,t)$ which saturates at time $t = t_c$ defined by

$$t_c = -\int_0^1 c(u_0(r))r^{N-1}dr, \tag{1.5}$$

as can be seen from conservation of mass:

$$\frac{d}{dt}\int_0^1 c(u(r))r^{N-1}dr = 1. \tag{1.6}$$

Moreover, the unsaturated zone is of the form

$$\mathcal{P} = \{r < s(t)\}. \tag{1.7}$$

At the free boundary of $\mathcal{P}$ we have, as is easily seen from solving Laplace's equation in the saturated zone,

$$u(s(t),t) = 0 \quad \text{and} \quad \frac{\partial u}{\partial r}(s(t),t)) = s(t)^{1-N}, \tag{1.8}$$

provided $s(t) < 1$. Moreover, $s(t) \to 0$ as $t \uparrow t_c$.

We are interested in the asymptotic behaviour of solutions as $t \to t_c$ and in particular in the continuity at $t = t_c$ of the solution. For dimension $N = 2$ this question was posed by Gilding, see [5]. In fact $N = 2$ is critical and three cases have to be distinguished, namely $N < 2$, $N = 2$ and $N > 2$. Our formal analysis, which is based on matched asymptotic expansions, indicates that for $N = 2$ the solution will be continuous, but only barely, the modulus of continuity being given by $1/(\log|\log t|)$. In [4] it was shown that (radial and nonradial) weak solutions of (1.1) have continuous saturation $c(u)$ provided the pressure u is bounded. Here we have that $u(1,t) = \int_{s(t)}^1 u_r(r,t)dr = \int_{s(t)}^1 \frac{1}{r}dr = -\log(s(t) \to \infty$ as $t \to t_c$, so the pressure becomes unbounded while the saturation remains continuous.

In the sections below we discuss each of the three cases. In addition to the analysis for constant flux and linear diffusion in the unsaturated zone, we also describe the asymptotics for the case that the flux is an (integrable) power function near $t = t_c$ and the diffusion in the unsaturated zone is nonlinear degenerate, where we restrict attention to the case that (1.2) is replaced by

$$c(s) = \min(0, |s|^{p-1}s). \tag{1.9}$$

Here we assume again that (1.7) holds, which, for slow diffusion ($p < 1$), may or may not be the case, depending on the initial data. We note that for the unsaturated-saturated flow model it is often assumed that c has some regularity, e.g. that c is a Lipschitz continuous or continuously differentiable function. This would exclude $p < 1$, however, recently, see [2], a one-dimensional model for groundwater has been proposed which is mathematically equivalent to our problem with $p < 1$, but where the initial conditions are such that (1.7) does not hold, due to a second free boundary on the inside where u and u_r are zero. This front may be overtaken by the outer boundary at $t = t_c$ before it has reached the fixed boundary $r = 0$. If this happens, the unsaturated zone is not of the form (1.7) near $t = t_c$. This kind of interior extinction (i.e. at some $r > 0$) is discussed in the last section.

2 $N < 2$

For $N < 2$ there exists a self-similar solution of the form

$$u(r,t) = (t_c - t)^{\frac{2-N}{2}} f(\frac{r}{\sqrt{t_c - t}}), \quad s(t) = b\sqrt{t_c - t}, \tag{2.1}$$

where the pair (f, b) satisfies

$$f''(\eta) + (\frac{N-1}{\eta} - \frac{\eta}{2})f'(\eta) + \frac{2-N}{2}f(\eta) = 0, \quad f'(0) = f(b) = 0, \quad f'(b) = 1. \tag{2.2}$$

In [6] it was proved that this solution describes to leading order the behaviour of (general) solutions near the focussing time in dimension $N = 1$. In [3] the similarity solution was shown to have a saddle point property in the selfsimilar variables, the unstable directions corresponding, as usual, to a shift in t of the focussing time.

In the limit $N \to 2$ we see that (2.1) reduces to a constant solution, which is of no use for the asymptotic analysis.

3 $N > 2$

It was shown in [8] that for $N > 2$ the free boundary satisfies

$$s(t) \sim (t_c - t)^{\frac{1}{N}} \quad \text{as } t \to t_c, \tag{3.1}$$

and consequently we can expect $u(0,t) \to -u_c$ as $t \uparrow t_c$, with $u_c > 0$ depending on the initial data. The asymptotic analysis reveals a two-layer structure here.

For $r = O(s)$ with $0 < r < s$, the outer region, we have $u \sim -u_c$ for some constant which depends on the initial data. The second region is an interior layer near the free boundary with

$$r = s(t) - \frac{z}{\dot{s}(t)}, \tag{3.2}$$

so that for $z = O(1)$ and $u = O(1)$ we have $u \sim u_0$ with the travelling wave balance

$$\frac{\partial u_0}{\partial z} = \frac{\partial^2 u_0}{\partial z^2}, \tag{3.3}$$

and $u_0 = 0$ at $z = 0$. Matching with $u_0 \to -u_c$ as $z \to -\infty$, we have

$$u_0 = -u_c(1 - \exp(z)). \tag{3.4}$$

Imposing the other free boundary condition we find that

$$-\dot{s}u_c \sim s^{1-N} \quad \Rightarrow \quad s(t) \sim (\frac{N(t_c - t)}{u_c})^{\frac{1}{N}} \text{ as } t \uparrow t_c. \tag{3.5}$$

In (3.2) we have $\frac{1}{s} << s$ as $t \to t_c$ provided $N > 2$ so the above scalings are consistent.

4 $N = 2$

This is the borderline case, with self-similar solutions of the form (2.1) failing because with $N = 2$ there is no solution of (2.2). We will find a matching based on the assumption that the asymptotic profile is 'almost' of the self-similar form (2.1) with $N = 2$. Introducing the corresponding similarity variables

$$u(r,t) = U(\eta,\tau), \quad \sigma(\tau) = \frac{s(t)}{\sqrt{t_c - t}}, \quad \eta = \frac{r}{\sqrt{t_c - t}}, \quad \tau = -\log(t_c - t), \tag{4.1}$$

we have

$$\frac{\partial U}{\partial \tau} + \frac{1}{2}\eta\frac{\partial U}{\partial \eta} = \frac{\partial^2 U}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial U}{\partial \eta}, \tag{4.2}$$

with

$$U = 0, \ \frac{\partial U}{\partial \eta} = \frac{1}{\sigma} \text{ at } \eta = \sigma, \quad \frac{\partial U}{\partial \eta} = 0 \text{ at } \eta = 0.$$

There are again two regions.

In the core, the outer region $\eta = O(\sigma)$ with $\eta < \sigma$, we seek an expansion of the form

$$U \sim \hat{U}(\tau) + \dot{\hat{U}}(\tau)V \text{ as } \tau \to \infty, \tag{4.3}$$

where $\hat{U}(\tau) = U(0,\tau)$ and V thus satisfies

$$1 + \frac{1}{2}\eta\frac{\partial V}{\partial \eta} = \frac{\partial^2 V}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial V}{\partial \eta}, \tag{4.4}$$

so that

$$V = \int_0^\eta \frac{2}{\eta'}(\exp(\frac{\eta'^2}{4}) - 1)d\eta'. \tag{4.5}$$

Note that the expansion (4.3) breaks down in the limit $\eta \to \infty$ because then

$$V \sim \frac{4}{\eta^2}\exp(\frac{\eta^2}{4}). \tag{4.6}$$

In the interior layer we write

$$\eta = \sigma(\tau) + \frac{z}{\sigma(\tau)}, \quad U = \hat{U}(\tau)W, \quad W \sim W_0 \ \text{ as } \ \tau \to \infty, \tag{4.7}$$

which gives (since we will have $\sigma(\tau) >> 1$ as $\tau \to \infty$)

$$\frac{1}{2}(W_0 - 1) = \frac{\partial W_0}{\partial z}, \tag{4.8}$$

so that

$$W_0 = 1 - \exp(\frac{z}{2}). \tag{4.9}$$

The second moving boundary condition implies then that

$$\sigma^2\hat{U} \sim -2 \ \text{ as } \ \tau \to \infty, \tag{4.10}$$

while matching exponential terms in the two regions requires

$$\frac{4}{\sigma^2}\exp(\frac{\sigma^2}{4})\dot{\hat{U}} \sim -\hat{U}. \tag{4.11}$$

Hence

$$\frac{8}{\sigma^3}\exp(\frac{\sigma^2}{4})\dot{\sigma} \sim 1 \ \text{ as } \tau \to \infty,$$

from which it follows that

$$\sigma \sim 2\sqrt{\log\tau} \ \text{ and } \hat{U} \sim -\frac{1}{2\log\tau},$$

or, in terms of the original variables,

$$s(t) \sim 2\sqrt{t_c - t}\sqrt{\log|\log(t_c - t)|} \ \text{ and } u(0,t) \sim -\frac{1}{2\log|\log(t_c - t)|}.$$

5 Nonlinear diffusion and nonconstant flux

A natural generalisation of the saturated-unsaturated flow model is to replace (1.2) by

$$c(s) = \min(0, |s|^{p-1}s). \tag{5.1}$$

This makes the diffusion in the unsaturated region nonlinear: when $p > 1$ the diffusion is fast, when $0 < p < 1$ it is slow. See e.g. [1] and [10]. We consider (1.1) with (5.1), posing the nonlinear flux boundary condition

$$\frac{\partial u}{\partial r} = (t_c - t)^{\alpha} \quad \text{in} \quad r = 1, \tag{5.2}$$

with $\alpha > -1$ for integrability, so that

$$\int_0^s r^{N-1} u(r,t)dr = -\frac{1}{\alpha+1}(t_c - t)^{\alpha+1}. \tag{5.3}$$

Here we assume that the unsaturated region is again given by (1.7).

Omitting all details we briefly summarise the results of the analysis in the subsections below.

5.1 $0 < p \leq 1$, slow diffusion

For $\alpha > \frac{N-2}{2}$ we have again selfsimilar behaviour

$$u \sim (t_c - t)^{\gamma} f\left(\frac{r}{(t_c - t)^{\beta}}\right), \quad s \sim b(t_c - t)^{\beta},$$

$$\gamma = \frac{2\alpha + 2 - N}{N - (N-2)p}, \quad \beta = \frac{\alpha + 1 - \alpha p}{N - (N-2)p},$$

while for $-1 \leq \alpha < \frac{N-2}{2}$ the solution is discontinous as $t \uparrow t_c$.

5.2 $N \leq 2$, $p > 1$ and $N > 2$, $1 < p < \frac{N}{N-2}$, fast diffusion

We restrict attention here to the range of p where the diffusion equation has mass conservation at infinity, see [10]. In this range we find that the similarity solution with inward moving interface exists only for $\frac{N-2}{2} < \alpha < \frac{1}{p-1}$. For $\alpha = \frac{1}{p-1}$ it is a separable solution. When α becomes larger, the generic behaviour is that the saturated region disappears completely before the saturation time, after which the solution decays to zero.

6 Elliptic-parabolic with slow diffusion: two interfaces

In [2] a new model for steady flows in porous media is proposed leading to a free boundary problem for the porous medium equation in one dimension,

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(u^\sigma \frac{\partial u}{\partial x}), \tag{6.1}$$

($\sigma > 0$) with free boundary conditions

$$u = u^\sigma \frac{\partial u}{\partial x} = 0 \text{ at } x = s(t) \text{ and } u = 0, \ u^\sigma \frac{\partial u}{\partial x} = J(t) \text{ at } x = q(t). \tag{6.2}$$

Here $q(t)$ is assumed to be smaller than $s(t)$.

In this model u is the height of a groundwater mound on an impermeable layer in a horizontal porous stratum. The right free boundary $s(t)$ is the boundary of the propagating water mound. Near $x = s(t)$ the solution may be described as a solution of the porous medium equation with expanding support. The free boundary $q(t)$ exists because of forced water drainage, represented by $J(t)$, on some fixed boundary on the left, assuming that at the bottom where there is no fluid mound, the hydraulic conductivity is infinite. How this is physically feasible is explained in [2]. Thus near $x = q(t)$ we may view u as the solution of (1.1) with $c(s) = ((\sigma+1)s)^{\frac{1}{\sigma+1}}$ for $s > 0$ and $c(s) = 0$ for $s < 0$. Note that (1.1) is now parabolic for positive u.

Putting another fixed zero flux boundary on the right, we may have that $s(t)$ reaches this boundary before the solution extinguishes, in which case we may expect extinction near this boundary as discussed in Section 5.1. However, it is also possible that $q(t)$ overtakes $s(t)$ at some point to the left of the fixed boundary. This can only happen at the extinction time $t = t_c$, which is again defined by

$$\int_0^{t_c} J(t)dt = \int_{q_0}^{s_0} u_0(x)dx, \tag{6.3}$$

where $u_0(x)$, q_0 and s_0 are the initial data for u, q and s at $t = 0$. Below we shall describe the solution near the extinction time under the assumption that this overtaking occurs. We treat again the more general case in which

$$J(t) \sim A(t_c - t)^p \text{ as } t \to t_c, \tag{6.4}$$

with $p = 0$ being the physical case in [2].

As in [2], we may seek selfsimilar behaviour as $t \to t_c$. For consistency with (6.2) and (6.4) we require

$$u \sim (t_c - t)^{\frac{2p+1}{\sigma+2}} f\Big(\frac{x-a}{(t_c-t)^{\frac{\sigma(p+1)+1}{\sigma+2}}}\Big), \tag{6.5}$$

$$q \sim a - q_c(t_c - t)^{\frac{\sigma(p+1)+1}{\sigma+2}}, \quad s \sim a - s_c(t_c - t)^{\frac{\sigma(p+1)+1}{\sigma+2}},$$

for some constants a, q_c and s_c. Such solutions exist for $p > \frac{1}{\sigma}$. For $p = \frac{1}{\sigma}$ however, (6.5) becomes

$$u \sim (t_c - t)^{\frac{1}{\sigma}} f\Big(\frac{x-a}{t_c - t}\Big) \ \text{ with } f(\eta) = (\sigma s_c(s_c - \eta))^{\frac{1}{\sigma}},$$

which does not exhibit a second moving interface and provides a lower bound on the range of p for which selfsimilar solutions describing overtaking exist.

It follows that for $p = 0$ overtaking cannot be expected to be asymptotically selfsimilar. We next show for p in the range $-1 < p < \frac{1}{\sigma}$, how the non-uniform behaviour as $t \to t_c$ is given by two regions.

We define $a = s(t_c)$ and $b = \dot{s}(t_c)$ to be, respectively the position and the speed of the right front at the extinction time, where both quantities can be expected to be bounded because of the finite speed of propagation property for the porous medium equation, see [1].

In the outer region we expect

$$u \sim (\sigma b(a - x))^{\frac{1}{\sigma}} \ \text{ as } t \to t_c\,; \tag{6.6}$$

this corresponds to the local behaviour at a moving front for the porous medium equation. We shall see that in the limit $t \to t_c$ the front $s(t)$ moves much more slowly than $q(t)$.

In the inner region, close to $x = q(t)$, we write

$$x = q(t) + z, \quad q \sim a - Q(t), \quad u \sim u_0 \tag{6.7}$$

From (6.1) and (6.2) we have at leading order

$$\dot{Q}(t)u_0 + J(t) = u_0^{\sigma} u_{0z} \ \text{ with } u_0 = 0 \ \text{ at } z = 0, \tag{6.8}$$

so that

$$u_0 \sim -\frac{J(t)}{\dot{Q}(t)} \ \text{ as } z \to \infty.$$

Matching this with (6.6) and using (6.4) requires that

$$(\sigma b Q)^{\frac{1}{\sigma}} \dot{Q} = -A(t_c - t),$$

so that

$$Q(t) = \left(\frac{\sigma+1}{\sigma}\frac{A}{p+1}\right)^{-\frac{\sigma}{\sigma+1}} (\sigma b)^{-\frac{1}{\sigma+1}} (t_c - t)^{\frac{\sigma(p+1)}{\sigma+1}}. \tag{6.9}$$

The solution to (6.8) thus scales in the selfsimilar manner

$$u_0 = (t_c - t)^{\frac{p+1}{\sigma+1}} g\left(\frac{z}{t_c - t}\right);$$

consistency of the expansions thus requires that $t_c - t << Q(t)$ as $t \to t_c$ and hence from (6.9) that $p < \frac{1}{\sigma}$, as expected.

We note that there is a second inner region at $x = s(t)$, but this is passive as far as the matching is concerned. We have

$$x = a + (t_c - t)\zeta, \quad s \sim a - b(t_c - t), \quad u \sim (t_c - t)^{\frac{1}{\sigma}} (\sigma b(b - \zeta))^{\frac{1}{\sigma}}.$$

The constants a and b cannot be determined by the local analysis.

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