Instabilities, bifurcation and saddle-points in some FBP's in combustion

Claude-Michel Brauner
Université Bordeaux I, Mathématiques Appliquées de Bordeaux
33405 Talence Cedex, France
E-mail: brauner@math.u-bordeaux.fr

Josephus Hulshof
Mathematical Department of the Leiden University
Niels Bohrweg 1, 2333 CA Leiden, The Netherlands
E-mail: hulshof@wi.leidenuniv.nl
WWW: http://www.wi.leidenuniv.nl/home/hulshof/

Alessandra Lunardi
Dipartimento di Matematica, Università di Parma
Via D'Azeglio 85/A, 43100 Parma, Italy
E-mail: lunardi@prmat.math.unipr.it

Claudine Schmidt-Lainé
CNRS UMR 128, Ecole Normale Supérieure de Lyon
69364 Lyon Cedex 07, France
E-mail: schmidt@umpa.ens-lyon.fr

1 Introduction

In the modelling of combustion and flames, the stability question is of major importance. Detonations are, by nature, unstable. Exchange of stability often arises in premixed flames, as shown by the classical thermo-diffusive model for Lewis number close to 1. In this framework, the famous Kuramoto-Sivashinsky equation exhibits cellular instabilities. As far as diffusion flames are concerned, chambered flames present S-shaped response curves for arbitrary Lewis number. Away from laminar combustion, turbulence may stabilize the flame and lead to extinction/reignition phenomena. Solid combustion should not be forgotten: clearly instabilities in the combustion of solid propellants can endanger the rocket propulsion.

Although there is a large literature on instabilities in combustion, most of the references are restricted to a rather formal approach, namely a linear stability analysis, based on the derivation of a so-called "dispersion relation". Such an analysis neglects all the curvature effects stemming from the corrugation of the flame front and is irrelevant, mathematically speaking. Among the few mathematical papers involving a deep nonlinear stability analysis, let us mention the bifurcation theorem in [10]. The authors give a full rigorous treatment of the thermo-diffusive model, when the activation energy is finite.

In the limit of high activation energy, the flame zone becomes a thin interface which can be viewed as a free boundary. At this front, several types of jump conditions may occur: usually the unknown (temperature, mass fraction, etc.) is continuous, in certain circumstances it is given, while the jump in the gradient is either given or determined by some nonlinear relation. This discontinuity concentrates the reaction term at the free boundary. As a basic example we mention the Near Equidiffusional Flame (NEF) theory, see [7]: the equidiffusional case gives rise to a one-phase paradigm model, see the survey [14]:

$$\begin{cases} u_t = \Delta u, \quad x < \xi(t,y),\ y \in \omega, \\ u = u_*,\ \frac{\partial u}{\partial n} = 1, \quad x = \xi(t,y),\ y \in \omega, \\ \frac{\partial u}{\partial n} = 0, \quad x < \xi(t,y),\ y \in \partial\omega, \\ u(-\infty, y, t) = 0, \quad y \in \omega. \end{cases} \tag{1.1}$$

Here u_* is a positive constant often normalized at 0. We denote by ω a bounded domain in $\mathbb{R}^{N-1}$ with smooth boundary $\partial\omega$.

Problem (1.1) admits as equilibria a family of $1D$-travelling waves (TW) solutions $u(x,y,t) = U(x+ct)$, which are (orbitally) stable, see [2]. In the context of spherical flames, the model problem is formulated in a bounded domain Ω_t:

$$\begin{cases} u_t = \Delta u, \quad \text{in } \Omega_t, \\ u = u_*,\ \frac{\partial u}{\partial n} = 1 \quad \text{on } \partial\Omega_t. \end{cases} \tag{1.2}$$

However problem (1.2) does not admit trivial equilibria. It is called a focusing problem, because the domain vanishes in finite time, see [14, 9, 12].

Our purpose is to present a general approach to the stability question for FBP's such as the TW problem and the problem in a bounded domain above, generalizing methods developed for special cases by the authors, see the reference list. Although we are motivated by combustion problems, we emphasize that our method can be applied to other FBP's besides those with "combustion type" free boundary conditions $u = 0$ and $\frac{\partial u}{\partial n} = g$ at the interface.

The principle of the method will be given in the next section, for simplicity in a bounded domain only, see [4] for details. It is based on a reduction to a fixed domain Ω and the elimination of the front thanks to the splitting

$$u = U + \Phi \cdot \nabla U + w, \tag{1.3}$$

where Φ measures the local $C^{2+\alpha}$-difference between the free boundary $\partial\Omega_t$ and $\partial\Omega$. The price to pay are nonlocal terms which make the equation for w fully nonlinear. However, the reformulation in terms of w enables us to apply abstract results from [13] and to establish the principle of linearized stability, bifurcation and existence of stable and unstable manifolds.

As examples we exhibit a saddle-point property for focusing flames in Section 3 and in Section 4 a two-phase version of the TW problem with a Burgers term, namely a Deflagration to Detonation Transition (DDT) model proposed by Stewart and Ludford [8]. In dimension $N = 2$, the planar TW becomes unstable at a critical value u_*^c of u_*, at which an infinite number of bifurcated branches of nonplanar TW's accumulate, see [1] for details.

2 A general procedure for nonlinear stability analysis

We consider free boundary problems of the form

$$u_t = \mathcal{L}u + f, \quad x = (x_1, \dots, x_N) \in \Omega_t, \quad t > 0, \tag{2.1}$$

with free boundary conditions

$$u = 0 \quad \text{and} \quad \frac{\partial u}{\partial n} = g \quad \text{on} \quad \partial\Omega_t. \tag{2.2}$$

Here Ω_t is a bounded domain in $\mathbb{R}^N$ with (free) boundary $\partial\Omega_t$, $\mathcal{L}$ is a uniformly elliptic operator on $\mathbb{R}^N$ with smooth coefficients, and f and g are given smooth functions defined on the whole of $\mathbb{R}^N$. We are interested in the stability properties of equilibria of (2.1)(2.2). We assume that the pair (Ω, U) is a smooth equilibrium, namely

$$\mathcal{L}u + f = 0 \ \text{ in } \Omega, \quad u = 0 \text{ and } \frac{\partial u}{\partial n} = g \ \text{ on } \partial\Omega, \tag{2.3}$$

with Ω bounded and $\partial\Omega$ in $C^{2+\alpha}$ and $U \in C^{2+\alpha}(\overline{\Omega})$. We consider (2.1)(2.2) with initial data Ω_0 and u_0 close (in the $C^{2+\alpha}$ sense) to Ω and U. The main structural condition is that g does not vanish on $\partial\Omega$.

Since $\partial\Omega$ is smooth, there exists a smooth outwards pointed normal

$$\nu : \partial\Omega \to S^{N-1}, \tag{2.4}$$

where $S^{N-1} = \{x \in \mathbb{R}^N : |x| = 1\}$. For $\delta > 0$ we define a map

$$X : \partial\Omega \times [-\delta, \delta] \to \mathbb{R}^N, \quad X(\phi, r) = \phi + r\nu(\phi). \tag{2.5}$$

If δ is sufficiently small, then (2.5) defines a bijection to a compact neighbourhood $N(\delta\Omega)$ of $\delta\Omega$.

Next we assume that, on some time interval I, $\partial\Omega_t$ is close to $\partial\Omega$ in the sense that, in terms of the coordinates r and ϕ defined by (2.5), $\partial\Omega_t$ is given by

$$\partial\Omega_t = \{x = \phi + r\nu(\phi) \in N(\delta\Omega) : r = s(\phi, t)\}, \tag{2.6}$$

where $s : \partial\Omega \times I \to [-\delta_1, \delta_1]$ with $0 < \delta_1 < \delta$ is a smooth function. Thus $r < s(\phi, t)$ in $\Omega_t \cap N(\partial\Omega)$.

For the coordinate transformation from Ω_t back to Ω we denote by ξ a point in Ω and by x a point in Ω_t. The time coordinate of the problem on Ω is denoted by τ and the local coordinates near $\partial\Omega$ of ξ are denoted by z and θ, i.e. $\xi = \theta + z\nu(\theta)$. We transform the free boundary problem by setting

$$x = \xi + \alpha(z)s(\theta, \tau)\nu(\theta) \ \ (\text{whence} \ \ r = z + \alpha(z)s(\theta, \tau) \ \text{ and } \ \phi = \theta\ ) \ \text{ and } \ t = \tau. \tag{2.7}$$

Here $\alpha : [-\delta, \delta] \to [0, 1]$ is a smooth mollifier. For the transformation to be a diffeomorphism s has to be restricted to a small interval, say $(-\epsilon, \epsilon)$, so as to ensure that the map $z \in [-\delta, \delta] \to z + \alpha(z)s \in [-\delta, \delta]$ is a diffeomorphism.

In what follows it will be convenient to write

$$\Phi(\xi, \tau) = \alpha(z)s(\theta, \tau)\nu(\theta). \tag{2.8}$$

This function Φ contains all the information about the $C^{2+\alpha}$-difference between the fixed boundary Ω and the free boundary Ω_t and remarkably will not appear in the final formulation. We see that x, t, ∇_x and $\frac{\partial}{\partial t}$ transform as

$$x = \xi + \Phi(\xi, \tau), \quad t = \tau, \quad \nabla_x = (I + \nabla_\xi \Phi)^{-1}\nabla_\xi, \quad \frac{\partial}{\partial t} = D_\tau = \frac{\partial}{\partial \tau} - \frac{\partial \Phi}{\partial \tau} \cdot (I + \nabla_\xi \Phi)^{-1}\nabla_\xi. \tag{2.9}$$

Here we denote by $\nabla\Phi$ a matrix which has the gradients of the components of Φ as columns, i.e. $\nabla\Phi$ is the transpose of the Jacobian matrix representing the derivative $D\Phi$ of Φ. If we write $u(x, t) = \hat{u}(\xi, \tau)$, we can rewrite the action of $\mathcal{L}$ on u as

$$(\mathcal{L}u)(x, t) = (\hat{\mathcal{L}}\hat{u})(\xi, \tau) = (\hat{\mathcal{L}}_0\hat{u})(\xi, \tau) + (\hat{\mathcal{L}}_1\hat{u})(\xi, \tau) + (\hat{\mathcal{L}}_2^r\hat{u})(\xi, \tau), \tag{2.10}$$

where $\hat{\mathcal{L}}_0$ is the original operator $\mathcal{L}$ with x replaced by ξ and $\hat{\mathcal{L}}_1$ is an operator with coefficients linear in Φ and its first and second order ξ-derivatives. The remainder term $\hat{\mathcal{L}}_2^r$ has coefficients bounded by a multiple of $|\Phi|^2 + |\nabla_\xi \Phi|^2 + |\nabla_\xi \nabla_\xi \Phi|^2$. This is easily seen from the expansion

$$(I + \nabla_\xi \Phi)^{-1} = I - \nabla_\xi \Phi + (\nabla_\xi \Phi)^2 (I + \nabla_\xi \Phi)^{-1}. \tag{2.11}$$

Likewise we get an expansion for the time derivative which reads

$$\frac{\partial u(x,t)}{\partial t} = D_\tau u(\xi,\tau) = \frac{\partial \hat{u}}{\partial \tau} - \frac{\partial \Phi}{\partial \tau} \cdot \nabla_\xi \hat{u} + \frac{\partial \Phi}{\partial \tau} \cdot (\nabla_\xi \Phi)(I + \nabla_\xi \Phi)^{-1} \nabla_\xi \hat{u}. \tag{2.12}$$

The transformation of Ω_t to Ω also acts on the equilibrium U itself and this has to be taken into account when we expand $\hat{u}$ near U. Observing that

$$U(\xi + \Phi(\xi,\tau)) = U(\xi) + (\nabla_\xi U(\xi)) \cdot \Phi(\xi,\tau) + R_2(\xi, \Phi(\xi,\tau)), \tag{2.13}$$

the expansion (2.13) leads us to look for a solution $u(x,t) = \hat{u}(\xi,\tau)$ of the form

$$u(x,t) = \hat{u}(\xi,\tau) = U(\xi) + (\nabla_\xi U(\xi)) \cdot \Phi(\xi,\tau) + w(\xi,\tau). \tag{2.14}$$

Comparing the zero order terms simply gives

$$\frac{\partial U(\xi)}{\partial \tau} - (\hat{\mathcal{L}}_0 U)(\xi) = f(\xi),$$

while for the first order terms

$$\left(\frac{\partial}{\partial \tau} - \hat{\mathcal{L}}_0\right)((\nabla_\xi U(\xi)) \cdot \Phi(\xi,\tau)) - \left(\frac{\partial \Phi}{\partial t} \cdot \nabla_\xi + \hat{\mathcal{L}}_1\right) U(\xi) = (\nabla_\xi f(\xi)) \cdot \Phi(\xi,\tau), \tag{2.15}$$

which reduces to

$$-(\hat{\mathcal{L}}_0 (\nabla_\xi U(\xi)) \cdot \Phi(\xi,\tau)) - \hat{\mathcal{L}}_1 U(\xi) = (\nabla_\xi f(\xi)) \cdot \Phi(\xi,\tau). \tag{2.16}$$

All terms of order zero and order one in Φ involving $U(\xi)$ and its derivatives cancel. The resulting expansion reads

$$\frac{\partial w}{\partial \tau} - \hat{\mathcal{L}}_0 w = \left(\frac{\partial \Phi}{\partial \tau} \cdot \nabla_\xi + \hat{\mathcal{L}}_1\right) w + \left(-\frac{\partial \Phi}{\partial \tau} \cdot (\nabla_\xi \Phi)(I + \nabla_\xi \Phi)^{-1} \nabla_\xi + \hat{\mathcal{L}}_2^r\right) w \tag{2.17}$$

$$+\left(\frac{\partial \Phi}{\partial \tau} \cdot \nabla_\xi + \hat{\mathcal{L}}_1\right)(\Phi \cdot U) + \left(-\frac{\partial \Phi}{\partial \tau} \cdot (\nabla_\xi \Phi)(I + \nabla_\xi \Phi)^{-1} \nabla_\xi + \hat{\mathcal{L}}_2^r\right)(U + \Phi \cdot \nabla_\xi U) + R_3(\xi, \Phi(\xi,\tau)).$$

The linear part (the left hand side) is now in the form we want, but the higher order terms contain derivatives of both Φ and w, including $\frac{\partial \Phi}{\partial \tau}$. To eliminate Φ from (2.17) we have to transform the free boundary conditions.

The first free boundary condition $u = 0$ on $\partial \Omega_t$ implies that

$$w(\xi,\tau) = -(\nabla_\xi U(\xi)) \cdot \Phi(\xi,\tau) = -g(\xi) s(\xi,\tau) \quad \text{for } \xi \in \partial \Omega. \tag{2.18}$$

For the normal derivative we have that

$$\frac{\partial}{\partial n} = n \cdot \nabla_x = \nu \cdot \nabla_\xi - \nabla_\xi s \cdot \nabla_\xi + (\nabla_\xi s \cdot \nabla_\xi \Phi + \rho \cdot)(I + \nabla_\xi \Phi)^{-1} \nabla_\xi. \tag{2.19}$$

Since g is expanded as

$$g(\xi + \Phi(\xi,\tau)) = g(\xi) + (\nabla_\xi g(\xi)) \cdot \Phi(\xi,\tau) + R_4(\xi, \Phi(\xi,\tau)), \tag{2.20}$$

with again R_4 smooth and bounded by a multiple of $\Phi(\xi,\tau)^2$, we can compute the condition following from $\frac{\partial u}{\partial n} - g = 0$. The new condition becomes

$$\frac{\partial w}{\partial \nu} + s\left(\frac{\partial^2 U}{\partial \nu^2} - \frac{\partial g}{\partial \nu}\right) w = R_5(\xi, s, \nabla_\xi s) + (\nabla_\xi s \cdot \nabla_\xi) w + R_6(\xi, s, \nabla_\xi s) \nabla_\xi w. \tag{2.21}$$

Now we can eliminate the free boundary terms s, Φ and their derivatives from the problem. We begin with equation (2.17) and evaluate it at $\partial\Omega$ using (2.18). In the resulting equation we isolate all the terms containing $\frac{\partial s}{\partial \tau}$ to obtain an equation of the form

$$-g(\xi) \frac{\partial s}{\partial \tau} = \nu(\xi) \frac{\partial s}{\partial \tau} R_7(\xi, w, \nabla_\xi w, \nabla_\xi \nabla_\xi w) - R_8(\xi, w, \nabla_\xi w, \nabla_\xi \nabla_\xi w), \tag{2.22}$$

where R_7 and R_8 are smooth with $R_7(\xi,0,0,0) = R_8(\xi,0,0,0) = 0$. Since we have assumed that g is nonzero, it follows that $\frac{\partial s}{\partial \tau}$ can be expressed as

$$\frac{\partial s}{\partial \tau} = \frac{R_8(\xi, w, \nabla_\xi w, \nabla_\xi \nabla_\xi w)}{R_7(\xi, w, \nabla_\xi w, \nabla_\xi \nabla_\xi w)\nu(\xi) + g(\xi)}, \quad \xi = \theta \in \partial\Omega.$$

Next we substitute this back into (2.17) to convert all the other terms with Φ and its ξ-derivatives into the traces of w and its (tangential) ξ-derivatives. This finally yields our final formulation: problem (2.1),(2.2) is rewritten for w as the fully nonlinear problem

$$w_t = \mathcal{L}w + \mathcal{F}(w, Dw, D^2w) \quad \text{on } \Omega, \quad t > 0, \tag{2.23}$$

with boundary conditions also fully nonlinear,

$$\mathcal{B}w = \mathcal{G}(w, Dw) \quad \text{on } \partial\Omega. \tag{2.24}$$

We derive that the local behaviour of solutions near an equilibrium (Ω, U) is determined by the *linearized* operator L defined by

$$Lv = \mathcal{L}v \quad \text{for} \quad v \in D(L) = \{v \in \cap_{p>1} W^{2,p}(\Omega),\, \mathcal{L}v \in C(\overline{\Omega}),\, \mathcal{B}v = 0 \quad \text{on } \partial\Omega\}, \tag{2.25}$$

where

$$\mathcal{B}v = \frac{\partial v}{\partial n} + \frac{1}{g}\Big(\frac{\partial g}{\partial n} - \frac{\partial^2 U}{\partial n^2}\Big)v \tag{2.26}$$

is the linearized boundary operator. The reformulation of (2.1)(2.2) as the fully nonlinear problem (2.23)(2.24) enables us to perform a *nonlinear* stability analysis of the free boundary problem.

Our first result concerns the local existence.

Theorem 2.1 *For every $T > 0$ there are r, $\rho > 0$ such that if w_0 satisfies the compatibility condition $\mathcal{B}w_0 = \mathcal{G}(w_0, Dw_0)$ and $\|w_0\|_{C^{2+\alpha}(\overline{\Omega})} \le \rho$, problem (2.23)-(2.24) with initial data $w = w_0$ has a solution $w \in C^{1+\alpha/2, 2+\alpha}([0,T] \times \overline{\Omega})$. Moreover w is the unique solution in $B(0,r) \subset C^{1+\alpha/2,2+\alpha}([0,T] \times \overline{\Omega})$.*

For the stability result, i.e. for the description of the stable and unstable manifolds we have to exclude the existence of eigenvalues with zero imaginary part:

$$\sigma(L) \cap i\mathbb{R} = \emptyset. \tag{2.27}$$

Let P be the spectral projection associated to the subset of $\sigma(L)$ with positive real part.

Theorem 2.2 *Assume that the spectrum of L contains elements with positive real part and that (2.27) holds.*
(i) There exists a unique local unstable manifold of the form

$$\varphi : B(0, r_0) \subset P(C(\overline{\Omega})) \mapsto (I - P)(C^{2+\alpha}(\overline{\Omega})),$$

φ Lipschitz continuous, differentiable at 0 with $\varphi'(0) = 0$.
(ii) There exist a unique local stable manifold of the form

$$\psi : B(0, r_1) \subset \{w_0 \in (I - P)(C^{2+\alpha}(\overline{\Omega})) : \mathcal{B}w_0 = \mathcal{G}(w_0, Dw_0)\} \mapsto P(C(\overline{\Omega})),$$

ψ Lipschitz continuous, differentiable at 0 with $\psi'(0) = 0$.

Clearly , if the spectrum is contained in $\{z : Rez < -\delta\}$ for some $\delta > 0$, then $P = 0$ and the stability theorem follows from (ii).

3 The focusing flame problem

The behaviour of solutions of the focusing problem from the introduction (with $u_* = 0$) is exhibited by selfsimilar solutions of the form

$$u(x,t) = \sqrt{T-t}\, f(\eta)\,,\ \eta = \frac{|x|}{\sqrt{T-t}}\,,\ \Omega_t = \{|x| < b\sqrt{T-t}\} \tag{3.1}$$

with $f(\eta)$ satisfying

$$f''(\eta) + \frac{N-1}{\eta} f'(\eta) + \frac{1}{2} f = \frac{1}{2}\eta f' \ \text{ for } 0 \le \eta \le b\,; f'(0) = f(b) = 0\ ,\ f'(b) = 1\,. \tag{3.2}$$

The stability analysis of the selfsimilar profile fits in the framework presented above. Transforming the problem to selfsimilar variables

$$\tilde{x} = \frac{x}{(T-t)^{\frac{1}{2}}},\ \tilde{t} = -\log(T-t),\ \tilde{u}(\tilde{x},\tilde{t}) = \frac{u(x,t)}{(T-t)^{\frac{1}{2}}},\ \tilde{\Omega}_{\tilde{t}} = \{\tilde{x} : x \in \Omega_t\}, \tag{3.3}$$

omitting the tildes, we arrive at

$$u_t = \mathcal{L}u = \Delta u - \frac{1}{2} x \cdot \nabla u + \frac{1}{2} u, \quad x \in \Omega_t \ \text{ with } \ u = 0, \quad \frac{\partial u}{\partial n} = 1 \ \text{ on } \partial\Omega_t. \tag{3.4}$$

The selfsimilar solution (3.1) is transformed by (3.3) into an equilibrium

$$U(x) = f(|x|),\ \Omega = B_b = \{x \in \mathbb{R}^N : |x| < b\}, \tag{3.5}$$

with $f(\eta)$ defined by (3.2). We cannot expect (3.5) to be stable because the original problem is invariant under translations in x and t. Thus if we apply a small shift to (3.5), we obtain another selfsimilar solution which is transformed by (3.3) into a solution which starts close to (3.5) but moves away from it. As a consequence we have that L defined by (2.25) with

$$\mathcal{L}v = \Delta v - \frac{1}{2} x \cdot \nabla v + \frac{1}{2} v \ \text{ on } B_b, \tag{3.6}$$

and (2.26) with

$$\mathcal{B}v = \frac{\partial v}{\partial n} + (\frac{N-1}{b} - \frac{b}{2})v = 0 \ \text{ on } \partial\mathcal{B}_b \tag{3.7}$$

must have unstable eigenvalues: one with a 1-dimensional eigenspace spanned by a radial eigenfunction corresponding to a shift in t, and another with an N-dimensional eigenspace corresponding to N shifts in x. Thus the equilibrium (3.5) is unstable.

Using powerseries it can be shown that these two eigenvalues are the only positive eigenvalues and that the remaining part of the spectrum consists of negative eigenvalues. This may be interpreted by saying that, although the equilibrium (3.5) is unstable, its profile does have a stability property: the unstable manifold consists exactly of the images under (3.3) of shifts in space and time of (3.5) and is therefore given by the parametrization

$$(\epsilon_1, \epsilon_2, t) \mapsto \sqrt{1 + \epsilon_2 e^t}\, U(\frac{x - \epsilon_1 e^{\frac{1}{2}t}}{\sqrt{\epsilon_2 e^t + 1}}). \tag{3.8}$$

The (local) image of (3.8) under the transformation of (2.1),(2.2) to (2.23),(2.24) by means of the transformation in Section 2 is the unstable manifold given by Theorem 2.2 for this particular case. This is very much like the travelling wave case where the center manifold of the equilibrium is given by the translates of the equilibrium.

4 Travelling waves for a two-phase problem

In the papers [5], [6] we have studied the stability of the planar travelling wave solutions to a free boundary problem with jump conditions at the interface defined by $\zeta = \xi(t, y)$,

$$\begin{cases} u_t(t,\zeta,y) = \Delta u(t,\zeta,y) + u(t,\zeta,y) u_\zeta(t,\zeta,y), \ \ t \ge 0,\ \zeta \ne \xi(t,y),\ y \in \omega, \\ u(t,\xi(t,y),y) = u_*, \ \ [\partial u/\partial n](t,\xi(t,y),y) = -1, \ \ t \ge 0,\ y \in \omega, \\ \partial\xi/\partial n(t,y) = 0, \ \ t \ge 0,\ y \in \partial\omega, \\ \partial u/\partial n(t,\zeta,y) = 0, \ \ t \ge 0,\ \zeta \ne \xi(t,y),\ y \in \partial\omega, \\ u(t,-\infty,y) = 0, \ \ u(t,\infty,y) = u_\infty, \ \ t \ge 0, \ \ y \in \omega, \end{cases} \tag{4.1}$$

where $\omega \subset \mathbb{R}^{N-1}$ is a bounded open set with regular boundary $\partial\omega$, and u_*, u_∞ are given positive numbers. By $[f](\xi)$ we mean $f(\xi^+) - f(\xi^-)$ at a point of possible discontinuity ξ.

N-dimensional TW solutions are special solutions of (4.1), such that $\xi(t, y) = -ct + s(y)$, $u(t, \zeta, y) = U(\zeta + ct, y)$. Therefore, setting $z = \zeta + ct$, the triplet (c, s, U) satisfies

$$\begin{cases} cU_z(z,y) = \Delta U(z,y) + U(z,y)U_z(z,y), \ \ z \neq s(y), \ y \in \omega, \\ U(s(y),y) = u_*, \ \ [\partial U/\partial n](s(y),y) = -1, \ \ y \in \omega, \\ \partial U/\partial n(x,y) = 0, \ \ x \neq s(y), \ y \in \partial\omega, \\ \partial s/\partial n(y) = 0, \ \ y \in \partial\omega, \\ U(-\infty,y) = 0, \ \ U(+\infty,y) = u_\infty, \ \ y \in \omega. \end{cases} \tag{4.2}$$

For certain values of the parameters (precisely, for $u_\infty > \sqrt{2}$, $2/u_\infty < u_* < 2/u_\infty + u_\infty$) problem (4.2) admits a (unique up to translations) $1D$ or "planar" TW solution $c = c_0$, $s \equiv 0$, $U(z, y) \equiv U_0(y)$. It is important to remark that the speed c_0 is independent of u_* and it is given by

$$c_0 = \frac{u_\infty}{2} + \frac{1}{u_\infty}, \tag{4.3}$$

while U_0 satisfies

$$\begin{cases} c_0 U_0'(z) = U_0''(z) + U_0(z)U_0'(z), \ \ z \neq 0, \\ U_0(0) = u_*, \ \ [U_0'](0) = -1, \\ U_0(-\infty) = 0, \ \ U_0(+\infty) = u_\infty. \end{cases} \tag{4.4}$$

This planar TW is orbitally stable under small (suitably smooth) perturbations iff

$$2/u_\infty < u_* \leq u_*^c = u_\infty/2 + 1/u_\infty + \sqrt{(u_\infty/2 + 1/u_\infty)^2 - 1}.$$

The instability for $u_* > u_*^c$ is due to a positive eigenvalue $\widetilde{\lambda}$ of the linear operator L associated to (4.1), such that $\lim_{u_* \downarrow u_*^c} \widetilde{\lambda} = +\infty$. Note that $\widetilde{\lambda}$ does not cross the imaginary axis as u_* crosses u_*^c.

We prove in [1] that in dimension $N = 2$ there exists a sequence of bifurcation points $u_*^k \downarrow u_*^c$ giving rise to branches of nonplanar travelling waves (c, s, U) bifurcating from the "trivial branch" $(c_0, 0, U_0)$. Their speeds c depend on the corrugation of their fronts: if $\omega = (-1, 1)$ then

$$c = c_0 + \frac{1}{2u_\infty} \int_{-1}^{1} \left(\sqrt{(1 + (s'(y))^2)} - 1 \right) dy > c_0. \tag{4.5}$$

The sequence u_*^k is determined by the relation $\widetilde{\lambda}(u_*^k) = \lambda_k$, where $-\lambda_k = -k^2\pi^2/4$ is the ordered sequence of the negative eigenvalues of the second order derivative in $(-1, 1)$ associated with the Neumann boundary condition.

The fact that a sequence of bifurcation points accumulates at u_*^c contributes to the understanding of the sharp instability phenomenon occurring at $u_* = u_*^c$.

The proof is based on two main tricks: first, like in Section 2 about the time depending case, we are able to eliminate the front s by an appropriate splitting of the unknown U; then another difficulty arises, namely at $u_* = u_*^k$, the kernel of the linearized operator around U_0 is $2D$ and its range has codimension 2. We take advantage of the translation invariance to reduce the dimension of the kernel, and simultaneously we eliminate c reducing the codimension of the range. So we turn our problem of bifurcation of fronts to a classical bifurcation problem from simple eigenvalues, which gives eventually infinitely many branches of nonplanar TW solutions.

References

[1] C.-M. Brauner & A. Lunardi: *Bifurcation of nonplanar travelling waves in a free boundary problem*, Preprint Math. Appl. Bordeaux, 97022 (1997).

[2] C.-M. Brauner, A. Lunardi & Cl. Schmidt-Lainé: *Multidimensional stability analysis of planar travelling waves*, Appl. Math. Letters **7** (1994), 1-4.

[3] C.M. Brauner, J. Hulshof & Cl. Schmidt-Lainé: *The saddle point property for focusing selfsimilar solutions of a combustion type free boundary problem: the radial case*, Proc. A.M.S., to appear.

[4] C.M. Brauner, J. Hulshof & A. Lunardi: *A general approach for stability in free boundary problems*, in preparation.

[5] C.M. Brauner, A. Lunardi & Cl. Schmidt-Lainé: *Stability of travelling waves with interface conditions*, Nonl. Anal. **19**, (1992) 455-474.

[6] C.-M. Brauner, A. Lunardi & Cl. Schmidt-Lainé: *Stability of travelling waves in a multidimensional free boundary problem*, Preprint Scuola Norm. Sup. Pisa 3 (1996).

[7] J.D. Buckmaster & G.S.S. Ludford: Theory of laminar flames, Cambridge University Press, Cambridge, 1982.

[8] D. S. Stewart & G. S. S. Ludford, *The acceleration of fast deflagration waves*, Z.A.M.M., **63** (1983), 291-302.

[9] V.A. Galaktionov, J. Hulshof & J.L. Vazquez: *Extinction and focusing behaviour of spherical and annular flames described by a free boundary problem,* Journal Math. Pures et Appl. **76**, (1997) 563-608.

[10] L. Glangetas & J.M. Roquejoffre: *Bifurcations of travelling waves in the thermo-diffusive model for flame propagation*, Arch. Rat. Mech. Anal., 134 (1996), pp. 341-402.

[11] D. Hilhorst & J. Hulshof: *An elliptic-parabolic problem in combustion theory: convergence to travelling waves*, Nonl. Anal. 17, pp 519-546, 1991

[12] D. Hilhorst & J. Hulshof: *A free boundary focusing problem,* Proc. AMS **121**, (1994) 1193-1202.

[13] A. Lunardi: Analytic Semigroups and Optimal Regularity in Parabolic Problems, Birkhäuser, Basel, 1995.

[14] J.L. Vazquez:*The Free Boundary Problem for the Heat Equation with fixed gradient condition,* Proc. Int. Conf. "Free Boundary Problems and Applications", Zakopane, Pitman Res. Notes Math. 363, Longman, 1996.