Some transformations related to the Thin Film Equation

by

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In a recent paper [BKO] the travelling wave problem for the Thin Film Equation (TFE),

$$u_t + (u^n u_{xxx})_x = 0, \quad (n > 0)$$

leads to the local analysis of solutions of the third order equation

$$u''' = u^{-a} \quad (a = n - 1),$$

which is subsequently reduced to a planar dynamical system.

In this letter we show how the reduction in [BKO] of the third order equation is related to some well known standard substitutions. First of all, for increasing solutions we may set

$$s = u, \quad y = u'^2,$$

which gives for y as a function of s the second order equation

$$y''(s) = 2s^{-a}y^{-1/2}.$$

Note that $y' = 2u''$. This transformation was applied in relation to the construction of source type solutions of TFE in [BPW]. We observe that this transformation is not restricted to to pure power type nonlinearities. Thus it is also useful for the global analysis of the travelling wave problem for TFE, see [BHP].

The y-equation may be reduced one step more by introducing (here the scaling invariance of the equation is essential)

$$\tau = \log s = \log u; \quad X = \frac{sy'(s)}{y(s)} = \frac{2uu''}{u'^2}; \quad Y = s^{2-a}y(s)^{-3/2} = u^{2-a}u'^{-3},$$

which leads to the quadratic system

$$X' = X - X^2 + 2Y; \quad Y' = Y(2 - a - \frac{3}{2}X).$$

This second transformation can already be found in [J] and was used in [H1] for the construction of all compactly supported similarity solutions of the porous medium equation with sign changes, see also the classical paper [LOT]. Apart from an inversion of y this quadratic system is almost the same as derived in [BKO].

Positive increasing (decreasing) solutions of the third order equation correspond to orbits of the quadratic system in the upper (lower) half plane. It is possible to give a complete rigorous global phase plane analysis of this system. This involves a local analysis of the three finite critical points, two of which are on the X-axis and have a very simple local structure. Orbits coming in to or out of these two points correspond to solutions with linear and quadratic behaviour as $u \to \infty$ and $u \to 0$ respectively. In [BKO] these points lie at infinity. The third critical point depends on the parameter a. Its local structure is slightly more complicated, but its stability properties are easily determined, see [BKO], where also the existence of limit cycles is excluded. This third point corresponds to an explicit power solution (exponent $\frac{3}{a+1}$) and orbits coming in to or out of it to solutions with (the same) powerlike behaviour as $u \to \infty$ and $u \to 0$ respectively.

Orbits escaping to infinity correspond to solutions of the third order equation with $u' = 0$ at some point. The analysis of orbits escaping to infinity is simpler if we change the definition of Y into

$$Y^2 = s^{2-a}y(s)^{-3/2} = u^{2-a}u'^{-3},$$

where we restrict to increasing solutions. This changes the quadratic system into

$$X' = X - X^2 + 2Y^2; \quad Y' = Y(1 - \frac{a}{2} - \frac{3}{4}X),$$

with $Y > 0$. The critical points at infinity are determined using the Poincaré transformation (see e.g. [P] for this technique and [H2, H3] for applications to similarity solutions)

$$X = \frac{\xi}{\zeta}; \quad Y = \frac{\eta}{\zeta}; \quad \xi^2 + \eta^2 + \zeta^2 = 1; \quad \zeta \geq 0.$$

One finds at infinity (i.e. $\zeta = 0$) six points, among them $(\xi, \eta, \zeta) = (\pm 1, 0, 0)$, one $(+)$ a stable node, the other $(-)$ an unstable node. This is consistent with the behaviour of solutions of the third order equation with $u' = 0$ and $u'' \neq 0$ at some point.

In addition to $(\xi, \eta, \zeta) = (\pm 1, 0, 0)$ one also has four saddles (only two of which are relevant) $(\xi, \eta, \zeta) = (\pm\frac{2}{3}\sqrt{2}, \pm\frac{1}{3})$ as critical points at infinity. This is consistent with the behaviour of solutions of the third order equation with $u' = u'' = 0$ at some point.

For decreasing solutions we change notation. Taking $-x$ as a new variable in the third order equation, we now look at increasing solutions of the equation

$$u''' = -u^{-a} \quad (a = n - 1),$$

so that the last quadratic system changes into

$$X' = X - X^2 - 2Y^2; \quad Y' = Y(1 - \frac{a}{2} - \frac{3}{4}X),$$

which only has $(\xi, \eta, \zeta) = (\pm 1, 0, 0)$ as critical points at infinity. (Note that $u' = u'' = 0$ at some point does not occur in this situation in view of the sign of u'''.)

As a is varied, all these systems undergo a qualitative change for distinct values of a, namely at

$$\frac{5}{4} - \frac{3}{4}\sqrt{3}, \quad \frac{1}{2}, \quad \frac{5}{4}, \quad 2, \quad \frac{5}{4} + \frac{3}{4}\sqrt{3}.$$

For $a = \frac{5}{4}$ the orbits are symmetric with respect to the line $X = \frac{1}{2}$. Moreover, if one defines a' by

$$a' + a = \frac{5}{2},$$

then the orbits for the systems with a replaced by a' maybe found by reflection in the line $X = \frac{1}{2}$. Thus there is a relation between solutions of the corresponding third order equations.

We conclude with the observation that all this is very similar to what happens in case of the radial semilinear Poisson equation

$$U''(r) + \frac{n-1}{r}U'(r) + U(r)^p = 0.$$

This equation was analysed by Fowler by transforming it into what is now called an Emden-Fowler equation, see [F], but one can also set

$$X = \frac{rU'(r)}{U(r)}; \quad Y = r^2 U(r)^{p-1}; \quad t = \log r,$$

which leads to the system (see [HV1])

$$X' = X(2 - n - X) - Y, \quad Y' = Y(2 + (p-1)X).$$

In this case the critical Sobolev exponent, $p = \frac{n+2}{n-2}$, gives a symmetry in the line $X = \frac{2-n}{2}$. In the same manner, this gives a relation between sub- and supercritical solutions. Recently, these methods have also been applied to systems of semilinear Poisson equations in [HV2].

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