A general approach to stability in free boundary problems *

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Abstract

Every solution of a linear elliptic equation on a bounded domain may be considered as an equilibrium of a free boundary problem. The free boundary problem consists of the corresponding parabolic equation on a variable unknown domain with free boundary conditions prescribing both Dirichlet and Neumann data. We establish a rigorous stability analysis of such equilibria, including the construction of stable and unstable manifolds. For this purpose we transform the free boundary problem to a fully nonlinear and nonlocal parabolic problem on a fixed domain with fully nonlinear lateral boundary conditions and we develop the general theory for such problems. As an illustration we give two examples, the second being the focussing flame problem in combustion theory.

AMS Subject Classification. 35K55, 35K65, 80A25.

Key Words and Phrases. Free boundary problems, focussing selfsimilar solutions, linearisation, analytic semigroups, fully nonlinear equations, stable and unstable manifolds, saddle point property.

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*We are grateful for the support of the University of Bordeaux-1 and of the HCM-project "Nonlinear PDE-'s" (ERBCHRXCT 940-618). J. Hulshof was partially supported by the TMR network Nonlinear Parabolic Partial Differential Equations: Methods And Applications, ERBFMRXCT980201. We thank J. Simon for fruitful comments.

1 Introduction

In this paper we consider free boundary problems of the form

$$u_t = \mathcal{L}u + f, \quad x = (x_1, \ldots, x_N) \in \Omega_t, \quad t > 0, \tag{1.1}$$

with free boundary conditions

$$u = 0 \quad \text{and} \quad \frac{\partial u}{\partial n} = g \quad \text{on} \quad \partial\Omega_t. \tag{1.2}$$

Here Ω_t is a bounded domain in $\mathbb{R}^N$ with (free) boundary $\partial\Omega_t$, $\mathcal{L}$ is a uniformly elliptic operator on $\mathbb{R}^N$ with smooth time independent coefficients, and f and g are given smooth functions defined on the whole of $\mathbb{R}^N$. We are interested in the stability properties of equilibria of (1.1),(1.2). We assume that the pair (Ω, U) is a smooth equilibrium, namely

$$\mathcal{L}U + f = 0 \ \text{ in } \Omega \ \text{ with } U = 0 \text{ and } \frac{\partial U}{\partial n} = g \ \text{ on } \partial\Omega, \tag{1.3}$$

with Ω bounded. We consider (1.1),(1.2) with initial data Ω_0 and u_0 close to Ω and U, in a sense to be made precise below. The only structural assumption is that g does not vanish on $\partial\Omega$, which we think of as a transversality condition. A simple example is

$$\Delta U = 1 \ \text{ in } \Omega \ \text{ with } U = 0 \text{ and } \frac{\partial U}{\partial n} = 1 \ \text{ on } \partial\Omega. \tag{1.4}$$

By a suitable change of coordinates, Problem (1.1),(1.2) is reduced to a problem in the fixed domain $\Omega \times [0, \infty)$. Then, using a generalisation of methods derived in the framework of travelling waves (see [4],[5],[6]), we get that the local behaviour of solutions near the equilibrium (Ω, U) is determined by the operator L defined by

$$Lv = \mathcal{L}v \ \text{ for } \ v \in D(L) = \{v \in \cap_{p>1} W^{2,p}(\Omega),\, \mathcal{L}v \in C(\overline{\Omega}),\, \mathcal{B}v = 0 \ \text{ on } \partial\Omega\}, \tag{1.5}$$

where

$$\mathcal{B}v = \frac{\partial v}{\partial n} + \frac{1}{g}\left(\frac{\partial g}{\partial n} - \frac{\partial^2 U}{\partial n^2}\right)v. \tag{1.6}$$

This boundary operator is strongly reminiscent of a 'Hadamard formula' derived in [19] in the context of shape optimisation problems.

More precisely, we set

$$u = U + \Phi \cdot \nabla U + w, \tag{1.7}$$

where Φ measures the local difference between the free boundary $\partial\Omega_t$ and the fixed boundary $\partial\Omega$, and will be defined in Section 2. We emphasise that the unknown Φ does not appear in the final formulation: Problem (1.1),(1.2) is rewritten for w as the fully nonlinear problem

$$w_t = \mathcal{L}w + F(w, Dw, D^2w) \ \text{ on } \Omega, \quad t > 0, \tag{1.8}$$

with boundary conditions also fully nonlinear,

$$\mathcal{B}w = G(w, Dw) \ \text{ on } \ \partial\Omega, \tag{1.9}$$

where G only depends on w and its tangential derivatives. The price to be paid for changing the free boundary problem to a fixed boundary problem are the nonlocal and fully nonlinear terms F and G.

As initial datum for w we take $w(x, 0) = w_0(x)$ for $x \in \overline{\Omega}$, with $w_0 \in C^{2+\alpha}(\overline{\Omega})$ close to 0 in this norm, satisfying the compatibility condition $\mathcal{B}w_0 = \mathcal{G}(w_0, Dw_0)$ on $\partial\Omega$. This will correspond to initial data Ω_0 and u_0 close (in the new coordinates) to Ω and U in the $C^{2+\alpha}$-norm, satisfying the two compatibility conditions

$$u_0 = 0 \quad \text{and} \quad \frac{\partial u_0}{\partial n} = g \quad \text{on} \quad \partial\Omega_0. \tag{1.10}$$

Thus, near an equilibrium, the free boundary problem is equivalent to a problem on a fixed domain: the solution of (1.1),(1.2) with initial data u_0 and Ω_0 can be retrieved from the solution of (1.8),(1.9) with initial datum w_0. Our first result concerns the local existence.

Theorem 1.1 *For every $T > 0$ there are r, $\rho > 0$ such that Problem (1.8)-(1.9) has a solution $w \in C^{1+\alpha/2,2+\alpha}([0,T] \times \overline{\Omega})$ if $\|w_0\|_{C^{2+\alpha}(\overline{\Omega})} \le \rho$. Moreover w is the unique solution in $B(0,r) \subset C^{1+\alpha/2,2+\alpha}([0,T] \times \overline{\Omega})$.*

The reformulation of (1.1),(1.2) as the fully nonlinear problem (1.8),(1.9) with initial datum w_0 also enables us to perform a stability analysis of the free boundary problem. The stability question for the original equilibrium (1.3) is rephrased as the stability of $w = 0$ for Problem (1.8),(1.9).

Theorem 1.2 *If all elements of the spectrum $\sigma(L)$ of the operator L have negative real part then $w = 0$ is a stable equilibrium of Problem (1.8),(1.9). If $\sigma(L)$ contains elements with positive real part then $w = 0$ is unstable.*

For the description of the stable and unstable manifolds we need the spectral projections P^+ and P^- associated to the subsets of $\sigma(L)$ with positive and negative real parts.

Theorem 1.3 *Assume that $\sigma(L)$ contains elements with positive real part.*
(i) There exists a unique local unstable manifold of the form

$$\varphi : B(0, r_0) \subset P^+(C^{2+\alpha}(\overline{\Omega})) \to (I - P^+)(C^{2+\alpha}(\overline{\Omega})),$$

φ Lipschitz continuous, differentiable at 0 with $\varphi'(0) = 0$.
(ii) There exist a unique local stable manifold of the form

$$\psi : B(0, r_1) \subset \{w_0 \in P^-(C^{2+\alpha}(\overline{\Omega})) : \mathcal{B}w_0 = G(w_0(\cdot))\} \to (I - P^-)(C^{2+\alpha}(\overline{\Omega})),$$

ψ Lipschitz continuous, differentiable at 0 with $\psi'(0) = 0$.

We note that in the special case that

$$\sigma(L) \cap i\mathbb{R} = \emptyset, \tag{1.11}$$

Theorem 1.3 is a generalisation of the classical saddle point theorem.

Our investigations are motivated by the mathematical modelling of combustion [16], where typically at the free boundary separating the fresh and the burnt regions one encounters jump conditions involving the normal derivative $\partial u/\partial n$. In particular we consider the model problem (see the survey [21])

$$u_t = \Delta u \ \text{ in } \Omega_t, \quad u = 0 \quad \text{and} \quad \frac{\partial u}{\partial n} = 1 \ \text{ on } \partial\Omega_t. \tag{1.12}$$

In the case of bounded domains, (1.12) does not allow nontrivial equilibria. It is called a focussing problem because typically the domain Ω_t vanishes in finite time, a behaviour exhibited by selfsimilar solutions. Transforming to selfsimilar variables the problem can be rewritten as

$$u_t = \mathcal{L}u = \Delta u - \frac{1}{2}x \cdot \nabla u + \frac{1}{2}u, \quad x \in \Omega_t \tag{1.13}$$

with the same boundary conditions. The stability analysis of the equilibrium corresponding to the selfsimilar profile fits in the framework presented above.

The paper is organised as follows. In Section 2 we describe the general method to transform (1.1),(1.2) into (1.8),(1.9). In fact, we consider a slightly more general problem, namely

$$u_t = \mathcal{L}u + f, \quad x = (x_1, \ldots, x_N) \in \Omega_t, \quad t > 0, \tag{1.14}$$

with

$$u = g_1 \quad \text{and} \quad \frac{\partial u}{\partial n} = g_2 \quad \text{on} \quad \partial\Omega_t. \tag{1.15}$$

In Section 3 we give the extension of the local existence and saddle point theorems in [17] to the case of (1.8),(1.9). This is needed because unlike in [3], where we had a linear boundary condition, we now arrive at the fully nonlinear boundary condition (1.9) and therefore a significant part of this paper will deal with adapting the functional analytic framework to this case. The proofs are based on fixed point arguments and rely on results for the asymptotic behaviour of linear inhomogeneous problems which we give in the appendix. In Section 4 we describe the application to the two examples mentioned above.

2 The general procedure for linearisation

We consider Problem (1.1),(1.15) from the introduction, where $\mathcal{L}$ is a uniformly elliptic operator with smooth coefficients, and f, g_1 and g_2 are given smooth functions defined on the whole of $\mathbb{R}^N$. We assume that the pair (Ω, U), with Ω bounded, $\partial\Omega$ and U smooth, is an equilibrium:

$$\mathcal{L}U + f = 0 \ \text{ in } \Omega \ \text{ with } U = g_1 \text{ and } \frac{\partial U}{\partial n} = g_2 \ \text{ on } \partial\Omega. \tag{2.1}$$

The structural assumption we need is that

$$\frac{\partial g_1}{\partial n} - g_2 \neq 0 \ \text{ at } \ \partial\Omega. \tag{2.2}$$

Thus, obstacle type problem are excluded.

The method consists in

- fixing the domain by transforming to new independent variables $\xi \in \Omega$ and τ;
- linearising around U in the new variables;
- rewriting the problem as a fully nonlinear problem in Ω.

2.1 Fixing the domain

We transform the problem on the variable domain Ω_t (space variable x, time variable t) to a problem on the fixed domain Ω (space variable ξ, time variable τ). The normal to $\partial\Omega_t$ being denoted by n, we will use ν for the normal on $\partial\Omega$.

For $\delta > 0$ we define a map

$$X : \partial\Omega \times [-\delta, \delta] \to \mathbb{R}^N, \quad X(\phi, r) = \phi + r\nu(\phi). \tag{2.3}$$

If δ is sufficiently small, then (2.3) defines a bijection to a compact neighbourhood $N(\partial\Omega)$ of $\partial\Omega$. In $N(\partial\Omega)$ every ξ can be written in a unique way as $\xi = X(\phi, r)$ with $\phi \in \partial\Omega$ and $r \in [-\delta, \delta]$. For convenience we shall write ξ' for ϕ. Clearly, if $\xi \in \partial\Omega$ then $\xi = \xi'$.

We will look for Ω_t close to Ω in some time interval I in the sense that $\partial\Omega_t$ will be given by

$$\partial\Omega_t = \{x = \xi' + s(\xi', t)\nu(\xi'),\, \xi' \in \partial\Omega\}, \tag{2.4}$$

where $s : \partial\Omega \times I \to [-\delta, \delta]$ is a smooth function which is an unknown of the problem. In what follows it will be convenient to write, for $\xi \in \partial\Omega$,

$$\Phi(\xi, t) = s(\xi, t)\nu(\xi). \tag{2.5}$$

We extend this field to the whole of $\mathbb{R}^N$ by setting

$$\Phi(\xi, t) = \begin{cases} \alpha(\xi)s(\xi', t)\nu(\xi') & \text{if } \xi \in N(\partial\Omega), \\ \\ 0 \quad \text{otherwise} . \end{cases} \tag{2.6}$$

Here $\alpha : \mathbb{R}^N \to [0, 1]$ is a smooth mollifier which is equal to 1 near $\partial\Omega$ and has compact support in the interior of $N(\partial\Omega)$. Thus, the field Φ is localised near $\partial\Omega$. It contains all the information about the difference between the fixed boundary $\partial\Omega$ and the free boundary $\partial\Omega_t$.

Now it is clear how to transform the problem on the variable domain Ω_t to a problem on the fixed domain Ω. We define a (bijective) coordinate transformation

$$x = \xi + \Phi(\xi, \tau),\, t = \tau. \tag{2.7}$$

From (2.7) and (2.5) we see that ∇_x and $\frac{\partial}{\partial t}$ transform as

$$\nabla_x = (I + A)^{-1}\nabla_\xi, \quad \frac{\partial}{\partial t} = \hat{D}_\tau = \frac{\partial}{\partial \tau} - \frac{\partial \Phi}{\partial \tau} \cdot (I + A)^{-1}\nabla_\xi, \tag{2.8}$$

where

$$A = A(\xi, \tau) = \nabla_\xi \Phi(\xi, \tau) \tag{2.9}$$

is the column gradient of Φ, i.e. the transposed Jacobian matrix of Φ (with respect to the ξ-variables only). The normal n on $\partial\Omega_t$ is related to the normal ν on $\partial\Omega$ by

$$n = \frac{(I + A)^{-1}\nu}{|(I + A)^{-1}\nu|}. \tag{2.10}$$

2.2 Expansion in the new variables

The transformation of Ω_t to Ω also acts on the equilibrium U itself and this has to be taken into account when we expand u near U in the new variables. It is convenient to extend U to a smooth function defined in the whole of $\mathbb{R}^N$. The extension of U outside $\overline{\Omega}$ will not appear in the final formulation. Observing that

$$U(\xi + \Phi(\xi, \tau)) = U(\xi) + (\nabla_\xi U(\xi)) \cdot \Phi(\xi, \tau) + R(\xi, \Phi(\xi, \tau)), \tag{2.11}$$

where here and in what follows $R(\xi, \Phi)$ is a nonspecified smooth function bounded by a multiple of $|\Phi|^2$, we are led to look for a solution the form

$$u(x, t) = \hat{u}(\xi, \tau) = U(\xi) + (\nabla_\xi U(\xi)) \cdot \Phi(\xi, \tau) + w(\xi, \tau). \tag{2.12}$$

This non-standard formula is crucial: it will allow us to rewrite the free boundary problem as a problem for w.

First we rewrite the action of $\mathcal{L}$ on u as

$$(\mathcal{L}u)(x, t) = (\hat{\mathcal{L}}\hat{u})(\xi, \tau) = (\hat{\mathcal{L}}_0\hat{u})(\xi, \tau) + (\hat{\mathcal{L}}_1\hat{u})(\xi, \tau) + (\hat{\mathcal{L}}_2^r\hat{u})(\xi, \tau), \tag{2.13}$$

where $\hat{\mathcal{L}}_0$ is the original operator $\mathcal{L}$ with x replaced by ξ and $\hat{\mathcal{L}}_1$ is an operator with coefficients linear in Φ and its first and second order ξ-derivatives. The remainder term $\hat{\mathcal{L}}_2^r$ has coefficients bounded by a multiple of $|\Phi|^2 + |D_\xi \Phi|^2 + |D_\xi^2 \Phi|^2$. This is easily seen from the expansion

$$(I + A)^{-1} = I - A + A^2(I + A)^{-1}, \quad A = \nabla_\xi \Phi. \tag{2.14}$$

Likewise we get an expansion for the time derivative which reads

$$\frac{\partial u(x,t)}{\partial t} = \hat{D}_\tau \hat{u}(\xi,\tau) = \frac{\partial \hat{u}}{\partial \tau} - \frac{\partial \Phi}{\partial \tau} \cdot \nabla_\xi \hat{u} + \frac{\partial \Phi}{\partial \tau} \cdot A(I + A)^{-1} \nabla_\xi \hat{u}. \tag{2.15}$$

Equation (1.1) for u thus transforms into an equation for $\hat{u}$,

$$(D_\tau - \hat{\mathcal{L}})(\hat{u}) = f(\xi + \Phi(\xi,\tau)) = f(\xi) + (\nabla_\xi f(\xi)) \cdot \Phi(\xi,\tau) + R(\xi, \Phi(\xi,\tau)), \tag{2.16}$$

in which we substitute (2.12). Next we expand the resulting equation using (2.13) and (2.15). We observe beforehand that the terms of zero and first order in $\Phi(\xi,\tau)$ and its derivatives coming from the first two terms of (2.12) must be equal on both sides and arrive at

$$\frac{\partial w}{\partial \tau} - \hat{\mathcal{L}}_0 w = \Big(\frac{\partial \Phi}{\partial \tau} \cdot \nabla_\xi + \hat{\mathcal{L}}_1 \Big) w + \Big(- \frac{\partial \Phi}{\partial \tau} \cdot A(I + A)^{-1} \nabla_\xi + \hat{\mathcal{L}}_2^r \Big) w \tag{2.17}$$

$$+ \Big(\frac{\partial \Phi}{\partial \tau} \cdot \nabla_\xi + \hat{\mathcal{L}}_1 \Big) (\Phi \cdot \nabla_\xi U) + \Big(- \frac{\partial \Phi}{\partial \tau} \cdot A(I + A)^{-1} \nabla_\xi + \hat{\mathcal{L}}_2^r \Big) (U + \Phi \cdot \nabla_\xi U) + R(\xi, \Phi(\xi,\tau)).$$

The linear part (the left hand side) is now in the form we want, but the higher order terms contain derivatives of both Φ and w, including $\frac{\partial \Phi}{\partial \tau}$. Equation (2.17) is of the type

$$\frac{\partial w}{\partial \tau} - \hat{\mathcal{L}}_0 w = \mathcal{F}_1(\xi, w, Dw, D^2 w, \Phi, D\Phi, D^2\Phi) + \frac{\partial \Phi}{\partial \tau} \cdot \mathcal{F}_2(\xi, Dw, \Phi, D\Phi), \tag{2.18}$$

with $\mathcal{F}_1$ consisting of second and higher order terms and $\mathcal{F}_2$ consisting of first and higher order terms in the w, Φ-dependent arguments.

To eliminate Φ and its derivatives from (2.18) we have to transform the free boundary conditions.

2.3 Expansion on the boundary: new boundary conditions

At the boundary the expansion (2.12) reduces to, using (1.15) and (2.5),

$$g_1(\xi + s(\xi,\tau)\nu(\xi)) = g_1(\xi) + s(\xi,\tau) g_2(\xi) + w(\xi,\tau), \quad \xi \in \partial\Omega. \tag{2.19}$$

Expanding the left hand side this gives

$$s(\xi,\tau) \Big(\frac{\partial g_1}{\partial \nu}(\xi) - g_2(\xi) \Big) + R(\xi, s(\xi,\tau)) = w(\xi,\tau), \tag{2.20}$$

where $R(\xi, s)$ is as before a nonspecified smooth function bounded by a multiple of s^2. In view of the structural assumption (2.2) we may invert this equation for w small to obtain

$$s(\xi,\tau) = \frac{w(\xi,\tau)}{\frac{\partial g_1}{\partial \nu}(\xi) - g_2(\xi)} + R(\xi, w(\xi,\tau)), \tag{2.21}$$

where $R(\xi, w)$ is again smooth and bounded by a multiple of w^2. In the special case that g_1 is constant this term is zero. Note that w and s (or Φ) are of the same (small) size. More importantly, (2.20) will allow us to decouple the system and obtain a problem for w only.

To transform the second free boundary condition in (1.15) we have to expand (2.10) using (2.14). We find, since

$$A\nu = (\nabla_\xi \Phi)\nu = \nabla_\xi s = \nabla_\xi^{tang} s \tag{2.22}$$

(not to be confused with $(D_\xi \Phi)\nu = 0$), whence $A\nu \cdot \nu = 0$,

$$n = \nu - \nabla_\xi s + R(A). \tag{2.23}$$

The vector valued function $R(A) = R(\xi, s, \nabla_\xi s)$ is bounded by a multiple of $||A||^2$ and hence, in view of (2.6) and (2.9), by a multiple of $s^2 + |\nabla_\xi s|^2$.

For the normal derivative we then have

$$\frac{\partial}{\partial n} = n \cdot \nabla_x = \nu \cdot \nabla_\xi - \nabla_\xi s \cdot \nabla_\xi + \nabla_\xi s \cdot A(I+A)^{-1}\nabla_\xi + R(A) \cdot (I+A)^{-1}\nabla_\xi,$$

which we can rewrite as

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial \nu} - \nabla_\xi^{tang} s \cdot \nabla_\xi^{tang} + B(\xi, s, \nabla_\xi s)\nabla_\xi, \tag{2.24}$$

with B a smooth matrix valued function bounded by a multiple of $s^2 + |\nabla_\xi s|^2$.

Applying (2.24) to (2.12), expanding $g_2(\xi + \Phi(\xi, \tau))$ as usual, we arrive at, omitting the subscripts ξ,

$$\frac{\partial w}{\partial \nu} + s\left(\frac{\partial^2 U}{\partial \nu^2} - \frac{\partial g_2}{\partial \nu}\right) - \nabla^{tang} s \cdot \nabla^{tang} g_1 = B(\xi, s, \nabla s)\nabla w + R(\xi, s, \nabla s), \tag{2.25}$$

with a slightly different matrix B. The derivation of this boundary conditon for w is independent of the partial differential equation to be satisfied in the interior. What we have done here is related to methods in domain optimisation problems where solutions of boundary value problems are differentiated with respect to the domain. See [18, 19, 10, 8].

2.4 The fully nonlinear problem

We recall that we have derived (2.18) for ξ in Ω and equations (2.20) and (2.25) for ξ on the boundary $\partial\Omega$. Note that (2.20) is equivalent to (2.21). Now we can eliminate the free boundary terms s, Φ and their derivatives from the problem.

We take the restriction of equation (2.18) to the boundary $\partial\Omega$ (where $\Phi(\xi, \tau) = s(\xi, \tau)\nu(\xi)$) and rewrite it as

$$\frac{\partial w}{\partial \tau} - \hat{\mathcal{L}}_0 w = \mathcal{F}_1(\xi, w, Dw, D^2 w, s, Ds, D^2 s) + \frac{\partial s}{\partial \tau}\mathcal{F}_2(\xi, Dw, s, Ds), \tag{2.26}$$

where now $\mathcal{F}_1$ and $\mathcal{F}_2$ are considered to be depending on s rather than on Φ. In (2.26) we use (2.21) to express s, Ds and $D^2 s$ in terms of w, Dw, $D^2 w$ and ξ. The important point is the elimination of $\frac{\partial s}{\partial \tau}$. For this we use (2.20) which we first differentiate with respect to τ to obtain

$$\frac{\partial w}{\partial \tau} - \left(\frac{\partial g_1}{\partial \nu} - g_2\right)\frac{\partial s}{\partial \tau} = R_s(\xi, s)\frac{\partial s}{\partial \tau} = \tilde{R}(\xi, w)\frac{\partial s}{\partial \tau}. \tag{2.27}$$

The partial derivative $R_s(\xi, s)$ is a smooth function bounded by a multiple of s and thus, using (2.21) again, $\tilde{R}(\xi, w)$ is smooth and bounded by a multiple of w. Using (2.27) to eliminate $\frac{\partial w}{\partial \tau}$ we obtain

$$\left(\frac{\partial g_1}{\partial \nu} - g_2 + \tilde{R}(\xi, w)\right)\frac{\partial s}{\partial \tau} - \hat{\mathcal{L}}_0 w = \mathcal{F}_1(\xi, w, Dw, D^2 w) + \frac{\partial s}{\partial \tau}\mathcal{F}_2(\xi, w, Dw, D^2 w), \tag{2.28}$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ are the same as before, with the s-dependence absorbed in w-dependence. Therefore, provided w, Dw and D^2w are small,

$$\frac{\partial s}{\partial \tau} = \frac{\hat{\mathcal{L}}_0 w + \mathcal{F}_1(\xi, w, Dw, D^2w)}{\frac{\partial g_1}{\partial \nu} - g_2 + \tilde{R}(\xi, w) - \mathcal{F}_2(\xi, w, Dw, D^2w)}. \tag{2.29}$$

We emphasise that (2.29) holds for ξ at the boundary $\partial\Omega$ only. Returning to (2.18) and using (2.6) we have for $\xi \in \overline{\Omega} \cap N(\partial\Omega)$ (where $\alpha(\xi)$ is supported) that the equation for w becomes of the form

$$\frac{\partial w}{\partial \tau} - \hat{\mathcal{L}}_0 w = \mathcal{F}(\xi, w(\xi,\tau), Dw(\xi,\tau), D^2w(\xi,\tau), w(\xi',\tau), Dw(\xi',\tau), D^2w(\xi',\tau)), \tag{2.30}$$

with $\mathcal{F}$ smooth, quadratic in the w-dependent variables. We recall that $\xi' \in \partial\Omega$ is the projection of $\xi \in N(\partial\Omega)$ on the boundary, see the beginning of Section 2.1. Thus the right hand side of (2.31) is fully nonlinear and nonlocal. Note that the right hand side, which contains the nonlocal terms, may be nonzero only for $\xi \in \overline{\Omega} \cap N(\partial\Omega)$, where $\alpha(\xi)$ is supported. Elsewhere in Ω it vanishes identically. Therefore, the final equation for w is

$$\frac{\partial w}{\partial \tau} - \hat{\mathcal{L}}_0 w = \begin{cases} \mathcal{F}(\xi, w(\xi,\tau), Dw(\xi,\tau), D^2w(\xi,\tau), w(\xi',\tau), Dw(\xi',\tau), D^2w(\xi',\tau)) \\ \text{for } \ \xi \in \overline{\Omega} \cap N(\partial\Omega), \ \tau \geq 0, \\ \\ 0 \quad \text{for } \ \xi \in \overline{\Omega} \setminus N(\partial\Omega), \ \tau \geq 0. \end{cases} \tag{2.31}$$

The boundary condition to be satisfied by w follows directly from (2.20), (2.21) and (2.25). It comes out as

$$\mathcal{B}w = \frac{\partial w}{\partial \nu} + \frac{w}{\frac{\partial g_1}{\partial \nu} - g_2}\left(\frac{\partial^2 U}{\partial \nu^2} - \frac{\partial g_2}{\partial \nu}\right) - \nabla^{tang}\left(\frac{w}{\frac{\partial g_1}{\partial \nu} - g_2}\right) \cdot \nabla^{tang} g_1 = \mathcal{G}(\xi, w, Dw), \tag{2.32}$$

In the special case that $g_1 \equiv 0$ and $g_2 = g$ the boundary operator defined by (2.32) reduces to $\mathcal{B}w$ defined by (1.6).

Now we have found a fully nonlinear problem for w, namely (2.31),(2.32), with initial datum $w(\xi, 0) = w_0(\xi)$ determined by Ω_0 and u_0 via (2.4) and (2.12). Since u_0 is assumed to satisfy the boundary conditions (1.15) at $t = 0$, it follows that w_0 satisfies $\mathcal{B}w_0 = \mathcal{G}(\cdot, w_0, Dw_0)$ at $\partial\Omega$. We shall solve this initial boundary value problem for w provided w_0 is sufficiently small by means of the general results in Section 3.

3 General theory

Problem (2.31),(2.32) is of the type

$$\begin{cases} w_t(\xi, t) = \mathcal{L}w + F(w(\cdot, t))(\xi), \ \ \xi \in \overline{\Omega}, \\ \\ \mathcal{B}w = G(w(\cdot, t))(\xi), \ \ \xi \in \partial\Omega, \end{cases} \tag{3.1}$$

with initial condition given by

$$w(\xi, 0) = w_0(\xi), \ \ \xi \in \overline{\Omega}, \tag{3.2}$$

where Ω is a bounded open set in $\mathbb{R}^N$ with regular boundary $\partial\Omega$, F and G are regular functions defined in a neighbourhood of 0 in $C^2(\overline{\Omega})$ with values in $C(\overline{\Omega})$ and $C^1(\partial\Omega)$ respectively and $\mathcal{L}$ and $\mathcal{B}$ are linear differential operators with regular coefficients. Moreover,

$$F(0) = 0, \ F'(0) = 0, \ \ G(0) = 0, \ G'(0) = 0,$$

so that $w \equiv 0$ is a solution of Problem (3.1) and the linearisation of (3.1) around the null solution is $w_t = \mathcal{L}w$ with boundary condition $\mathcal{B}w = 0$.

Below we state precise regularity assumptions for the local existence and uniqueness of a regular solution to (3.1),(3.2) and the construction of the stable and unstable manifolds of the null solution. Such assumptions are easily seen to be satisfied in the case of Problem (2.31),(2.32).

H1. Ω is a bounded open set in $\mathbb{R}^N$ with $C^{2+\alpha}$ boundary, $0 < \alpha < 1$.

H2. $F : B(0,R) \subset C^2(\overline{\Omega}) \to C(\overline{\Omega})$ is continuously differentiable with Lipschitz continuous derivative, $F(0) = 0$, $F'(0) = 0$ and the restriction of F to $B(0,R) \subset C^{2+\alpha}(\overline{\Omega})$ has values in $C^{\alpha}(\overline{\Omega})$ and is continuously differentiable; $G : B(0,R) \subset C^1(\overline{\Omega}) \to C(\partial\Omega)$ is continuously differentiable with Lipschitz continuous derivative, $G(0) = 0$, $G'(0) = 0$ and the restriction of G to $B(0,R) \subset C^{2+\alpha}(\overline{\Omega})$ has values in $C^{1+\alpha}(\partial\Omega)$ and is continuously differentiable too.

H3. $\mathcal{L} = a_{ij}D_{ij} + b_iD_i + c$ is a uniformly elliptic operator with coefficients a_{ij}, b_i, c in $C^{\alpha}(\overline{\Omega})$ and $\mathcal{B} = \beta_iD_i + \gamma$ is a nontangential operator with coefficients β_i, γ in $C^{1+\alpha}(\partial\Omega)$.

H4. $w_0 \in C^{2+\alpha}(\overline{\Omega})$ satisfies the compatibility condition

$$\mathcal{B}w_0 = G(w_0(\cdot))(\xi), \quad \xi \in \partial\Omega. \tag{3.3}$$

A local existence and uniqueness result for Problem (3.1),(3.2) may be shown using a standard linearisation procedure, which works also in the fully nonlinear case thanks to optimal regularity results and estimates for linear problems. It gives existence of the solution for arbitrarily long time intervals, provided the initial datum is small enough. This may be seen as continuous dependence of the solution on the initial datum at $w_0 = 0$.

We shall use the functional spaces $C^{\alpha,\alpha/2}(\overline{\Omega}\times I)$, $C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times I)$, $C^{1+\alpha,1/2+\alpha/2}(\partial\Omega\times I)$, I being a real interval, with the usual meanings and norms. We recall that a function w belongs to $C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times I)$ if and only if $t \to w(\cdot,t)$ is in $C^{1+\alpha/2}(I; C(\overline{\Omega})) \cap B(I; C^{2+\alpha}(\overline{\Omega}))$ (B stands for bounded) and there is $K_\alpha > 0$ such that

$$\|w\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times I)} \le K_\alpha(\|w\|_{C^{1+\alpha/2}(I;C(\overline{\Omega}))} + \|w\|_{B(I;C^{2+\alpha}(\overline{\Omega}))}).$$

Similarly, $v \in C^{1+\alpha,1/2+\alpha/2}(\partial\Omega \times I)$ if and only if $t \to v(\cdot,t)$ is in $C^{1/2+\alpha/2}(I; C(\partial\Omega)) \cap B(I; C^{1+\alpha}(\partial\Omega))$ and there is $C_\alpha > 0$ (independent of I) such that

$$\|v\|_{C^{1+\alpha,1/2+\alpha/2}(\partial\Omega\times I)} \le C_\alpha(\|v\|_{C^{1/2+\alpha/2}(I;C(\partial\Omega))} + \|v\|_{B(I;C^{1+\alpha}(\partial\Omega))}).$$

Theorem 3.1 *Under the above assumptions, for every $T > 0$ there are r, $\rho > 0$ such that (3.1),(3.2) has a solution $w \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0,T])$ if $\|w_0\|_{C^{2+\alpha}(\overline{\Omega})} \le \rho$. Moreover w is the unique solution in $B(0,r) \subset C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0,T])$.*

Proof — Let $0 < r \le R$, and set

$$K(r) = \sup\{\|F'(\varphi)\|_{L(C^{2+\alpha}(\overline{\Omega}),C^{\alpha}(\overline{\Omega}))} : \varphi \in B(0,r) \subset C^{2+\alpha}(\overline{\Omega})\},$$

$$H(r) = \sup\{\|G'(\varphi)\|_{L(C^{2+\alpha}(\overline{\Omega}),C^{1+\alpha}(\partial\Omega))} : \varphi \in B(0,r) \subset C^{2+\alpha}(\overline{\Omega})\}.$$

Since $F'(0) = 0$ and $G'(0) = 0$, $K(r)$ and $H(r)$ go to 0 as $r \to 0$. Let $L > 0$ be such that, for all φ, $\psi \in B(0,r) \subset C^2(\overline{\Omega})$ with small r,

$$\|F'(\varphi) - F'(\psi)\|_{L(C^2(\overline{\Omega}),C(\overline{\Omega}))} \le L\|\varphi - \psi\|_{C^2(\overline{\Omega})},$$

$$\|G'(\varphi) - G'(\psi)\|_{L(C^1(\overline{\Omega}),C(\partial\Omega))} \le L\|\varphi - \psi\|_{C^1(\overline{\Omega})}.$$

For every $0 \le s \le t \le T$ and for every $w \in B(0,r) \subset C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0,T])$ with r so small that $K(r), H(r) < \infty$, we have

$$\|F(w(\cdot,t))\|_{C^\alpha(\overline{\Omega})} \le K(r)\|w(\cdot,t)\|_{C^{2+\alpha}(\overline{\Omega})}, \quad \|F(w(\cdot,t)) - F(w(\cdot,s))\|_{C(\overline{\Omega})} \le$$

$$Lr\|w(\cdot,t) - w(\cdot,s)\|_{C^2(\overline{\Omega})} \le Lr|t-s|^{\alpha/2}\|w\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])},$$

and similarly

$$\|G(w(\cdot,t))\|_{C^{1+\alpha}(\partial\Omega)} \le H(r)\|w(\cdot,t)\|_{C^{2+\alpha}(\overline{\Omega})} \le H(r)\|w\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])},$$

$$\|G(w(\cdot,t)) - G(w(\cdot,s))\|_{C(\partial\Omega)} \le$$

$$Lr\|w(\cdot,t) - w(\cdot,s)\|_{C^1(\overline{\Omega})} \le Lr|t-s|^{1/2+\alpha/2}\|w\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])}.$$

Therefore, $(\xi,t) \to F(w(\cdot,t))(\xi)$ belongs to $C^{\alpha,\alpha/2}(\overline{\Omega}\times[0,T])$, $(\xi,t) \to G(w(\cdot,t))(\xi)$ belongs to $C^{1+\alpha,1/2+\alpha/2}(\partial\Omega \times [0,T])$ and

$$\|F(w)\|_{C^{\alpha,\alpha/2}(\overline{\Omega}\times[0,T])} \le (K(r) + Lr)\|w\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])},$$

$$\|G(w)\|_{C^{1+\alpha,1/2+\alpha/2}(\partial\Omega\times[0,T])} \le C_\alpha(2H(r) + Lr)\|w\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])}.$$

So, if $\|w_0\|_{C^{2+\alpha}(\overline{\Omega})}$ is small enough, we define a nonlinear map

$$\Gamma : \{w \in B(0,r) \subset C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0,T]) : w(\cdot,0) = w_0\} \to C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0,T]),$$

by $\Gamma w = v$, where v is the solution of

$$\begin{cases} v_t(x,t) = \mathcal{L}v + F(w(\cdot,t))(x), \quad 0 \le t \le T,\ x \in \overline{\Omega}, \\ \mathcal{B}v = G(w(\cdot,t))(x), \quad 0 \le t \le T,\ x \in \partial\Omega, \\ v(x,0) = w_0(x). \end{cases}$$

Actually, thanks to the compatibility condition $\mathcal{B}w_0 = G(w_0)$ and the regularity of $F(w)$ and $G(w)$, the range of Γ is contained in $C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0,T])$. Moreover, there is $C = C(T) > 0$, independent of r, such that (see e.g. [15, Thm. 5.3] or [17, Thm. 5.1.20])

$$\|v\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])} \le$$

$$C(\|w_0\|_{C^{2+\alpha}(\overline{\Omega})} + \|F(w)\|_{C^{\alpha,\alpha/2}(\overline{\Omega}\times[0,T])} + \|G(w)\|_{C^{1+\alpha,1/2+\alpha/2}(\partial\Omega\times[0,T])}),$$

so that

$$\|\Gamma(w)\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])} \le$$

$$C(\|w_0\|_{C^{2+\alpha}(\overline{\Omega})} + (K(r) + Lr + C_\alpha(2H(r) + Lr))\|w\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])}).$$

Therefore, if r is so small that

$$C(K(r) + Lr + C_\alpha(2H(r) + Lr)) \le 1/2, \tag{3.4}$$

and w_0 is so small that

$$\|w_0\|_{C^{2+\alpha}(\overline{\Omega})} \le Cr/2,$$

Γ maps the ball $B(0,r)$ into itself. Let us check that Γ is a $1/2$-contraction. Let w_1, $w_2 \in B(0,r)$, $w_i(\cdot,0) = w_0$. Writing $w_i(\cdot,t) = w_i(t)$, $i = 1,2$, we have

$$\|\Gamma w_1 - \Gamma w_2\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])} \leq$$

$$C(\|F(w_1) - F(w_2)\|_{C^{\alpha,\alpha/2}(\overline{\Omega}\times[0,T])} + \|G(w_1) - G(w_2)\|_{C^{1+\alpha,1/2+\alpha/2}(\partial\Omega\times[0,T])}),$$

and, arguing as above, for $0 \leq t \leq T$,

$$\|F(w_1(t)) - F(w_2(t))\|_{C^{\alpha}(\overline{\Omega})} \leq$$

$$K(r)\|w_1(t) - w_2(t)\|_{C^{2+\alpha}(\overline{\Omega})} \leq K(r)\|w_1 - w_2\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])},$$

$$\|G(w_1(t)) - G(w_2(t))\|_{C^{1+\alpha}(\partial\Omega)} \leq$$

$$H(r)\|w_1(t) - w_2(t)\|_{C^{2+\alpha}(\overline{\Omega})} \leq H(r)\|w_1 - w_2\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])},$$

while for $0 \leq s \leq t \leq T$

$$\|F(w_1(t)) - F(w_2(t)) - F(w_1(s)) - F(w_2(s))\|_{C(\overline{\Omega})} =$$

$$\Big\|\int_0^1 F'(\sigma w_1(t) + (1-\sigma)w_2(t)(w_1(\cdot,t) - w_2(t))$$

$$-F'(\sigma w_1(s) + (1-\sigma)w_2(s))(w_1(s) - w_2(s))d\sigma\Big\|_{C(\overline{\Omega})}$$

$$\leq \int_0^1 \|(F'(\sigma w_1(t) + (1-\sigma)w_2(t) - F'(\sigma w_1(s) + (1-\sigma)w_2(s))(w_1(t) - w_2(t))d\sigma\|_{C(\overline{\Omega})}$$

$$+\int_0^1 \|F'(\sigma w_1(s))(w_1(t) - w_2(t) - w_1(s) + w_2(s))\|_{C(\overline{\Omega})}$$

$$\leq \frac{L}{2}(\|w_1(t) - w_1(s)\|_{C^2(\overline{\Omega})} + \|w_2(t) - w_2(s)\|_{C^2(\overline{\Omega})})\|w_1(t) - w_2(t)\|_{C^2(\overline{\Omega})}$$

$$+Lr\|w_1(t) - w_2(t) - w_1(s) + w_2(s))\|_{C^2(\overline{\Omega})}$$

$$\leq 2Lr(t-s)^{\alpha/2}\|w_1 - w_2\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])},$$

and similarly

$$\|G(w_1(\cdot,t)) - G(w_2(\cdot,t)) - G(w_1(\cdot,s)) - G(w_2(\cdot,s))\|_{C(\partial\Omega)}$$

$$\leq 2Lr(t-s)^{1/2+\alpha/2}\|w_1 - w_2\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])}.$$

Therefore,

$$\|\Gamma w_1 - \Gamma w_2\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])}$$

$$\leq C(K(r) + Lr + C_\alpha(2H(r) + Lr))\|w_1 - w_2\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])}$$

$$\leq \frac{1}{2}\|w_1 - w_2\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])},$$

the last inequality being a consequence of (3.4). The statement follows. ∎

As in Section 1 we define the realisation of $\mathcal{L}$ with homogeneous boundary conditions in $X = C(\overline{\Omega})$ by

$$D(L) = \{v \in \cap_{p\geq 1} W^{2,p}(\Omega) : \mathcal{L}v \in X,\ \mathcal{B}v = 0 \text{ in } \partial\Omega\}, \tag{3.5}$$

$$Lv = \mathcal{L}v, \quad v \in D(L).$$

We note that, due to [20], L is a sectorial operator and since Ω is bounded, the resolvent $(\lambda I - L)^{-1}$ is a compact operator for every λ in the resolvent set $\rho(L)$. Therefore $\sigma(L)$ consists of a sequence of isolated eigenvalues.

We shall state the principle of linearised stability in terms of the spectrum $\sigma(L)$ of L.

Theorem 3.2 *Let Ω satisfy assumption* H1 *and let F, G, $\mathcal{L}$, $\mathcal{B}$ satisfy* H2, H3.

(i) If all the elements of $\sigma(L)$ have negative real part then $w = 0$ is a stable equilibrium of Problem (3.1) with respect to the $C^{2+\alpha}(\overline{\Omega})$ norm. More precisely, for every $\omega \in (0, \max\{\mathrm{Re}\,\lambda : \lambda \in \sigma(L)\})$, there are C, $r > 0$ such that for every w_0 satisfying H4 *and $\|w_0\|_{C^{2+\alpha}(\overline{\Omega})} \leq r$, the solution of (3.1) with initial datum w_0 exists in the large and satisfies*

$$\|w(\cdot,t)\|_{C^{2+\alpha}(\overline{\Omega})} \leq Ce^{-\omega t}\|w_0\|_{C^{2+\alpha}(\overline{\Omega})}, \quad t \geq 0.$$

(ii) If $\sigma(L)$ contains elements with positive real part then $w = 0$ is unstable in $C^{2+\alpha}(\overline{\Omega})$.

Proof — The proof of statement (i) follows the proof of Theorem 9.1.2 of [17], replacing the space Y used in [17] by the space of functions w such that $(\xi,t) \to e^{-\omega t}w(\xi,t) \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0,\infty))$. This is possible thanks to Theorem 0.1 in the appendix.

The proof of (ii) follows the proof of Theorem 9.1.3 of [17], replacing the space $C^{\alpha}((-\infty,0]; D,\omega)$ used in [17] by the space of the functions v such that $(\xi,t) \to e^{-\gamma t}v(\xi,t) \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times (-\infty,0])$, with any $\gamma \in (0, \min\{\mathrm{Re}\,\lambda : \lambda \in \sigma(L),\ \mathrm{Re}\,\lambda > 0\})$. ■

In the unstable case we shall construct the stable and unstable manifolds arguing as in Theorems 9.1.3 and 9.1.4 of [17]. To do this we shall use asymptotic behaviour results for forward and backward linear problems, whose precise statements and proofs we defer to the Appendix.

We need some notation. We denote by $\sigma^+(L)$, $\sigma^0(L)$ and $\sigma^-(L)$ the subsets of $\sigma(L)$ respectively consisting of elements with positive, zero and negative real parts. Since $\sigma(L)$ is discrete, $\sigma^+(L)$, $\sigma^0(L)$ and $\sigma^-(L)$ are spectral sets. Let P^+ and P^- be the spectral projections associated to $\sigma^+(L)$ and $\sigma^-(L)$.

Theorem 3.3 *Let Ω satisfy assumption $H1$ and let F, G, $\mathcal{L}$, $\mathcal{B}$ satisfy $H2$, $H3$.*

(i) Assume that $\sigma^+(L) \neq \emptyset$ and fix $\omega \in (0, \min\{Re\lambda : \lambda \in \sigma_+(L)\})$. Then there exist R_0, $r_0 > 0$ and a Lipschitz continuous function

$$\varphi : B(0,r_0) \subset P^+(C(\overline{\Omega})) = P^+(C^{2+\alpha}(\overline{\Omega})) \to (I - P^+)(C^{2+\alpha}(\overline{\Omega})),$$

differentiable at 0 with $\varphi'(0) = 0$, such that for every w_0 belonging to the graph of φ, Problem (3.1) has a unique backward solution v such that $\tilde{v}$ defined by $\tilde{v}(\xi,t) = e^{-\omega t}v(\xi,t)$, belongs to $C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times (-\infty,0])$ and satisfies

$$\|\tilde{v}\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times(-\infty,0])} \leq R_0. \tag{3.6}$$

Moreover, for every $\omega' \in (0, \min\{Re\lambda : \lambda \in \sigma_+(L)\})$ we have $(\xi,t) \to e^{-\omega' t}v(\xi,t) \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times (-\infty,0])$. Conversely, if Problem (3.1) has a backward solution v which satisfies (3.6) and $\|P^+v(0)\| \leq r_0$, then $v(0) \in graph\ \varphi$.

(ii) Fix $\omega \in (0, -\max\{Re\lambda : \lambda \in \sigma^-(L)\})$. Then there exist R_1, $r_1 > 0$ and a Lipschitz continuous function

$$\psi : B(0, r_1) \subset \{w_0 \in P^-(C^{2+\alpha}(\overline{\Omega})) : \mathcal{B}w_0 = G(w_0(\cdot))\} \to (I - P^-)(C^{2+\alpha}(\overline{\Omega})),$$

differentiable at 0 with $\psi'(0) = 0$, such that for every w_0 belonging to the graph of ψ, Problem (3.1) has a unique solution w such that $\tilde{w}$ defined by $\tilde{w}(\xi, t) = e^{\omega t} w(\xi, t)$ belongs to $C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0, \infty))$ and

$$\|\tilde{w}\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,\infty))} \leq R_1. \tag{3.7}$$

Moreover, for every $\omega' \in (0, -\max\{Re\lambda : \lambda \in \sigma^-(L)\})$ we have $(\xi, t) \to e^{\omega' t} w(\xi, t) \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0, \infty))$. Conversely, if Problem (3.1) has a forward solution w which satisfies (3.7) and $\|P^- w(\cdot, 0)\|_{C^{2+\alpha}(\overline{\Omega})} \leq r_1$, then $w(\cdot, 0) \in$ graph ψ.

Proof — The proof closely follows the proof of Theorems 9.1.3 and 9.1.4 of [17] using as main tools Theorems 0.1 and 0.2. Of course the spaces used in Theorems 9.1.3 and 9.1.4 of [17] have to be replaced by the spaces of the functions w such that $(\xi, t) \to e^{-\omega t} w(\xi, t) \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times (-\infty, 0])$ and $(\xi, t) \to e^{\omega t} w(\xi, t) \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times (0, \infty))$. Note that Theorems 9.1.3 and 9.1.4 of [17] are stated under the assumption that the spectrum of L does not intersect the imaginary axis. Here, since we only have a discrete spectrum, this assumption may be dropped. ■

4 Applications

We recall that we have transformed Problem (1.14),(1.15) with initial data u_0 and $\partial\Omega_0$ to the fully nonlinear Problem (3.1) with initial condition (3.2), where w_0 is given by

$$w_0(\xi) = u_0(\xi + \Phi_0(\xi)) - U(\xi) - \nabla_\xi U(\xi) \cdot \Phi_0(\xi). \tag{4.1}$$

Thus w_0 depends on u_0 as well as on $\partial\Omega_0$, which determines $\Phi_0(\xi) = \alpha(\xi)s(\xi', 0)\nu(\xi')$ through (2.4) and (2.6) with $t = \tau = 0$.

We solved Problem (3.1),(3.2) under the assumption that w_0 is small in $C^{2+\alpha}(\overline{\Omega})$ and satisfies the compatibility condition (3.3). In view of the smoothness of the data U and $\partial\Omega$, the smallness condition on w_0 is equivalent to the assumption that $\partial\Omega_0$ is $C^{2+\alpha}$-close to $\partial\Omega$ and that u_0 is $C^{2+\alpha}$-close to U (through the inverse of the $C^{2+\alpha}$-bijection $\xi \in \overline{\Omega} \to \xi + \Phi_0(\xi) \in \Omega_0$). Moreover, the compatibility condition (3.3) is satisfied provided (1.10) holds, which, for Problem (1.14),(1.15), reads

$$u_0 = g_1 \quad \text{and} \quad \frac{\partial u_0}{\partial n} = g_2 \quad \text{on} \quad \partial\Omega_0. \tag{4.2}$$

Having solved Problem (3.1),(3.2), we recover the solution $(u(\cdot, t), \Omega_t)$ of the original free boundary problem, using (2.19) rewritten as (2.21) to express s in terms of w, then defining the free boundary $\partial\Omega_t$ by (2.4), returning to the original space variables by (2.7) and finally recovering $u(x, t)$ by (2.12).

We consider the two specific examples mentioned in the introduction to illustrate the ideas and the theory developed in Sections 2 and 3. In these two cases we are able to verify the spectral hypotheses of Theorems 1.2 and 1.3.

4.1 A simple example

The first example we discuss is the case that, in the notation of (1.1),(1.2), $\mathcal{L} = \Delta$, $f = -1$ and $g = 1$, i.e.

$$u_t = \Delta u - 1 \quad \text{in } \Omega_t, \quad u = 0 \quad \text{and} \quad \frac{\partial u}{\partial n} = 1 \quad \text{on } \partial\Omega_t. \tag{4.3}$$

The unique (up to translations) negative radial equilibrium solution is

$$U(x) = \frac{|x|^2}{2N} - \frac{N}{2}, \qquad \Omega = B_N = \{x \in \mathbb{R}^N : |x| < N\}. \tag{4.4}$$

The operator $L : D(L) \to C(\overline{\Omega})$ is defined by

$$Lv = \Delta v \quad \text{for} \quad v \in D(L) = \{v \in \cap_{p>1} W^{2,p}(\Omega),\ \Delta v \in C(\overline{\Omega}),\ \mathcal{B}v = 0 \ \text{ on } \partial\Omega\}, \tag{4.5}$$

with

$$\mathcal{B}v = \frac{\partial v}{\partial n} - \frac{1}{N}v. \tag{4.6}$$

It is a standard exercise in spectral analysis to show that the spectrum $\sigma(L)$ consists entirely of real eigenvalues which may be found, using separation of variables, from the radial ordinary differential equation

$$\psi'' + \frac{N-1}{r}\psi' - \frac{n(n+N-2)}{r^2}\psi = \mu\psi, \quad \psi(r) \sim r^n \ \text{ as } r \to 0, \tag{4.7}$$

with boundary condition

$$\psi'(N) = \frac{1}{N}\psi(N). \tag{4.8}$$

The angular part of the eigenfunction is then a harmonic polynomial of degree n.

We have for each $n = 0, 1, 2, \ldots$ a sequence of eigenvalues

$$\mu_1^{(n)} > \mu_2^{(n)} > \mu_3^{(n)} > \cdots \downarrow -\infty, \tag{4.9}$$

with corresponding solutions $\psi_1^{(n)}, \psi_2^{(n)}, \psi_3^{(n)}, \ldots$. Each $\psi_j^{(n)}$ has exactly $j-1$ sign changes in the interval $(0, N)$. It follows from Sturm's Comparison Theorem that the double sequence $\mu_j^{(n)}$ is also decreasing in j.

In view of the translation invariance we may expect to have zero as an eigenvalue. Indeed, the functions $\frac{\partial U}{\partial x_i}$, $i = 1, \ldots, N$, are easily seen to be eigenfunctions with eigenvalue 0. They correspond to $n = j = 1$, i.e. $\mu_1^{(1)} = 0$ with $\psi_1^{(1)}(r) = r$. As a consequence of the monotonicity properties of the $\mu_j^{(n)}$ we then have that $\mu_1^{(0)} > 0$, so that U is unstable, thanks to Theorem 1.2. We note that except for $\mu_1^{(0)} > 0$ and $\mu_1^{(1)} = 0$, all the $\mu_j^{(n)}$ are negative. To see this it is sufficient to show that $\mu_2^{(0)} < 0$. This is clear because a solution of (4.7) with $n = 0$ and $\mu \geq 0$ cannot change sign while $\psi_2^{(0)}(r)$ does.

4.2 The focussing problem

The study of (1.12), i.e.

$$u_t = \Delta u \quad \text{in } \Omega_t, \quad u = 0 \quad \text{and} \quad \frac{\partial u}{\partial n} = 1 \quad \text{on } \partial\Omega_t, \tag{4.10}$$

is motivated by the mathematical modelling of combustion [16, 21]. Problem (4.10) can be seen as the physical high activation limit of the regularising problems $u_t = \Delta u + f_\epsilon(u)$, where f_ϵ has support in a small interval $[-\epsilon, 0]$ and is of the form $f_\epsilon(s) = f_1(s/\epsilon)/\epsilon$. In

[9] this regularisation is used to prove an existence result for (4.10) under appropriate conditions on the initial data. The same technique has previously been used in [1] for stationary problems and especially for the travelling wave case.

As far as we know the uniqueness question for Problem (4.10) has not been settled yet allthough it is clear that additional assumptions have to be made. Examples of nonuniqueness may be constructed even in dimension $N = 1$: taking initial data with an internal zero we can choose between a solution for which the free domain splits up in two domains or a solution for which the zero disappears instantaneously. Under additional assumptions avoiding such counterexamples, wellposedness results have been obtained in [14] and [11] for radial and one-dimensional problems.

Equilibria of (4.10) with Ω bounded do not exist. Thus solutions of (4.10) cannot be expected to stabilise. Typically the domain Ω_t vanishes in finite time, a behaviour exhibited by selfsimilar solutions of the form

$$u(x,t) = \sqrt{T-t}\, f(\eta)\,,\ \eta = \frac{|x|}{\sqrt{T-t}}\,,\quad \Omega_t = \{|x| < b\sqrt{T-t}\} \tag{4.11}$$

with $f(\eta)$ satisfying

$$f''(\eta) + \frac{N-1}{\eta} f'(\eta) + \frac{1}{2} f = \frac{1}{2}\eta f' \ \text{ for } 0 \le \eta \le b\,; f'(0) = f(b) = 0\ ,\ f'(b) = 1\,. \tag{4.12}$$

This uniquely determines the "free boundary" $b = b_N$ for which exactly one solution f of (4.12) with $f < 0$ on $[0, b)$ exists, and yields a similarity solution of (4.10) which "focusses" in the origin at $t = T$.

In [14] and [11] it is shown that radial solutions focus at the origin in finite time and that they are asymptotically selfsimilar. In one dimension the latter is also true for nonsymmetric initial data, in which case the asymptotic profile is symmetric but the focussing point does not have to be the origin. A natural question is to ask whether in every space dimension all solutions are asymptotically selfsimilar. However, posed in such generality, the answer is negative since it was proved in [11] that radial annular solutions focus on a sphere, provided the inner hole does not shrink to a point. On the other hand, since it is tempting to speculate, we conjecture that such a behaviour is unstable under nonradial perturbations causing the domain to change its topology. Further speculations, in particular on necessary and sufficient conditions on the initial data in terms of e.g. the shape of the domain for asymptotic selfsimilarity, are left to the reader. Here we restrict ourselves to a local affirmative answer given by the local (linearised) stability analysis of (4.11). To perform this analysis we follow [14] and [11] and transform the problem to selfsimilar variables

$$\tilde{x} = \frac{x}{(T-t)^{\frac{1}{2}}},\ \tilde{t} = -\log(T-t),\ \tilde{u}(\tilde{x},\tilde{t}) = \frac{u(x,t)}{(T-t)^{\frac{1}{2}}},\ \tilde{\Omega}_{\tilde{t}} = \{\tilde{x} : x \in \Omega_t\}. \tag{4.13}$$

Omitting the tildes, we arrive at

$$u_t = \Delta u - \frac{1}{2} x \cdot \nabla u + \frac{1}{2} u, \quad x \in \Omega_t \tag{4.14}$$

with boundary conditions

$$u = 0, \quad \frac{\partial u}{\partial n} = 1 \ \text{ on } \partial\Omega_t. \tag{4.15}$$

The selfsimilar solution (4.11) is transformed by (4.13) into an equilibrium

$$U(x) = f(|x|),\ \Omega = B_b = \{x \in \mathbb{R}^N : |x| < b\}, \tag{4.16}$$

for (4.14),(4.15).

In a previous work with C. Schmidt-Lainé [3] we have established a saddle point property of (4.11) in the class of radial solutions. The local analysis in [3] is done by scaling the radial r-variable to fix the free boundary and using a splitting which is essentially equivalent to (1.7). In that case the boundary condition (1.9) comes out to be linear, i.e. $\mathcal{G}(w) = 0$. As announced in [3] the scaling trick can also be used for nonradial perturbations of (4.16) but it leads to singularities at $r = 0$ in the higher order terms. To avoid this problem one has to localise the transformation which fixes the free boundary, and this may be done using a mollifier. Rather than presenting such an adaptation which works for radial equilibria only, we have chosen to develop the general approach of Section 2, which does not require any symmetry and is basically a localisation of the method in [4, 5, 6]. The application to the focussing problem and the spectral analysis below extend the result of [3] and give a saddle point property of (4.16) in the class of all (radial and nonradial) solutions.

We cannot expect (4.16) to be stable because the original problem is invariant under translations in x and t. Thus if we apply a small shift to (4.16), we obtain another selfsimilar solution which is transformed by (4.13) into a solution which starts close to (4.16) but moves away from it. The unstable manifold of (4.16) must therefore contain the images under (4.13) of shifts in space and time of (4.16). These are given by

$$\sqrt{1+\epsilon_2 e^t}\, U\Big(\frac{x-\epsilon_1 e^{\frac{1}{2}t}}{\sqrt{\epsilon_2 e^t + 1}}\Big), \tag{4.17}$$

where $\epsilon_1 \in \mathbb{R}^N$ and $\epsilon_2 \in \mathbb{R}$. As a consequence we have that $L : D(L) \to C(\overline{B_b})$ defined by

$$Lv = \Delta v - \frac{1}{2} x \cdot \nabla v + \frac{1}{2} v, \tag{4.18}$$

for

$$v \in D(L) = \{v \in \cap_{p>1} W^{2,p}(B_b),\ \Delta v \in C(\overline{B}_b),\ \mathcal{B}v = 0 \ \text{ on } \partial B_b\}, \tag{4.19}$$

where

$$\mathcal{B}v = \frac{\partial v}{\partial n} + \Big(\frac{N-1}{b} - \frac{b}{2}\Big) v = 0 \ \text{ on } \partial B_b \tag{4.20}$$

must have unstable eigenvalues: one with a 1-dimensional eigenspace spanned by a radial eigenfunction corresponding to a shift in t, and another with an N-dimensional eigenspace corresponding to shifts in $x_1, x_2, \ldots, x_N$. We will show below by means of power series developments that except for these two unstable eigenvalues, $\sigma(L)$ consists entirely of negative eigenvalues. Therefore the unstable manifold provided by Theorem 1.3 consists *only* of the images of (4.17) under the transformation in Section 2. Every orbit in the unstable manifold has just the same selfsimilar profile, disguised by the transformation to selfsimilar variables which depends on an arbitrary choice of the focussing point and time. This may be interpreted by saying that, although the equilibrium (4.16) is unstable, the profile itself is stable. This is very much like the travelling wave case where the translates of the equilibrium form a center manifold.

It remains to show that the spectrum of (4.18) with boundary condition (4.20) has the desired properties. As in the example in Section 4.1, the spectrum consists only of real eigenvalues which now are to be found from

$$\psi'' + \Big(\frac{N-1}{r} - \frac{r}{2}\Big)\psi' + \Big(\frac{1}{2} - \frac{n(n+N-2)}{r^2}\Big)\psi = \mu\psi, \quad \psi(r) \sim r^n \ \text{ as } r \to 0, \tag{4.21}$$

with boundary condition

$$\psi'(b) + \Big(\frac{N-1}{b} - \frac{b}{2}\Big)\psi(b) = 0. \tag{4.22}$$

Note that the solution of (4.21) is a multiple of the power series (using the Pochhammer symbols $(k)_j$ defined by $(k)_0 = 1$ and $(k)_{j+1} = (k+j)(k)_j$)

$$\psi(r) = \sum_{j=0}^{\infty} \frac{(\mu + \frac{n-1}{2})_j}{4^j j!(n+\frac{N}{2})_j} r^{n+2j} = r^n + \frac{\mu + \frac{n-1}{2}}{4(n+\frac{N}{2})} r^{n+2} +$$

$$\frac{(\mu + \frac{n-1}{2})(\mu + \frac{n-1}{2}+1)}{4^2\, 2!(n+\frac{N}{2})(n+\frac{N}{2}+1)} r^{n+4} + \frac{(\mu + \frac{n-1}{2})(\mu + \frac{n-1}{2}+1)(\mu + \frac{n-1}{2}+2)}{4^3\, 3!(n+\frac{N}{2})(n+\frac{N}{2}+1)(n+\frac{N}{2}+2)} r^{n+6} + \cdots \quad (4.23)$$

We number the eigenvalues again by $\mu_j^{(n)}$ $(j = 1, 2 \ldots)$ with corresponding solutions $\psi_j^{(n)}$. The monotonicity properties of $\mu_j^{(n)}$ and the sign change properties of $\psi_j^{(n)}$ are the same as in Section 4.1, i.e. $\mu_1^{(n)} > \mu_2^{(n)} > \mu_3^{(n)} > \cdots \downarrow -\infty$, $\psi_j^{(n)}$ has $j-1$ sign changes in the interval $(0, b)$ and the double sequence $\mu_j^{(n)}$ is also decreasing in j.

The eigenvalues due to shifts in the original variables are

$$\mu_1^{(0)} = 1 \ \text{ with } \psi_1^{(0)} = f'' + \frac{N-1}{r} f' \ \text{ and } \mu_1^{(1)} = \frac{1}{2} \ \text{ with } \psi_1^{(1)} = f'. \quad (4.24)$$

We have to show that these are the only nonnegative eigenvalues. In view of the monotonicity properties it suffices to show that $\mu_2^{(0)}$ and $\mu_1^{(2)}$ are negative.

By the radial analysis in [3] we have that $\mu_2^{(0)} < 0$. We recall that this is because f solves (4.21) with $\mu = 0$ and has a first sign change at $r = b$. Therefore the solution ψ of (4.21) with $n = 0$ and $\mu \geq 0$ cannot have a sign change before $z = b$. Since $\psi_2^{(0)}$ must change sign, it follows that $\mu_2^{(0)} < 0$.

It remains to show that $\mu_1^{(2)}$ is negative and this is more involved. From here on let $n = 2$. We will show that $\mu_1^{(2)}$ must lie in the interval $(-1, 0)$. To do so we first observe that for $\mu \geq -1$ the function ψ defined by (4.23) is positive on $(0, b]$. Indeed we have for $\mu = -1$ that

$$\psi(r) = \sum_{j=0}^{\infty} \frac{(-\frac{1}{2})_j}{4^j j!(2+\frac{N}{2})_j} r^{2+2j}, \quad (4.25)$$

whence, using

$$g(b) = \sum_{j=0}^{\infty} \frac{(-\frac{1}{2})_j}{4^j\, j!\, (\frac{N}{2})_j} b^{2j} = 0 \quad (4.26)$$

to simplify expressions,

$$\psi(b) = \psi(b) - b^2 g(b) = \sum_{j=0}^{\infty} \frac{(-\frac{1}{2})_j}{4^j j!(2+\frac{N}{2})_j} b^{2+2j} - b^2 \sum_{j=0}^{\infty} \frac{(-\frac{1}{2})_j}{4^j j!(\frac{N}{2})_j} b^{2j} > 0,$$

so that, because only the first term in (4.25) has a positive coefficient, $\psi(r) > 0$ on $(0, b]$. Increasing μ from $\mu = -1$ we only make ψ larger on $(0, b]$ so we conclude that $\psi(r) > 0$ on $(0, b]$ for all $\mu \geq -1$.

To finish we prove that $(\mathcal{B}\psi)(b)$ goes from positive to negative when μ goes from 0 to -1. From (4.23) we derive that

$$(\mathcal{B}\psi)(b) = (N+1)b + \sum_{j=1}^{\infty} \frac{(\mu + \frac{1}{2})_{j-1}}{4^j\, j!\, (\frac{N}{2}+2)_j} (\mu(2j+N+1) - 2j - \frac{N+1}{2}) b^{1+2j}.$$

Thus, when $\mu = 0$, we have, using (4.26),

$$(\mathcal{B}\psi)(b) = (N+1)b - \sum_{j=1}^{\infty} \frac{(\frac{1}{2})_{j-1}}{4^j\, j!\, (\frac{N}{2}+2)_j} (2j + \frac{N+1}{2}) b^{1+2j} - (N+1)bg(b)$$

$$\sum_{j=1}^{\infty} \frac{(\frac{1}{2})_{j-1}}{4^j\, j!}\Big(\frac{\frac{N+1}{2}}{(\frac{N}{2})_j} - \frac{2j+\frac{N+1}{2}}{(\frac{N}{2}+2)_j}\Big)b^{2j} = \sum_{j=1}^{\infty} \frac{(\frac{1}{2})_{j-1}}{4^j\,(j-1)!}\frac{Nj+j+1}{2(\frac{N}{2})_{j+2}}b^{2j} > 0.$$

For $\mu = -1$ we have, using (4.26) again,

$$(\mathcal{B}\psi)(b) = (N+1)b - \sum_{j=1}^{\infty} \frac{(-\frac{1}{2})_{j-1}}{4^j\, j!\,(\frac{N}{2}+2)_j}(4j+3\frac{N+1}{2})b^{1+2j} + (\frac{N^2+3N-2}{2N(N+4)}b^2 - N - 1)bg(b) =$$

$$\sum_{j=1}^{\infty} \frac{4(N^2+3N-2)j^2 + 4(N+3)(N^2+2N-2)j + N^4+8N^3+15N^2-12N-16}{2N(N+4)(j-1)!\,4^{j+1}}\frac{(-\frac{1}{2})_j}{(\frac{N}{2})_{j+3}}b^{1+2j} < 0,$$

because $(-\frac{1}{2})_j < 0$ and the first numerator in the latter expression is, setting $j = 1$, larger then $(N+4)(N^3+8N^2+7N-12) > 0$.

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Appendix: Asymptotic behavior in linear problems

Throughout the appendix we use the notation of Section 3. Let Ω be a bounded open set in $\mathbb{R}^N$ with $C^{2+\alpha}$ boundary, $0<\alpha<1$. Consider the linear problem

$$\begin{cases} u_t = \mathcal{L}w + f(\xi,t), \;\; t\ge 0,\; \xi\in\overline{\Omega}, \\ \mathcal{B}w = g(\xi,t), \;\; t\ge 0,\; \xi\in\partial\Omega, \\ u(\xi,0)=u_0(\xi), \;\; \xi\in\overline{\Omega}, \end{cases} \tag{0.1}$$

where the elliptic operator $\mathcal{L}$ and the nontangential operator $\mathcal{B}$ satisfy assumption H3, and $u_0\in C^{2+\alpha}(\overline{\Omega})$.

The realisation L of $\mathcal{L}$ with homogeneous boundary conditions in $X=C(\overline{\Omega})$, defined in (3.5), is a sectorial operator thanks to [20]. Moreover if f, g and u_0 are regular enough the unique solution of (0.1) is given by the extension of the Balakrishnan formula (see e.g. [17, p.200])

$$\begin{aligned} u(\cdot,t) &= e^{tL}(u_0-G(\cdot,0)) + \int_0^t e^{(t-s)L}[f(\cdot,s)+\mathcal{L}G(\cdot,s)]ds \\ &\quad -L\int_0^t e^{(t-s)L}[G(\cdot,s)-G(\cdot,0)]ds + G(\cdot,0) \\ &= e^{tL}u_0 + \int_0^t e^{(t-s)L}[f(\cdot,s)+\mathcal{L}G(\cdot,s)]ds - L\int_0^t e^{(t-s)L}G(\cdot,s)ds, \;\; 0\le t\le T. \end{aligned} \tag{0.2}$$

Here $G(\cdot,t)=\mathcal{N}g(\cdot,t)$, the operator $\mathcal{N}$ being any lifting operator such that

$$\begin{cases} \mathcal{N}\in L(C^{\theta}(\partial\Omega), C^{\theta+1}(\overline{\Omega})), \;\; 0\le\theta\le\alpha+1, \\ \mathcal{B}\mathcal{N}g = g, \;\; g\in C(\partial\Omega). \end{cases} \tag{0.3}$$

For instance, we can take as $\mathcal{N}$ the operator given in Theorem 0.3.2 of [17].

Theorem 0.1 *Let $0<\omega<-\max\{Re\lambda : \lambda\in\sigma^-(L)\}$. Let f be such that $(\xi,t)\to e^{\omega t}f(\xi,t)\in C^{\alpha,\alpha/2}(\overline{\Omega}\times[0,\infty))$, let g be such that $(\xi,t)\to e^{\omega t}g(\xi,t)\in C^{1+\alpha,1/2+\alpha/2}(\partial\Omega\times[0,\infty))$ and let $u_0\in C^{2+\alpha}(\overline{\Omega})$. Then $v(\xi,t)=e^{\omega t}u(\xi,t)$ is bounded in $[0,+\infty)\times\overline{\Omega}$ if and only if*

$$(I-P^-)u_0 = -\int_0^{+\infty} e^{-sL}(I-P^-)[f(\cdot,s)+\mathcal{L}\mathcal{N}g(\cdot,s)]ds + L\int_0^{\infty} e^{-sL}(I-P^-)\mathcal{N}g(\cdot,s)ds. \tag{0.4}$$

In this case, u is given by

$$\begin{aligned} &u(\cdot,t) = \\ &e^{tL}P^-u_0 + \int_0^t e^{(t-s)L}P^-[f(\cdot,s)+\mathcal{L}\mathcal{N}g(\cdot,s)]ds - L\int_0^t e^{(t-s)L}P^-\mathcal{N}g(\cdot,s)ds \\ &-\int_t^{+\infty} e^{(t-s)L}(I-P^-)[f(\cdot,s)+\mathcal{L}\mathcal{N}g(\cdot,s)]ds + L\int_t^{+\infty} e^{(t-s)L}(I-P^-)\mathcal{N}g(\cdot,s)ds. \end{aligned} \tag{0.5}$$

Moreover, if the compatibility condition

$$\mathcal{B}u_0 = g(\cdot,0) \;\text{ in }\; \partial\Omega$$

holds, v is regular up to $t=0$. To be precise, $v\in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,\infty))$ and

$$\begin{aligned} &\|v\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,\infty))} \\ &\le C(\|u_0\|_{C^{2+\alpha}(\overline{\Omega})} + \|e^{\omega t}f\|_{C^{\alpha,\alpha/2}(\overline{\Omega}\times[0,\infty))} + \|e^{\omega t}g\|_{C^{1+\alpha,1/2+\alpha/2}(\partial\Omega\times[0,\infty))}). \end{aligned}$$

Proof — The proof follows [17, §4.3, §5.1]. The novelty with respect to [17] is the nonzero boundary condition which gives additional terms whose asymptotic behaviour has to be taken into account.

Using the estimates (which hold for small $\varepsilon > 0$ and for $t > 0$)

$$\|P^- e^{tL}\|_{L(X)} \leq Ce^{-(\omega+\varepsilon)t}, \ \ \|LP^- e^{tL}\|_{L(X)} \leq \frac{Ce^{-(\omega+\varepsilon)t}}{t}, \ \ \|(I-P^-)e^{-tL}\|_{L(X)} \leq Ce^{(\omega-\varepsilon)t},$$

and arguing as in [17], it is easy to check that the function u given by (0.5) is bounded by $Ce^{-\omega t}$.

Due to (0.2) we have $u = u_1 + u_2$, where u_1 is the function in the right hand side of (0.5) and

$$\begin{aligned}
&u_2(\cdot,t) = \\
&e^{tL}\left((I-P^-)u_0 + \int_0^\infty e^{-sL}(I-P^-)(f(\cdot,s) + \mathcal{L}G(\cdot,s))ds - L\int_0^\infty e^{-sL}(I-P^-)G(\cdot,s)ds\right) \\
&= e^{tL}y, \ \ t \geq 0,
\end{aligned}$$

y being an element of $(I-P^-)(X)$. Thus $e^{\omega t}u_2(\cdot,t)$ is bounded in $[0,\infty)$ with values in X (which means that v is bounded) if and only if $y = 0$, i.e. if and only if (0.4) holds.

Let us prove that $v = e^{\omega t}u \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0,\infty))$. Observe that v satisfies (0.1) with $\mathcal{L}$ replaced by $\widetilde{\mathcal{L}} = \mathcal{L} + \omega I$, and f and g replaced respectively by $\widetilde{f} = fe^{\omega t}$ and $\widetilde{g} = ge^{\omega t}$. In the following we shall set

$$\|\widetilde{f}\| = \|\widetilde{f}\|_{C^{\alpha,\alpha/2}(\overline{\Omega}\times[0,\infty))}, \ \ \|\widetilde{g}\| = \|\widetilde{g}\|_{C^{1+\alpha,1/2+\alpha/2}(\partial\Omega\times[0,\infty))}.$$

Thanks to the compatibility condition $\mathcal{B}u_0 = g(\cdot,0)$ and to the regularity of the data, v belongs to $C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0,1])$ and

$$\|v\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,1])} \leq C(\|u_0\|_{C^{2+\alpha}(\overline{\Omega})} + \|\widetilde{f}\| + \|\widetilde{g}\|).$$

So we have to check that $v \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [1,\infty))$ and that its norm may be estimated in terms of the norms of the data.

Due to the choice of ω the spectrum of $\widetilde{L} = L + \omega I$ does not intersect the imaginary axis and the projection $(I - P^-)$ is the spectral projection associated to the unstable part of $\sigma(\widetilde{L})$. Therefore the following estimates hold for some $\gamma > 0$:

$$\|\widetilde{L}^k P^- e^{t\widetilde{L}}\|_{L(X)} \leq \frac{C_k e^{-\gamma t}}{t^k}, \ \ \|\widetilde{L}^k (I-P^-)e^{-t\widetilde{L}}\|_{L(X)} \leq C_k e^{-\gamma t}, \ \ t > 0, \ k \in \mathbb{N}. \tag{0.6}$$

It is convenient to split $v(t) = v(\cdot,t)$ as $v = \sum_{i=1}^3 v_i$, where

$$\begin{aligned}
v_1(t) &= e^{t\widetilde{L}}P^-u_0 - (e^{t\widetilde{L}} - I)P^-\mathcal{N}\widetilde{g}(\cdot,0) \\
&+ \int_0^t e^{(t-s)\widetilde{L}}P^-[\widetilde{f}(\cdot,s) + \widetilde{\mathcal{L}}\mathcal{N}\widetilde{g}(\cdot,s)]ds - \int_t^{+\infty} e^{(t-s)\widetilde{L}}(I-P^-)[\widetilde{f} + \widetilde{\mathcal{L}}\mathcal{N}\widetilde{g}]ds,
\end{aligned}$$

$$v_2(t) = -\widetilde{L}\int_0^t e^{(t-s)\widetilde{L}}P^-\mathcal{N}[\widetilde{g}(\cdot,s) - \widetilde{g}(\cdot,0)]ds,$$

$$v_3(t) = +\widetilde{L}\int_t^{+\infty} e^{(t-s)\widetilde{L}}(I-P^-)\mathcal{N}\widetilde{g}(\cdot,s)ds.$$

The function v_1 may be estimated arguing as in [17]. In fact, both $t \to \widetilde{f}(\cdot,t)$ and $t \to \widetilde{\mathcal{L}}\mathcal{N}\widetilde{g}(\cdot,t)$ belong to $C^{\alpha/2}([0,\infty);X) \cap B([0,\infty);C^\alpha(\overline{\Omega}))$. Therefore,

$$\|v_1\|_{C^{1+\alpha/2}([1,\infty);X)} + \|v_1\|_{B([1,\infty);D_L(1+\alpha/2,\infty))} \leq C(\|u_0\|_X + \|\widetilde{f}\| + \|\widetilde{g}\|).$$

Let us consider v_2. Since $t \to \widetilde{g}(\cdot,t) \in C^{1/2+\alpha/2}([0,\infty);C(\partial\Omega))$ we have $\mathcal{N}\widetilde{g} \in C^{1/2+\alpha/2}([0,\infty);$ $C^1(\overline{\Omega}))$. Moreover $C^1(\overline{\Omega})$ is continuously embedded in $D_L(1/2,\infty)$, see e.g. [17, Thm. 3.1.31]. Applying [17, Thm. 4.3.16] with $\theta = 1/2 + \alpha/2$, $\beta = \alpha/2$, we get $v_2 \in C^{1+\alpha/2}([0,T];X)$ and

$t \to v_2(t) - (I-P)(\mathcal{N}\widetilde{g}(\cdot,t) - \mathcal{N}\widetilde{g}(\cdot,0)) \in B([0,T]; D_L(1+\alpha/2,\infty))$ for every $T>0$. Looking at the proof of Theorem 4.3.16 and of the previous Theorem 4.3.1(iii) one sees that

$$[v_2']_{C^{\alpha/2}([0,T];X)} + \sup_{0\le t\le T}[L(v_2(t) - P^-(\mathcal{N}\widetilde{g}(t) - \mathcal{N}\widetilde{g}(0))]_{D_L(\alpha/2,\infty)}$$
$$\le C\|\mathcal{N}\widetilde{g}\|_{C^{1/2+\alpha/2}([0,\infty);C^1(\overline{\Omega}))},$$

whith constant C independent of T. A similar estimate holds for the lower order norms, as is easily seen using (0.6). Recalling that $D_L(1+\alpha/2,\infty)$ is continuously embedded in $C^{2+\alpha}(\overline{\Omega})$ (see [17, Thm. 3.1.34(ii)]) and that $P^-\mathcal{N}\widetilde{g}$ is bounded with values in $C^{2+\alpha}(\overline{\Omega})$, we get $v_2 \in C^{1+\alpha/2}([1,\infty);X) \cap B([1,\infty);C^{2+\alpha}(\overline{\Omega}))$ and

$$\|v_2\|_{C^{1+\alpha/2}([1,\infty);X)} + \|v_2\|_{B([1,\infty);C^{2+\alpha}(\overline{\Omega}))} \le C\|\widetilde{g}\|.$$

Finally let us consider v_3. By estimates (0.6) it is obviously bounded with values in $D(L^k)$ for every $k \in \mathbb{N}$. Moreover $v_3' = \widetilde{L}v_3 - \widetilde{L}(I-P^-)\mathcal{N}\widetilde{g}$ is Hölder continuous with values in X and

$$\|v_3\|_{C^{1+\alpha/2}([1,\infty);X)} + \|v_3\|_{B([1,\infty);D_L(1+\alpha/2,\infty))} \le C\|\widetilde{g}\|.$$

Summing up and recalling once again that $D_L(1+\alpha/2,\infty) \subset C^{2+\alpha}(\overline{\Omega})$ we get

$$\|v\|_{C^{1+\alpha/2}([1,\infty);X)} + \|v\|_{B([1,\infty);C^{2+\alpha}(\overline{\Omega}))} \le C(\|u_0\|_X + \|\widetilde{f}\| + \|\widetilde{g}\|).$$

It follows that $v \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[1,\infty))$, and

$$\|v\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[1,\infty))} \le C(\|u_0\|_X + \|\widetilde{f}\| + \|\widetilde{g}\|),$$

which finishes the proof. ■

Let us now consider the backward problem

$$\begin{cases} u_t = \mathcal{L}u + f(\xi,t), & t\le 0,\ \xi \in \overline{\Omega}, \\ \mathcal{B}u = g(\xi,t), & t \le 0,\ \xi \in \partial\Omega, \\ u(\xi,0) = u_0(\xi), & \xi \in \overline{\Omega}. \end{cases} \tag{0.7}$$

Theorem 0.2 *Let $\sigma^+(L) \neq \emptyset$ and $0 < \omega < \min\{Re\lambda : \lambda \in \sigma^+(L)\}$. Let f be such that $(\xi,t) \to e^{-\omega t}f(\xi,t) \in C^{\alpha,\alpha/2}(\overline{\Omega}\times(-\infty,0])$ and let g be such that $(\xi,t) \to e^{-\omega t}g(\xi,t) \in C^{1+\alpha,1/2+\alpha/2}(\partial\Omega\times(-\infty,0])$, $u_0 \in C^{2+\alpha}(\overline{\Omega})$.*

Then Problem (0.7) has a solution u such that $v(\xi,t) = e^{-\omega t}u(\xi,t)$ is bounded in $(-\infty,0]\times\overline{\Omega}$ if and only if

$$(I-P^+)u_0 = \int_{-\infty}^0 e^{-sL}(I-P^+)[f(\cdot,s) + \mathcal{L}\mathcal{N}g(\cdot,s)]ds - L\int_{-\infty}^0 e^{-sA}(I-P^+)\mathcal{N}g(\cdot,s)ds. \tag{0.8}$$

In this case, u is given by

$$\begin{aligned} u(\cdot,t) &= e^{tL}P^+u_0 + \int_0^t e^{(t-s)L}P^+[f(\cdot,s) + \mathcal{L}\mathcal{N}g(\cdot,s)]ds - L\int_0^t e^{(t-s)L}P^+\mathcal{N}g(\cdot,s)ds \\ &+ \int_{-\infty}^t e^{(t-s)L}(I-P^+)[f(\cdot,s) + \mathcal{L}\mathcal{N}g(\cdot,s)]ds - L\int_{-\infty}^t e^{(t-s)L}(I-P^+)\mathcal{N}g(\cdot,s)ds, \quad t\le 0. \end{aligned} \tag{0.9}$$

Moreover, v belongs to $C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times(-\infty,0])$ and

$$\|v\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times(-\infty,0])}$$
$$\le C(\|u_0\|_{C(\overline{\Omega})} + \|e^{-\omega t}f\|_{C^{\alpha,\alpha/2}(\overline{\Omega}\times(-\infty,0])} + \|e^{-\omega t}g\|_{C^{1+\alpha,1/2+\alpha/2}(\partial\Omega\times(-\infty,0])}).$$

Proof — Let $\widetilde{L} = L - \omega I$. For every $\gamma > 0$ we have

$$\|\widetilde{L}^k P^+ e^{-t\widetilde{L}}\|_{L(X)} \le C_k e^{-\gamma t}; \quad \|\widetilde{L}^k (I - P^+) e^{t\widetilde{L}}\|_{L(X)} \le C_k \frac{e^{-\gamma t}}{t^k},\ t > 0,\ k \in \mathbb{N}. \tag{0.10}$$

Using (0.10) it is not hard to see that, if u is the function defined by (0.9), $v(t) = e^{-\omega t}u(\cdot, t)$ is bounded. Proving that condition (0.8) is necessary for v to be bounded is similar to [17, Thm. 4.4.6] and is left to the reader. Concerning P^+v, we remark that it satisfies (see [17, Thm. 5.1.18])

$$\begin{cases} (P^+v)'(t) = \widetilde{L}P^+v(t) + P^+(\widetilde{f}(\cdot,t) - \widetilde{\mathcal{L}}\mathcal{N}\widetilde{g}(\cdot,t)), \ \ t \le 0, \\ \\ P^+v(0) = P^+u_0, \end{cases}$$

with $\widetilde{f} = fe^{\omega t}$ and $\widetilde{g} = ge^{\omega t}$, so that

$$P^+v(t) = e^{t\widetilde{L}}P^+u_0 + \int_0^t e^{(t-s)\widetilde{L}}P^+[\widetilde{f} + \widetilde{\mathcal{L}}\mathcal{N}\widetilde{g}]ds - \widetilde{L}\int_0^t e^{(t-s)L}P^+\mathcal{N}\widetilde{g}ds, \ \ t \le 0,$$

and (0.9) holds.

Let us prove that $v \in C^{1+\alpha/2}((-\infty, 0]; X) \cap B((-\infty, 0]; C^{2+\alpha}(\overline{\Omega}))$. Using estimates (0.10) it is easy to see that P^+v is bounded in $(-\infty, 0]$ with values in $D(\widetilde{L}^k)$ for every $k \in \mathbb{N}$. Moreover, since $t \to \widetilde{f}(\cdot, t)$ and $t \to \widetilde{\mathcal{L}}\mathcal{N}\widetilde{g}(\cdot, t)$ are in $C^{\alpha/2}((-\infty, 0]; X)$ (see the proof of Theorem 0.1), we have $P^+v' \in C^{\alpha/2}((-\infty, 0]; X)$. It follows that $P^+v \in C^{1+\alpha/2}((-\infty, 0]; X) \cap B((-\infty, 0]; D_L(1 + \alpha/2, \infty)) \subset C^{1+\alpha/2}((-\infty, 0]; X) \cap B((-\infty, 0]; C^{2+\alpha}(\overline{\Omega}))$ and

$$\|P^+v\|_{C^{1+\alpha/2}((-\infty,0];X)} + \|P^+v\|_{B((-\infty,0];C^{2+\alpha}(\overline{\Omega}))} \le C(\|u_0\|_X + \|\widetilde{f}\| + \|\widetilde{g}\|).$$

Let us split $(I - P^+)v$ as $(I - P^+)v = v_1 + v_2$, where

$$v_1(t) = \int_{-\infty}^t e^{(t-s)\widetilde{L}}(I - P^+)[\widetilde{f} + \widetilde{\mathcal{L}}\mathcal{N}\widetilde{g}]ds, \ \ t \le 0,$$

$$v_2(t) = -\widetilde{L}\int_{-\infty}^t e^{(t-s)\widetilde{L}}(I - P^+)\mathcal{N}\widetilde{g}ds, \ \ t \le 0.$$

Since $t \to \widetilde{f}(\cdot, t)$ and $t \to \widetilde{\mathcal{L}}\mathcal{N}\widetilde{g}(\cdot, t)$ belong to $C^{\alpha/2}((-\infty, 0]; X) \cap B((-\infty, 0]; D_L(\alpha/2, \infty))$, by [17, Prop. 4.4.5(ii)(iii)] v_1 belongs to $C^{1+\alpha/2}((-\infty, 0]; X) \cap B((-\infty, 0]; D_L(1 + \alpha/2, \infty)) \subset C^{1+\alpha/2}((-\infty, 0]; X) \cap B((-\infty, 0];\ C^{2+\alpha}(\overline{\Omega}))$ and

$$\|v_1\|_{C^{1+\alpha/2}((-\infty,0];X)} + \|v_1\|_{B((-\infty,0];C^{2+\alpha}(\overline{\Omega}))} \le C(\|\widetilde{f}\| + \|\widetilde{g}\|).$$

To estimate v_2 we recall that $\mathcal{N}\widetilde{g} \in C^{1/2+\alpha/2}((-\infty, 0]; C^1(\overline{\Omega})) \subset C^{1/2+\alpha/2}((-\infty, 0]; D_L(1/2, \infty))$ (see again the proof of Theorem 0.1). By [17, Prop. 4.4.5(ii)] and [17, Prop. 2.2.12(i)], applied with X replaced by $D_L(1/2, \infty)$, the function

$$z(t) = -\int_{-\infty}^t e^{(t-s)\widetilde{L}}(I - P^+)\mathcal{N}\widetilde{g}(s)ds$$

is such that z' is bounded with values in $D_L(1+\alpha/2, \infty) \subset C^{2+\alpha}(\overline{\Omega})$. Since $z' = v_2 - (I - P^+)\mathcal{N}\widetilde{g}$, and $(I - P^+)\mathcal{N}\widetilde{g}$ is bounded with values in $C^{2+\alpha}(\overline{\Omega})$, we get $v_2 \in B((-\infty, 0]; C^{2+\alpha}(\overline{\Omega}))$ and

$$\|v_2\|_{B((-\infty,0];C^{2+\alpha}(\overline{\Omega}))} \le C\|\widetilde{g}\|.$$

Let now $a < 0$ and consider the restriction of v_2 to $[a, 0]$. It is equal to $\widetilde{L}w(t)$, where

$$w(t) = -e^{(t-a)\widetilde{L}}\int_{-\infty}^a e^{(a-s)\widetilde{L}}(I - P^+)\mathcal{N}\widetilde{g}(s)ds - \int_a^t e^{(t-s)\widetilde{L}}(I - P^+)\mathcal{N}\widetilde{g}(s)ds$$

$$= e^{(t-a)\widetilde{L}}z(a) - \int_a^t e^{(t-s)\widetilde{L}}(I - P^+)\mathcal{N}\widetilde{g}(s)ds, \ \ a \le t \le 0.$$

We have just proved that $\widetilde{L}z(a) - (I - P^+)\mathcal{N}\widetilde{g}(a,\cdot) = z'(a) \in D_L(1+\alpha/2,\infty)$, with norm independent of a. By [17, Thm 4.3.16] and the same arguments used in the proof of Theorem 0.1, we get $v_2 \in C^{1+\alpha/2}((-\infty,0];X)$ and

$$\|v_2\|_{C^{1+\alpha/2}((-\infty,0];X)} \le C\|\widetilde{g}\|.$$

Summing up, we find that $v \in C^{1+\alpha/2}((-\infty,0];X) \cap B((-\infty,0];C^{2+\alpha}(\overline{\Omega}))$ so that $v \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times(-\infty,0])$ and

$$\|v\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times(-\infty,0])} \le C(\|u_0\|_X + \|\widetilde{f}\| + \|\widetilde{g}\|),$$

which finishes the proof. ∎