Dipoles and similarity solutions of the thin film equation in the half-line *

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Abstract

We consider non-negative solutions on the half-line of the thin film equation $h_t + (h^n h_{xxx})_x = 0$, which arises in lubrication models for thin viscous films, spreading droplets and Hele-Shaw cells. We present a discussion of the boundary conditions at $x = 0$ on the basis of formal and modelling arguments when $x = 0$ is an edge over which fluid can drain. We apply this discussion to define some similarity solutions of the first and the second kind. Depending on the boundary conditions, we introduce mass preserving solutions of the first kind ($0 < n < 3$), "anomalous dipoles" of the second kind ($0 < n < 2$, $n \neq 1$) and a standard dipole solution of the first kind for $n = 1$. For solutions of the first kind we prove results on existence, uniqueness and asymptotic behaviour, both at $x = 0$ and at the free boundary. For solutions of the second kind we briefly present some qualitative properties.

AMS classification scheme numbers: 35K55, 35K65, 76D08, 34C35

*We gratefully acknowledge the support of the Dutch Organisation for Scientific Research (NWO), the British Council, the Leverhulme Trust, the D.G.E.S. (M.E.C., Spain) Project PB97-0052 and the TMR-network Nonlinear Parabolic Partial Differential Equations: Methods And Applications, ERBFMRXCT980201

1 Introduction

In this paper we consider, on the half-line $\{x \geq 0\}$, the thin film equation

$$h_t + (h^n h_{xxx})_x = 0, \tag{1.1}$$

which arises in lubrication models for thin viscous films, spreading droplets and Hele-Shaw cells, see the survey papers [17], [11], [16], [42]. Hele-Shaw cells correspond to $n = 1$: [1], [22], [24], [27], [45]. We study non-negative similarity solutions of the form

$$h(x,t) = t^{-\alpha} f(\eta), \quad \eta = xt^{-\beta}, \tag{1.2}$$

when $x = 0$ is an edge over which fluid can drain. Imposing that (1.2) is a solution of (1.1) we obtain that the parameters α and β satisfy the relation

$$n\alpha + 4\beta = 1, \tag{1.3}$$

and the function $f = f(\eta)$ is a solution of the ordinary differential equation

$$(f^n f''')' = \alpha f + \beta \eta f'. \tag{1.4}$$

Since the draining configuration imposes the boundary condition $h(0,t) = 0$, the experience of the porous media equation [3], [7], [4] may suggest that we are going to find dipole solutions. (See below for a discussion on the concept of dipole and dipole solution). However, the situation is more involved. First of all, the thin film equation (1.1) does not preserve the moment

$$\int_0^\infty x h(x,t)\, dx,$$

except in the special case $n = 1$ and for appropriate boundary conditions. Secondly, since Equation (1.1) is of fourth order, we have to prescribe another boundary condition at $x = 0$. In Section 2 we analyse this question and find out several choices, depending on n, and give plausible reasons to concentrate on two pairs of boundary conditions at $x = 0$, namely:

$$\text{(I) } h = h^n h_{xxx} = 0 \quad (0 < n < 3) \qquad \text{and} \qquad \text{(II) } h = h_x = 0 \quad (0 < n < 2).$$

(In addition, the boundary conditions $h = h_{xx} = 0$ at $x = 0$ are very briefly considered at the end of Section 6).

The arguments and results of this paper strongly support the emergence of $n = 2$ as a critical exponent, in the sense that for $2 \leq n < 3$ the conditions (I) are the only relevant boundary conditions at the draining line $x = 0$.

We assume that the solutions are "strong" solutions for $t > 0$ and $x > 0$ (see details in Section 2), confined to a finite region. Hence $n < 3$ and they have a right free boundary or contact line where $h = h_x = h^n h_{xxx} = 0$ holds. (The function $h(x,t)$ is identically zero beyond this free boundary).

In the case (I) we have conservation of mass, cf. (2.15), hence $\alpha = \beta = 1/(n+4)$ and the fourth order ODE (1.4) can be reduced to a third order equation. This leads us to consider the following boundary value problem (singular when $n > 1$) for the *profile* $f = f(\eta)$:

$$\begin{cases} f''' = \frac{1}{n+4}\eta f^{1-n} & \text{for } 0 < \eta < a, \\ f(0) = f(a) = f'(a) = 0, & \int_0^a f(\eta)\, d\eta = M, \\ f \in C^3(0,a) \cap C^1[0,a], & f(\eta) > 0 \quad \text{for } 0 < \eta < a, \end{cases} \tag{1.5}$$

where M is a given positive number, while a is a positive number to be determined as a part of the problem. The zero flux conditions $f^n(\eta)f'''(\eta) \to 0$ as $\eta \to a$ and as $\eta \to 0$ are satisfied automatically by Problem (1.5). We will prove that this problem has a unique solution for all $n \in (0,3)$. For $n \geq 3$ Problem (1.5) has no solutions in view of the non-existence proof given in [15]. Clearly, conservation of mass imposes the same exponents α and β as the source-type solutions on the whole line, cf. [15]. Also, the solution of (1.1) associated to (1.5) satisfies $h(x,t) \to M\delta(x)$ as $t \to 0^+$, in the sense that

$$\lim_{t \to 0^+} \int_0^\infty h(x,t)\varphi(x)\, dx = M\varphi(0) \quad \text{for all } \varphi \in C([0,\infty)),$$

where δ stands for the Dirac mass. This situation may seem surprising, but for fourth (and higher) order parabolic equations it is feasible to have mass preserving solutions in the half-line which satisfy, in addition, $h(0,t) = 0$.

The unique solution of Problem (1.5) for $n = 1$ is

$$f(\eta) = \frac{1}{120}\eta(a-\eta)^2(\eta + 2a) \quad \text{with} \quad a^5 = 600M.$$

We do not know explicit formulas for other values of n.

The boundary conditions (II) lead to a self-similar problem *of the second kind*. This means that both the exponents α, β and the function f have to be determined simultaneously as a solution of the *nonlinear eigenvalue problem*

$$\begin{cases} (f^n f''')' = \alpha f + \beta\eta f' & \text{for } 0 < \eta < a, \\ f(0) = f'(0) = f(a) = f'(a) = 0, & f^n(\eta)f'''(\eta) \to 0 \quad \text{as } \eta \to a, \\ f \in C^4(0,a) \cap C^1[0,a], & f(\eta) > 0 \quad \text{for } 0 < \eta < a, \\ \int_0^a \eta^{k-1} f(\eta)\, d\eta = M_{k-1}, \quad k = \alpha/\beta, \end{cases} \tag{1.6}$$

where α and β are linked by (1.3). Notice that (1.6) has five boundary conditions (rather than four). This overdetermination is expected to select the appropriate values of α, β. Solutions $h(x,t)$ of (1.1) associated to solutions of (1.6) satisfy

$$\int_0^\infty x^{k-1} h(x,t)\, dx = \text{ Constant } = M_{k-1},$$

but this conservation law does not hold for general (non-self-similar) solutions of (1.1), except for $n = 1$ and $k = 2$, see Problem (1.7) below. For $n \neq 1$ the solutions of (1.6) are known as "anomalous dipole solutions" and the associated exponents α, β as "anomalous exponents": [6], [2]. (This is a generalisation of the concept of dipole solution explained below). In the Afterword of the book [4] Barenblatt raises the question of dipoles for (1.1) and says that "a self-similarity of the second kind can be expected here".

Let us remark that in this paper the self-similar solutions of the second kind are *exact solutions of the parabolic equation* (1.1), not approximate solutions or solutions to leading order, a terminology which is also found in the literature.

The analysis of Section 2 suggests that Problem (1.6) has a solution only if $n < 2$. Indeed, in Section 6 we will prove that (1.6) has no solution if $n \geq 2$, and that any solution of (1.6) with $n < 2$ has positive draining, i.e. decreasing mass (which in turn is equivalent to $\alpha > \beta$).

In the case $n = 1$ the conservation of the first moment, cf. (2.16), related to the exact derivative $ff''' = (ff'' - (f')^2/2)'$, gives $\alpha = 2\beta$, hence $\beta = 1/6$. (This conservation law was already observed in [39]). Then the above problem becomes

$$\begin{cases} (ff''')' = \frac{1}{6}(2f + \eta f') & \text{for } 0 < \eta < a, \\ f(0) = f'(0) = f(a) = f'(a) = 0, & f(\eta)f'''(\eta) \to 0 \quad \text{as } \eta \to a, \\ f \in C^4(0,a) \cap C^1[0,a], & f(\eta) > 0 \text{ for } 0 < \eta < a, \\ \int_0^a \eta f(\eta)\, d\eta = M_1, \end{cases} \tag{1.7}$$

where M_1 is a given positive number, while a is a positive number to be determined as a part of the problem. As we will see, this fourth order ODE can be reduced to a third order one and the overdetermination in the boundary conditions can be circumvented. Hence, the case $n = 1$ will lead to a self-similar problem of the first kind. The solution of (1.7) yields a dipole solution of equation (1.1) with $n = 1$ in the standard sense of a self-similar solution which preserves the first moment. Furthermore, let $h(x,t)$ be the solution of (1.1) associated to (1.7) and let $\hat{h}$ be the odd extension of h with respect to x. Then $\hat{h}(x,t) \to -2M_1\delta'(x)$ as $t \to 0^+$, in the sense that

$$\lim_{t \to 0^+} \int_{-\infty}^{+\infty} \hat{h}(x,t)\varphi(x)\, dx = 2M_1\varphi'(0) \quad \text{for all } \varphi \in C^1(\mathbb{R}).$$

Hence $\hat{h}(x,t)$ has indeed a dipole (i.e. a derivative of a Dirac mass) as initial condition.

We shall prove the following theorems on Problems (1.5) and (1.7).

Theorem 1.1 *If $0 < n < 3$ there exists a unique solution f of Problem (1.5). This solution satisfies that $\sqrt{f}$ is strictly concave in $(0,a)$ and hence f has a unique maximum m.*

(In a forthcoming work we will prove, in addition, that $m < a/2$).

Theorem 1.2 *Let $f(\eta)$ be the solution of Problem (1.5). Then there exist positive constants A_0, A_1, A_2 and A_3 such that $f(\eta) \sim A_0\eta$ as $\eta \to 0$ and $-\infty < f''(0^+) < 0$ for all $n \in (0,3)$, while as $\eta \to a^-$:*

$$\begin{array}{lll} \textit{For } 0 < n < 3/2 & f(\eta) \sim A_1(a-\eta)^2 \,. & \\ \textit{For } n = 3/2 & f(\eta) \sim A_2(a-\eta)^2 \left|\log(a-\eta)\right|^{2/3} \,. & \qquad (1.8) \\ \textit{For } 3/2 < n < 3 & f(\eta) \sim A_3(a-\eta)^{3/n} \,. & \end{array}$$

Theorem 1.3 *There exists a unique solution f of Problem (1.7). Furthermore, $\sqrt{f}$ is strictly concave and hence f has a unique maximum m.*

Theorem 1.4 *If $f(\eta)$ is the solution of Problem (1.7) then $f'''(\eta) \to a/6$ as $\eta \to a^-$ and there exist positive constants B_1, B_2 and B_3 such that $f(\eta)f'''(\eta) \to -B_1$ as $\eta \to 0$,*

$$f(\eta) \sim B_2\eta^{3/2} \ \textit{as } \eta \to 0 \quad \textit{and} \quad f(\eta) \sim B_3(a-\eta)^2 \ \textit{as } \eta \to a^- .$$

In this paper we present three approaches to prove Theorems 1.1 and 1.3. The first (Section 3) is an adaptation of the Green's function method in [26]. This method is applied to the mass conservation case when $1 < n < 3$, i.e. when the problem is singular. The second method (Section 4) is a shooting method which relies on [14] and is used to prove existence of the dipole solution for $n = 1$. We do not know a uniqueness proof by this method. The third method (Section 5) is based on a dynamical systems approach inspired in [34]-[35], by which we show that for a related autonomous system a unique connection between two critical points exists which corresponds to the unique dipole solution for $n = 1$. The phase space of Section 5 is 3-dimensional. This method could also be used to give an alternative proof for the mass conservation case.

The results on asymptotic behaviour (Theorems 1.2 and 1.4) can be quickly guessed by formal methods (see e.g. [8]). Rigorous proofs follow by combining [46] and a device used in [15].

Finally, in Section 6 we briefly present some qualitative properties of two self-similar problems of the second kind: the problem (1.6) and the zero pressure problem.

2 Overview and formal ideas

This section deals with general (not necessarily self-similar) solutions defined in the positive half-line and having bounded support. All the other sections deal with self-similar solutions. For $n < 3$ we consider always "strong solutions" (at least for $t > 0$ and $x > 0$), i.e. solutions which satisfy the definition of weak solution of [13] and, *in addition*, the entropy estimates of [9] and [18] (at least away from $t = 0$ and $x = 0$). Strong solutions are C^1 in x for almost every t and have a free boundary or contact line (the right free boundary in the introduction) if $h(x,0)$ has bounded support. Strong solutions can be viewed as zero contact angle solutions which satisfy

$$h = h_x = h^n h_{xxx} = 0$$

at the free boundary. This free boundary comes out *a posteriori* as a consequence of the construction of strong solutions of the Cauchy problem or other initial-boundary value problems: see [12], [36], [20], [23] and references therein. Weak solutions in the sense of [13] exist for all $n > 0$. Strong solutions are known to exist for $0 < n < 3$ and formal asymptotics indicates that they do not exist for $n \geq 3$. For similarity solutions we have conclusive non-existence results: As already noted, Problem (1.5) has no solution for $n \geq 3$ because of the proof given in [15], while we prove in Section 6 below that Problem (1.6) has no solution if $n \geq 2$. Hence, the case $n \geq 3$ does not appear in the other sections and is only mentioned in this section for the sake of completeness. We expect that the self-similar solutions of this paper provide the large-time behaviour of general strong solutions having the same boundary conditions at $x = 0$. For the Cauchy problem and the source-type solutions of [15] this is already proved: see [12], [36], [23] and references therein. In the case of dipole solutions for the porous media equation this was proved in [38] and [37]. It is worth observing that so far the results on large-time behaviour for fourth order parabolic equations are less precise than their second order counterparts.

Other very recent —and very different— approaches to the contact line and the contact angle can be found in [5], [19] and [44].

2.1 Physical background

In terms of thin film flows, the problems we address can be interpreted as representing fluid spreading due to surface tension over a flat semi-infinite substrate $0 < x < +\infty$, with $x = 0$ being an edge over which fluid can drain, so that the height $h(x,t)$ of the fluid droplet satisfies

$$h = 0 \text{ at } x = 0. \tag{2.1}$$

We take the fluid to be confined to a finite region, with initial data

$$h = h_0(x) \text{ at } t = 0, \tag{2.2}$$

where $h_0(x) \equiv 0$ for $x \geq s_0$.

As explained above appropriate conditions at the front (contact line) $x = s(t)$, whereby $h \equiv 0$ for $x \geq s(t)$ with $s(0) = s_0$, are

$$h = h_x = h^n h_{xxx} = 0 \quad \text{for } n < 3, \tag{2.3}$$

(zero contact angle), and these are the conditions we henceforth adopt. See Section 2.3 below for the case $n \geq 3$.

2.2 Local behaviour at the origin

We now classify the possible types of local behaviour as $x \to 0^+$ of solutions to (1.1) which satisfy (2.1); since (1.1) is fourth order, a second boundary condition is evidently needed at $x = 0$ and our main purpose here is to investigate what conditions are admissible. Our tools are formal asymptotic expansions. Further developments can be found in [40].

(a) No-flux solutions. Here the second condition is

$$\text{at } x = 0 \quad h^n h_{xxx} = 0. \tag{2.4}$$

Local expressions for h which have the requisite two degrees of freedom are then as follows.

Case (i): $n < 4$. The first three terms in the local expansion are

$$h \sim A(t)x + B(t)x^2 - \frac{A(t)^{-n}A'(t)}{2(3-n)(4-n)(5-n)}\, x^{5-n} \quad \text{as } x \to 0^+ \quad (n \neq 3), \tag{2.5}$$

where $A(t)$ and $B(t)$ are arbitrary; the constraint $n < 4$ follows from the requirement that the third term in (2.5) be smaller than the first. For $n = 3$ a term in $x^2 \log(1/x)$ appears.

Case (ii): $n > 4$. Here

$$h \sim \left(\frac{n^3}{8(n-4)(n-2)(n+4)}\, \frac{x^4}{t} \right)^{1/n} \tag{2.6}$$

provides an acceptable local balance. For $n = 4$ one has a term in $x(\log(1/x))^{1/4}$.

(b) Non-zero flux solutions. We define the flux

$$J(t) = -\left. h^n h_{xxx} \right|_{x=0}, \tag{2.7}$$

which represents the rate of fluid drainage across $x = 0$, and take $J \neq 0$. Notice that $J > 0$ means that the fluid leaves the semi-infinite substrate.

Case (i): Non-zero contact angle, $n < 2$. The expression

$$h \sim A(t)x + B(t)x^2 - \frac{A(t)^{-n}J(t)}{(1-n)(2-n)(3-n)}\, x^{3-n} \quad \text{as } x \to 0^+ \quad (n \neq 1), \tag{2.8}$$

represents the first three terms in a consistent local balance of (1.1) with $n < 2$; for $n = 1$ a term $x^2 \log(1/x)$ arises. This solution is underspecified because $A(t)$, $B(t)$ and $J(t)$ are all arbitrary; in practical problems it may be appropriate to prescribe the contact angle $A(t)$, but henceforth in this section we shall consider circumstances in which the zero contact angle condition

$$\text{at } x = 0 \quad h_x = 0 \tag{2.9}$$

provides the appropriate second boundary condition. Notice that here the zero contact angle condition is at the draining edge, not at the free boundary. We then have the following.

Case (ii): Zero contact angle, $n < 1/2$. In this case

$$h \sim B(t)x^2 - \frac{B(t)^{-n}J(t)}{2(1-2n)(1-n)(3-2n)}\,x^{3-2n} \quad \text{as } x \to 0^+ \tag{2.10}$$

gives the local expansion, with $B(t)$ and $J(t)$ being arbitrary. In this case $J(t)$ can have either sign. In the drainage applications with which we are concerned it is necessary that J be non-negative and the appropriate conditions are then (2.1), (2.9) if $J \geq 0$. Otherwise there exists a left free boundary $x = S(t)$ and (cf. (2.3))

$$\text{at } x = S(t) \quad h = h_x = h^n h_{xxx} = 0, \tag{2.11}$$

where the left free boundary $S(t) > 0$ is determined as part of the solution, with $h \equiv 0$ for $0 \leq x \leq S(t)$. One has

$$J(t)S(t) = 0 \quad \text{with} \quad J(t) \geq 0 \text{ and } S(t) \geq 0. \tag{2.12}$$

For all sufficiently large t, $S(t) = 0$. (This has been proved in [12], [36]; see also [23] and references therein). It is also possible for $J(t) < 0$ to occur in (2.8), if, say, one prescribes too large a contact angle $A(t)$. The remedy is again the introduction of a left free boundary as above. The self-similar solutions presented in the introduction do not have a left free boundary.

Case (iii): Zero contact angle, $n = 1/2$. It is worth spelling out the behaviour in this critical case. We have

$$h \sim \left(\frac{3}{4}J(t)\log(1/x)\right)^{2/3} x^2 \quad \text{as } x \to 0^+ . \tag{2.13}$$

For $2 > n \geq 1/2$ and zero contact angle it necessarily follows from the form of the local behaviour that $J(t) \geq 0$.

Case (iv): Zero contact angle, $1/2 < n < 2$. In this case

$$h \sim \left(\frac{(n+1)^3 J(t)}{3(2n-1)(2-n)}\,x^3\right)^{1/(n+1)} \quad \text{as } x \to 0^+ . \tag{2.14}$$

In this range of n, it is straightforward to show that the local expansion for h contains precisely one further degree of freedom.

We can thus summarize the admissible boundary conditions at $x = 0$, as follows.

(I) $n \geq 2$. Here the only relevant possibility is (2.1), (2.4) and the mass

$$\int_0^{s(t)} h(x,t)\,dx = M \equiv \int_0^{s_0} h_0(x)\,dx \tag{2.15}$$

is conserved, with no drainage of fluid being possible.

(II) $n < 2$. Here, in view of (2.8), there is an infinite number of possibilities. The three natural quantities to prescribe in (2.8) are the contact angle $A(t)$, the pressure, which is proportional to $-B(t)$, or the flux $J(t)$. Of course, one could also prescribe some relation between them. However, the two "extreme" cases of solutions with $J(t) \geq 0$ are given by (2.1), (2.9) (minimal, i.e. zero, contact angle) and (2.1), (2.4) (maximal contact angle, in the sense that if at any stage one switches from (2.1), (2.4) to prescribing the value of $A(t)$ in (2.8), then, if $A(t)$ is larger than the $h_x|_{x=0}$ determined using (2.1), (2.4), we expect that $J(t) < 0$ in (2.8)). Moreover, for (2.1), (2.4) we have minimal (zero) fluid drainage ($J \equiv 0$), while we conjecture that (2.1), (2.9) maximises $J(t)$. Similarity solutions for these two "extreme" cases are considered throughout the paper. Notice that for similarity solutions of the form (1.2) the fluid drainage is characterized by $\alpha - \beta$, since the mass decreases with time as $1/t^{\alpha-\beta}$, provided that $\alpha > \beta$.

Another interesting possibility is to prescribe zero pressure at $x = 0$, $B(t) = 0$ in (2.8). This leads to the boundary conditions $h = h_{xx} = 0$ at $x = 0$. Then (2.8) requires $3 - n > 2$, i.e. $n < 1$. Similarity solutions with $h = h_{xx} = 0$ at $x = 0$ are very briefly considered in Section 6, Remark 6.4.

As already noted, for (2.1), (2.4) the mass (2.15) is preserved; by contrast, for (2.1), (2.9) one seems to have a conservation law only for $n = 1$ (cf. [39])

$$\frac{d}{dt}\int_0^{s(t)} xh(x,t)\,dx = -\left[xhh_{xxx} - hh_x + \frac{1}{2}h_x^2\right]_0^{s(t)} = 0,$$

from which follows conservation of the first moment

$$\int_0^{s(t)} xh(x,t)\,dx = M_1 \equiv \int_0^{s_0} xh_0(x)\,dx. \tag{2.16}$$

2.3 The case $n \geq 3$

For the sake of completeness we briefly mention the case $n \geq 3$. Formal asymptotics strongly indicates that the conditions at the free boundary are

$$h = h^n h_{xxx} = 0, \quad s(t) = \text{ Constant } = s_0 \quad \text{for } n \geq 3,$$

(fixed front), and hence the same conditions apply to the draining edge $x = 0$. Mass is conserved. (A proof of the fact that $s(t)$ is constant is known only for $n \geq 4$: see [9]). Furthermore, formal asymptotics also suggests that the large-time behaviour is given by the steady-state solution:

$$h \sim \frac{6M}{s_0^3}\,x(s_0 - x)_+ \quad \text{as } t \to \infty, \quad (n \geq 3).$$

3 Mass preserving similarity solutions ($n < 3$)

In this section we consider Problem (1.5) and prove Theorems 1.1 and 1.2. The main proofs require a separate treatment for $n > 1$ and $n < 1$. For $n = 1$ the unique solution has a simple explicit formula given in the introduction.

3.1 Auxiliary results and asymptotic behaviour

We begin with a maximum principle which allows singular behaviour at the end points of the interval, so that it can be applied to a rescaled version of Problem (1.5).

Proposition 3.1 (Maximum principle). *Assume that*

$$u \in C^3(0,1) \cap C^1[0,1] \quad , \quad u(0) = u(1) = u'(1) = 0,$$

$u''' \geq 0$ *in* $(0,1)$ *and* $u \not\equiv 0$. *Then* $u > 0$ *in* $(0,1)$, u *has a unique maximum,* $\sqrt{u}$ *is concave,* $u'(0) > 0$ *and*

$$u''(0^+) < 0 \quad , \quad u''(1^-) > 0, \tag{3.1}$$

where the limits in (3.1) may not be finite. If, in addition, $u'''(x) > 0$ *near* $x = 1$ *then* $\sqrt{u}$ *is strictly concave in* $(0,1)$.

Proof. Since $u''' \geq 0$ we have that u' is convex, u'' is increasing and the limits $u''(0^+)$ and $u''(1^-)$ exist. By Rolle's theorem u' and u'' have at least one zero in $(0,1)$. If u' has two or more zeros in $(0,1)$, then (by convexity) $u' \geq 0$ in $(0,1)$ and hence $u \equiv 0$. This contradiction proves that u' has exactly one zero in $(0,1)$ and hence u has a unique maximum. Now (3.1), $u'(0) > 0$ and $u > 0$ follow easily. Setting $g = uu'' - (u')^2/2$ we have that

$$2(\sqrt{u})'' = \frac{d}{dx}\left(\frac{u'}{\sqrt{u}}\right) = u^{-3/2}\, g \quad \text{and} \quad g' = uu''' \geq 0.$$

Therefore g is increasing and $\lim_{x\to 1} g(x)$ exists. Since this limit is equal to $\lim_{x\to 1} u(x)u''(x)$, both limits are zero. (If they are not zero, u' is unbounded). Thus $g \leq 0$ in $(0,1)$ and $\sqrt{u}$ is concave. If, in addition, $u'''(x) > 0$ near $x = 1$ then $g < 0$ in $(0,1)$ and $\sqrt{u}$ is strictly concave. □

Next we consider some properties of a Green's function $G(x,t)$ for the third derivative. For simplicity of notation we denote the partial x-derivatives of G by $G'(x,t), G''(x,t), \ldots$ and define G by

$$\begin{cases} G'''(x,t) = \delta(x-t) \quad , \quad 0 < x < 1\,,\ 0 < t < 1, \\ G(0,t) = G(1,t) = G'(1,t) = 0 \quad , \quad 0 < t < 1. \end{cases} \tag{3.2}$$

By explicit computation we find that

$$G(x,t) = \begin{cases} (t-t^2)x - \frac{1}{2}(1-t^2)x^2 & \text{if} \quad x < t \\ \frac{1}{2}t^2(1-x)^2 & \text{if} \quad x > t\,. \end{cases} \tag{3.3}$$

We proceed to list some properties of $G(x,t)$. To begin with, and as expected from Proposition 3.1, $G(x,t) > 0$ for $x, t \in (0,1)$. Secondly,

$$G(x,t) \leq Ct^2(1-t) \quad \text{and} \quad |G'(x,t)| \leq Ct(1-t) \quad \text{for all } x,t \in [0,1], \tag{3.4}$$

where C is a positive constant. Finally, we derive from (3.3) two more relations, which will be used to obtain *lower* bounds in Section 3.2:

$$\int_0^x tG(x,t)\,dt = \frac{1}{8}x^4(1-x)^2\,, \tag{3.5}$$

$$\int_x^1 tG(x,t)\,dt = \frac{x}{24}(1-x)^3(3x^2+3x+2) \geq \frac{x}{12}(1-x)^3 \text{ if } 0<x<1\,. \tag{3.6}$$

Proof of the asymptotic behaviour (Theorem 1.2). As $\eta \to a$ the proof is identical to that in [15]. By the scaling indicated below for Problem (3.7), we can apply Proposition 3.1 to Problem (1.5). Hence $f'(0)>0$ and $f(\eta) \sim f'(0)\eta$ as $\eta \to 0$. By (3.1) $f''(0^+)<0$ and $f'''(\eta) \sim C\eta^{2-n}$ implies that $f''(0^+) \neq \infty$ for all $n<3$. Notice that the asymptotic behaviour has been proved without assuming uniqueness of the solution. □

3.2 Existence of solution for $1<n<3$

Performing the scaling $u(x) = cf(\eta)$, $\eta = ax$ with $c^n a^4 = n+4$, we reduce Problem (1.5) to the following equivalent problem.

$$\begin{cases} u''' = xu^{1-n} & \text{for } 0<x<1, \\ u(0)=u(1)=u'(1)=0, & \\ u \in C^3(0,1) \cap C^1[0,1], & u(x)>0 \quad \text{for } 0<x<1. \end{cases} \tag{3.7}$$

Proof of existence

Step 1: Approximating problems. For each positive integer k we consider the (nonsingular) problem

$$\begin{cases} u_k''' = xu_k^{1-n} & \text{for } 0<x<1, \\ u_k(0)=u_k(1)=\frac{1}{k}\,, & u_k'(1)=0, \\ u_k \in C^3[0,1], & u_k(x)>0 \quad \text{for } 0\leq x\leq 1. \end{cases} \tag{3.8}$$

Existence of solution for Problem (3.8). Consider the closed convex set

$$A = \{v \in C[0,1] : v \geq 1/k \text{ in } (0,1)\}$$

and the nonlinear operator T_k defined by

$$(T_k v)(x) = \frac{1}{k} + \int_0^1 G(x,t)tv(t)^{1-n}dt\,,$$

where G is the Green's function defined by (3.2). Recall that $G>0$ and $n>1$. It is clear that T_k maps A into A and that T_k is continuous with respect to the maximum norm topology. Furthermore, $T_k(A)$ is (for each k) a bounded subset of $C^3[0,1]$ and hence a relatively compact subset of A. By Schauder's fixed point theorem, there exists $u_k \in T_k(A)$ such that $T_k u_k = u_k$. This is the desired solution of Problem (3.8). Notice that u_k satisfies

$$u_k(x) = \frac{1}{k} + \int_0^1 G(x,t) u_k'''(t)\, dt, \quad u_k'(x) = \int_0^1 G'(x,t) u_k'''(t)\, dt. \tag{3.9}$$

By Proposition 3.1 (applied to $u_k - 1/k$) there exists a unique point m_k in which the maximum of u_k is attained. Therefore

$$u_k \text{ is increasing in } (0, m_k) \text{ and decreasing in } (m_k, 1). \tag{3.10}$$

Step 2: Estimates. In the sequel C stands for a positive constant *independent* of k; this constant may be different in different occurrences. The main idea is to obtain a *lower* bound of u_k. Since $G > 0$ we have that

$$u_k(x) - \frac{1}{k} = \int_0^1 G(x,t) t u_k(t)^{1-n}\, dt \geq \int_0^x G(x,t) t u_k(t)^{1-n}\, dt$$

and from (3.10) and (3.5) it follows that if $0 < x < m_k$

$$u_k(x) \geq u_k(x)^{1-n} \int_0^x tG(x,t)\, dt \geq \frac{1}{8} x^4 (1-x)^2 u_k(x)^{1-n}\,.$$

Hence

$$u_k(x) \geq C x^{4/n} (1-x)^{2/n} \quad \text{if } 0 < x < m_k\,. \tag{3.11}$$

Writing $\int_0^1 \geq \int_x^1$ and using (3.10) and (3.6), a similar argument gives

$$u_k(x) \geq C x^{1/n} (1-x)^{3/n} \quad \text{if } m_k < x < 1\,. \tag{3.12}$$

¿From (3.11)-(3.12) the desired lower bound follows:

$$u_k(x) \geq C x^{4/n} (1-x)^{3/n} \quad \text{for all } x \in (0,1). \tag{3.13}$$

Next we deduce from the differential equation that, for all $x \in (0,1)$,

$$x(1-x) u_k'''(x) = x^2 (1-x) u_k^{1-n}(x) \leq C x^{-2+4/n} (1-x)^{-2+3/n}. \tag{3.14}$$

This estimate is strong enough to pass to the limit and complete the existence proof, as we show below. Observe that $n < 3$ implies that

$$-2 + 4/n > -1 \quad \text{and} \quad -2 + 3/n > -1\,, \tag{3.15}$$

and hence the right hand side of (3.14) is an integrable function.

Step 3: Passing to the limit. From (3.9), (3.4) and (3.14)-(3.15) it follows that u_k is bounded in $C^1[0,1]$. Therefore there exist a subsequence, again denoted by u_k and a function u such that $u_k \to u$ uniformly on $[0,1]$.

Thus $u(0) = u(1) = 0$ and, by (3.13), $u > 0$ in $(0,1)$. Hence, for each compact subset K of $(0,1)$ we have that

$$u_k'''(x) = xu_k(x)^{1-n} \;\to\; xu(x)^{1-n} \quad \text{in } C(K),$$

and, by (3.14)-(3.15) and Lebesgue's dominated convergence theorem,

$$x(1-x)u_k'''(x) \;\to\; x^2(1-x)u(x)^{1-n} \quad \text{in } L^1(0,1) \tag{3.16}$$

Since $u_k''' \to u'''$ in the distribution sense, it follows that

$$u'''(x) = xu(x)^{1-n} \quad \text{in } (0,1), \tag{3.17}$$

i.e., u satisfies the differential equation. Furthermore, from (3.16), (3.17), (3.9) and (3.4) we deduce that $u_k \to u$ in $C^1[0,1]$ and hence u also satisfies $u'(1) = 0$. This completes the existence proof for $1 < n < 3$. □

Remark 3.2 We recall that for $n \geq 3$ it is known from [15] that Problem (1.5) and Problem (3.7) have no solution.

3.3 Existence of solution for $n \leq 1$

We are going to perform a shooting argument. Consider the initial-value problem

$$\begin{cases} u''' = xu^{1-n} & x > 0, \\ u(0) = 0, \; u'(0) = 1, \; u''(0) = -p, \end{cases} \tag{3.18}$$

where p is the shooting parameter. We look for a value of p such that u_p has a first zero a_p and $u_p'(a_p) = 0$. Then a scaling transformation gives a solution of Problem (1.5), as explained below for Problem (3.19).

Since u^{1-n} is not Lipschitz continuous at $u = 0$, the uniqueness of solution is not standard. However, by the theorem of [14] Problem (3.18) has indeed a unique solution. Since uniqueness holds, continuity with respect the parameter p also holds: see Hartman's book [30], Theorem V 2.1, p. 94. Consider the sets

$$S^+ = \{p \in \mathbb{R} \;:\; u_p(x) > 0 \text{ for all } x > 0\},$$

$$S^- = \{p \in \mathbb{R} \;:\; u_p \text{ has a first positive zero } a_p \text{ and } u_p'(a_p) < 0\}.$$

By continuity in the parameter p both sets are open. Any $p \leq 0$ belongs to S^+. We proceed to prove that S^- is nonempty. Take $p > 0$. Then $u_p'' \leq 0$ in some maximal interval $[0, c_p]$. Hence

$$u_p'(x) \leq 1, \quad u_p(x) \leq x \quad \text{and} \quad u_p'''(x) \leq x^{2-n} \quad \text{if } x \in [0, c_p].$$

Then $u_p''(x) \leq -p + x^{3-n}/(3-n)$ and

$$x_p \equiv ((3-n)p)^{1/(3-n)} \quad \text{is in} \quad [0, c_p].$$

¿From the bound $u_p'''(x) \le x^{2-n}$ and Taylor's formula it follows that

$$u_p(x_p) \le x_p - \frac{p}{2}x_p^2 + \frac{1}{(3-n)(4-n)(5-n)}x_p^{5-n}.$$

Computing the right-hand side we obtain

$$u_p(x_p) \le x_p\left(1 - \frac{6-n}{2(4-n)(5-n)}((3-n)p)^{(4-n)/(3-n)}\right),$$

which becomes negative for p large enough. Hence there exists p such that u_p has a first zero a_p. If $u_p'(a_p) = 0$ then u_p is the desired solution. And if not the set S^- is nonempty and there exists q such that $q \notin S^+$ and $q \notin S^-$. Therefore u_q has a first zero a_q and $u_q'(a_q) = 0$. This completes the existence proof for $n \le 1$.

Remark 3.3 The above proof in fact shows that Problem (1.5) has a solution if $-4 < n \le 1$. The restriction $n > -4$ appears in the scaling transformation, which requires $n + 4 > 0$.

3.4 Uniqueness

Proof of uniqueness for $1 < n < 3$**.** We consider Problem (3.7), which, as already noted, is equivalent to Problem (1.5). Recall that the asymptotic behaviour (Theorem 1.2) is already proved. Let u_1 and u_2 be two solutions of Problem (3.7) and set $v = u_1 - u_2$. Since $v''' = x(u_1^{1-n} - u_2^{1-n})$ and the function $u \mapsto u^{1-n}$ is decreasing we deduce that $vv''' \le 0$ and therefore $g(x) = v(x)v''(x) - (v'(x))^2/2$ is non-increasing in $(0,1)$. ¿From the asymptotic behaviour of the solutions at $x = 0$ and at $x = 1$ it follows that $g(0^+) \le 0$ and $g(1^-) = 0$. Hence $g \equiv 0$, $vv''' \equiv 0$ and $v \equiv 0$. This completes the uniqueness proof for $1 < n < 3$. □

Proof of uniqueness for $n \le 1$**.** Performing the scaling $u(x) = cf(\eta)$, $\eta = bx$ with $c^n b^4 = n + 4$ and $cbf'(1) = 1$, we reduce Problem (1.5) to the following equivalent problem.

$$\begin{cases} u''' = xu^{1-n} & \text{for } 0 < x < a/b, \\ u(0) = u(a/b) = u'(a/b) = 0, & u'(0) = 1 \\ u \in C^3(0,a/b) \cap C^1[0,a/b], & u(x) > 0 \quad \text{for } 0 < x < a/b. \end{cases} \tag{3.19}$$

Let u_1, a_1 and u_2, a_2 be two solutions of Problem (3.19) and set $v = u_1 - u_2$. Notice that the interval of positivity $(0, a_i)$ of each solution may, in principle, be different. Assume for contradiction that $u_1 \ne u_2$. Then, by the uniqueness theorem of [14], $u_1''(0) \ne u_2''(0)$ and we may assume that $u_1''(0) > u_2''(0)$. Because the function $u \mapsto u^{1-n}$ is non-decreasing we deduce that $u_1 > u_2$ as long as both are positive. Hence $a_2 < a_1$. We also deduce that in $(0, a_2)$ all the three functions v, v' and v'' are increasing and positive. Then

$$0 < v'(a_2) = u_1'(a_2) \tag{3.20}$$

$$\text{and} \quad 0 < v''(a_2^-) = u_1''(a_2) - u_2''(a_2^-).$$

By Proposition 3.1 $u_2''(a_2^-) > 0$ and hence $u_1''(a_2) > 0$. Since $u''' > 0$ it follows that $u_1'' > 0$ in (a_2, a_1) and hence $u_1'(a_2) < 0$. This contradicts (3.20) and proves that $u_1 \equiv u_2$. Furthermore, this also completes the proof of Theorem 1.1. □

Remark 3.4 The above proof actually shows that the solution of Problem (1.5) is unique if $-4 < n \leq 1$. (The scaling transformation leading to (3.19) requires $n + 4 > 0$). In conclusion, we have proved that the solution of Problem (1.5) exists and is unique for all $n \in (-4, 3)$.

4 The standard dipole solution for $n = 1$

4.1 A singular initial value problem

The following problem will play an important role in the study of Problem (1.7).

$$\begin{cases} u(x)u'''(x) = -\int_x^1 u(s)\,ds + xu(x)\,, & x < 1, \\ u(1) = u'(1) = 0\,, & u''(1) = b, \\ u \in C^3[1-\varepsilon, 1) \cap C^2[1-\varepsilon, 1], & u(x) > 0 \quad \text{for } 1 - \varepsilon \leq x < 1. \end{cases} \tag{4.1}$$

Theorem 4.1 *Let $b > 0$. Then there exists $\varepsilon > 0$ such that Problem (4.1) has a unique solution.*

The method of proof is inspired by [14]. Using Taylor's formula

$$u(x) = \frac{b}{2}(1-x)^2 - \frac{1}{2}\int_x^1 (t-x)^2 u'''(t)\,dt, \tag{4.2}$$

and the relations

$$\int_x^1 (t-x)^2 \frac{dt}{u(t)} \int_t^1 u(s)\,ds = \iint_{x<t<s<1} (t-x)^2 \frac{u(s)}{u(t)}\,ds\,dt, \tag{4.3}$$

$$\frac{1}{2}\int_x^1 (t-x)^2 t\,dt = (1-x)^3\,\frac{x+3}{24}, \tag{4.4}$$

we rewrite Problem (4.1) as the integral equation

$$u(x) = c(1-x)^2 + \frac{1}{2}\iint_{x<t<s<1} (t-x)^2 \frac{u(s)}{u(t)}\,ds\,dt - (1-x)^3\,\frac{x+3}{24}, \tag{4.5}$$

where $c = b/2$. Now we consider the new unknown function

$$y(x) = u(x)/(1-x)^2.$$

The function $y(x)$ is continuous up to $x = 1$ if we set $y(1) = c$. Hence Problem (4.1) is reduced to finding a function $y \in C[1-\varepsilon, 1]$ such that $y(1) = c$ and y satisfies the integral equation

$$y(x) = (Ty)(x), \tag{4.6}$$

where T is defined by

$$(Ty)(x) = c + \iint_{x<t<s<1} g(x,s,t)\frac{y(s)}{y(t)}\, ds\, dt - (1-x)\frac{x+3}{24}, \tag{4.7}$$

$$g(x,s,t) = \frac{1}{2}\frac{(t-x)^2(1-s)^2}{(1-x)^2(1-t)^2}.$$

Notice that

$$g(x,s,t) \leq 1/2 \quad \text{if } x < t < s < 1. \tag{4.8}$$

Next we consider the metric space

$$E = C(\,[1-\varepsilon, 1]\,;\,[c/2, 3c/2]\,)$$

with the usual distance:

$$d(y,z) = \sup_{1-\varepsilon<x<1} |y(x) - z(x)|.$$

Lemma 4.2 *For $\varepsilon > 0$ small enough T maps E into E.*

Proof. Let $y \in E$. From (4.7)-(4.8), $y(s) \leq 3c/2$ and $y(t) \geq c/2$ it follows that Ty is continuous in $[1-\varepsilon, 1]$ and

$$|(Ty)(x) - c| \leq \frac{3}{2}\iint_{x<t<s<1} ds\, dt + \frac{1}{6}(1-x).$$

Notice that this double integral is equal to $(1-x)^2/2 \leq \varepsilon^2/2$. Hence for $\varepsilon > 0$ small enough $|(Ty)(x) - c| \leq c/2$ and $Ty \in E$. □

Lemma 4.3 *For $\varepsilon > 0$ small enough $T : E \to E$ is a contraction mapping.*

Proof. Let $y, z \in E$. From (4.7) and (4.8) we obtain

$$|(Ty)(x) - (Tz)(x)| \leq \frac{1}{2}\iint_{x<t<s<1} \left|\frac{y(s)}{y(t)} - \frac{z(s)}{z(t)}\right| ds\, dt.$$

Observing that $y(t)z(t) \geq c^2/4$ and

$$|y(s)z(t) - y(t)z(s)| \leq y(s)|z(t) - y(t)| + y(t)|y(s) - z(s)| \leq 3c\, d(y,z),$$

we obtain $d(Ty, Tz) \leq (3/c)\varepsilon^2 d(y, z)$. $\square$

Proof of Theorem 4.1 (conclusion). By the Picard-Banach fixed point theorem, it follows that for ε small enough the integral equation (4.6)-(4.7) has a unique solution $y \in C[1-\varepsilon, 1]$ with $y(1) = c$. This completes the proof of Theorem 4.1. $\square$

Since uniqueness holds, continuity with respect to b also holds: see Hartman's book [30], Theorem V 2.1, p. 94. More specifically, let $(\omega(b), 1]$ be the left maximal interval of existence for the solution $u = u_b$ of Problem (4.1). Then $u_b(x)$ is continuous in (x, b) in the set defined by $b > 0$ and $\omega(b) < x \leq 1$. Furthermore, $\omega(b)$ is upper semicontinuous. Therefore, the set $S = \{b > 0 : \omega(b) < 0\}$ is open.

4.2 A priori observations and asymptotic behaviour

By setting $u(x) = cf(\eta)$, $\eta = ax$ with $ca^4 = 6$, we reduce Problem (1.7) to the following equivalent problem:

$$\begin{cases} (uu''')' = 2u + xu' & \text{for } 0 < x < 1, \\ u(0) = u'(0) = u(1) = u'(1) = 0, & u(x)u'''(x) \to 0 \quad \text{as } x \to 1, \\ u \in C^4(0,1) \cap C^1[0,1], & u(x) > 0 \quad \text{for } 0 < x < 1, \end{cases} \tag{4.9}$$

By integrating once we obtain

$$u(x)u'''(x) = -\int_x^1 u(s)\,ds + xu(x). \tag{4.10}$$

Recalling that $uu''' = (uu'' - (u')^2/2)'$ a further integration yields

$$u(x)u''(x) - \frac{1}{2}(u'(x))^2 = -x\int_x^1 u(s)\,ds, \tag{4.11}$$

where we take into account that $\lim_{x\to1} u(x)u''(x)$ exists and hence is zero. (If it is not zero, u' is unbounded). Observing that

$$2(\sqrt{u})'' = \frac{d}{dx}\left(\frac{u'}{\sqrt{u}}\right) = u^{-3/2}(uu'' - \frac{1}{2}(u')^2), \tag{4.12}$$

the following lemma holds.

Lemma 4.4 *If u is a solution of Problem 4.9 then u satisfies (4.10) and (4.11), $\sqrt{u}$ is strictly concave and $u'/\sqrt{u}$ is strictly decreasing in $(0,1)$. Hence u attains its maximum at a unique point $m \in (0,1)$.*

Consider (4.10) with $m < x < 1$. Since u is decreasing in $(m, 1)$ we have that $u(s) \leq u(x)$ and

$$\frac{1}{u(x)} \int_x^1 u(s)\, ds \leq 1 - x \quad \text{if } x \in (m, 1). \tag{4.13}$$

Therefore

$$u'''(1^-) = 1 \quad \text{and} \quad 0 < u''(1^-) < \infty, \tag{4.14}$$

and we have proved the following:

Lemma 4.5 *If u is a solution of Problem (4.9) then (4.13) and (4.14) hold.*

The above arguments suggest to perform a shooting argument with respect to the parameter b of Problem (4.1) in order to prove the existence of solution of Problem (4.9). This will be done in Section 4.3.

Proof of the asymptotic behaviour (Theorem 1.4). Setting $u''(1^-) = b$, by (4.14) and (4.10) we have that

$$u(x) \sim \frac{b}{2}(1-x)^2 \text{ as } x \to 1 \quad \text{and} \quad \lim_{x \to 0} u(x) u'''(x) = -\int_0^1 u(s)\, ds.$$

On the other hand, from (4.11) and (4.12) it follows that the function $g = \sqrt{u}$ satisfies

$$2(g(x))^3 g''(x) = -x \int_x^1 u(s)\, ds,$$

which implies that $g(x) \sim C x^{3/4}$ by a result of Taliaferro [46]. Finally, notice that by scaling (4.14) implies that $f'''(\eta) \to a/6$ as $\eta \to a$. □

4.3 Existence of the dipole solution: shooting method

This section is devoted to prove the existence assertion of Theorem 1.3. (Uniqueness will be proved in Section 5.5). We perform a shooting argument with respect to the parameter b of Problem (4.1). As explained at the end of Section 4.1, the set $S = \{b > 0 : \omega(b) < 0\}$ is open. The main difficulty is that we cannot directly assure that the set $S^* = \{b > 0 : \omega(b) > 0\}$ is open, because in general we only know that $\omega(b)$ is upper semicontinuous. (See also the remark at the end of this section). We proceed along a series of lemmas.

Lemma 4.6 *If $b > 1/4$ then $b \in S$.*

Proof. ¿From (4.5) we have

$$u(x) \geq \frac{b}{2}(1-x)^2 - (1-x)^3 \frac{x+3}{24} = \frac{1}{24}(1-x)^2(x^2 + 2x + 12b - 3), \tag{4.15}$$

whence $u > 0$ in $[0, 1)$ if $b > 1/4$. □

Lemma 4.7 *Let u_b be the solution of Problem (4.1). Then*

$$u_b'''(x) \geq x - \frac{1-x}{x^2} \quad \textit{for all } b \in S \textit{ and all } x \in (0,1).$$

Proof. We drop the subindex b for simplicity. Let $0 \leq m < 1$ be the unique point where the maximum of u in $[0,1]$ is attained. Notice that the case $m = 0$ is also considered. Taking into account (4.10), it is enough to prove that

$$\frac{1}{u(x)} \int_x^1 u(s)\,ds \leq \frac{1-x}{x^2} \text{for all } x \in (0,1).$$

If $x \in (m,1)$ this follows from (4.13). Furthermore, by the concavity of $\sqrt{u}$,

$$\sqrt{u(x)} \geq \sqrt{u(m)}\, x/m \geq \sqrt{u(m)}\, x \quad \text{if } x \in (0,m). \tag{4.16}$$

This and $u(s) \leq u(m)$ imply the desired inequality when $0 < x < m$. □

Lemma 4.8 *Let $b_0 = \inf S$. Then $b_0 > 0$.*

Proof. ¿From Lemma 4.7, (4.2), (4.4) and

$$\frac{1}{2} \int_x^1 (t-x)^2 \, \frac{1-t}{t^2}\, dt \leq \frac{1}{2x^2} \int_x^1 (t-x)^2 (1-t)\, dt = (1-x)^4 \frac{1}{24x^2}\,,$$

it follows that

$$u_b(x) \leq \frac{b}{2}(1-x)^2 - (1-x)^3 \frac{x+3}{24} + (1-x)^4 \frac{1}{24x^2} \quad \text{for all } b \in S \text{ and all } x \in (0,1).$$

Since $u_b > 0$ in $(0,1)$ for $b \in S$, this implies

$$\frac{b}{2} \geq (1-x)\left(\frac{x+3}{24} - \frac{1-x}{24x^2}\right) \quad \text{for all } b \in S \text{ and all } x \in (0,1).$$

Setting e.g. $x = 1/2$ we obtain $b \geq 3/48$ for all $b \in S$. □

Lemma 4.9 *Let $M_b = \max u_b = u_b(m_b)$ and $M_0 = \inf_{b \in S} M_b$. Then $M_0 > 0$.*

Proof. By (4.15), if $b \geq 1$ then $M_b \geq u_b(0) \geq 3/8$, while if $0 < b < 1$ again from (4.15) it follows that $M_b \geq u(1-b) \geq b^3/3 + b^4/24 > b^3/3$. Now Lemma 4.9 is a consequence of Lemma 4.8. □

Lemma 4.10 *Let $M_0 > 0$ be the number defined in the former lemma. Then*

$$u_b(x) \geq M_0 x^2 (1-x)^2 \quad \textit{for all } b \in S \textit{ and all } x \in (0,1).$$

Proof. The concavity of $\sqrt{u}$ implies (4.16) and

$$\sqrt{u(x)} \geq \sqrt{u(m)}\,(1-x)/(1-m) \geq \sqrt{u(m)}\,(1-x) \quad \text{if } x \in (m,1).$$

The conclusion of the lemma follows from the definition of M_0 and the inequalities $x \geq x(1-x)$ and $1-x \geq x(1-x)$ for $0 < x < 1$. □

Lemma 4.11 *Let* $u_0 = u_b$ *for* $b = b_0$. *Then* u_0 *is defined in* $(0,1]$, $u_0 \in C^1[0,1]$, $u_0(0) = 0$ *and* $u_0'(0) = 0$. *Therefore* u_0 *is a solution of Problem (4.9).*

Proof. Let $(\omega_0, 1]$ be the left maximal interval of existence of u_0. Since S is an open set, $b_0 \notin S$ and $\omega_0 \geq 0$. By Lemma 4.10 we have $\omega_0 = 0$ and u_0 satisfies the conclusion of this lemma. The concavity of $\sqrt{u_0}$ implies that the limit $u_0(0^+)$ exists and is finite, hence it is zero. From (4.10) we know that u_0''' is negative and u_0' concave near $x = 0$. Hence the limit $u_0'(0^+)$ exists, is finite and non-negative. Suppose for contradiction that $u_0'(0^+) > 0$. Then from (4.11) it follows that $\lim_{x\to 0} u_0(x)u_0''(x) > 0$, which implies that u_0' is unbounded. This contradiction proves that $u_0'(0^+) = 0$ and completes the proof of the lemma. Hence, this also proves the existence assertion of Theorem 1.3. □

Remark 4.12 . The strategy of this section can be reworded in the following way. Lemma 4.8 implies that the set $S^* = \{b > 0 : \omega(b) > 0\}$ is nonempty. In turn, Lemma 4.10 implies that the complementary set of S^* is closed and hence S^* is open. Since S and S^* are two nonempty and disjoint open sets, there exists $b > 0$ such that $\omega(b) = 0$, which gives the desired solution.

5 A dynamical systems approach for the dipole solution when $n = 1$

In this section we give another, very different proof of the existence of solution for Problem (1.7), as well as the uniqueness proof for the same problem (cf. Theorem 1.3). Rewriting the ODE for f in (1.7) as

$$\left(\eta f f''' - f f'' + \frac{1}{2} f'^2 - \frac{1}{6}\eta^2 f\right)' = 0,$$

integration gives, using $f = f' = ff''' = 0$ at the free boundary $\eta = a$,

$$f''' = \frac{f''}{\eta} - \frac{f'^2}{2\eta f} + \frac{\eta}{6}\,. \tag{5.1}$$

We look for a solution $f(\eta)$ defined on an η-interval $[0, a]$ satisfying

$$f(0) = f'(0) = 0; \quad f(\eta) > 0 \text{ if } 0 < \eta < a; \quad f = f' = ff''' = 0 \text{ in } \eta = a. \tag{5.2}$$

Notice that the scale invariance of Equation (5.1) corresponds to the translation invariance of the autonomous system (5.4) below. The additional condition $\int_0^a \eta f(\eta)\, d\eta = M_1$, cf. Problem (1.7), fixes the loss of uniqueness caused by these invariance groups.

5.1 The quadratic system

Following [34] and [35] we introduce

$$x = \frac{\eta f'}{f}; \quad y = \frac{\eta^2 f''}{f}; \quad z = \frac{\eta^4}{6f}; \quad t = \log\eta, \tag{5.3}$$

where x and t are not to be confused with the independent variables above, and obtain the quadratic 3-dimensional dynamical system

$$\dot{x} = x(1-x) + y; \quad \dot{y} = y(3-x) - \frac{1}{2}x^2 + z; \quad \dot{z} = z(4-x). \tag{5.4}$$

The solution f we look for will correspond to a particular connection in the phase space of this system between two critical points, with one of the two lying at infinity.

5.2 The finite critical points

The finite critical points (i.e. the critical points of (5.4) in $\mathbb{R}^3$) are

$$(0,0,0), \quad (3/2, 3/4, 0), \quad (2,2,0) \text{ and } (4,12,20). \tag{5.5}$$

We denote the matrix of the linearised system around a critical point P by $A(P)$. Stable and unstable manifolds are denoted by $W^s(P)$ and $W^u(P)$.

The eigenvalues of $A(0,0,0)$ are 1, 3 and 4 and so $(0,0,0)$ is a source. Orbits coming out of it correspond to solutions $f(\eta)$ with $f(0) > 0$ and $f'(0)$ and $f''(0)$ arbitrary.

The eigenvalues of $A(3/2,3/4,0)$ are $5/2$ and $(-1 \pm \sqrt{13})/4$. Thus $\dim W^s(3/2,3/4,0) = 1$ and $\dim W^u(3/2,3/4,0) = 2$. Notice that $5/2$ is the largest eigenvalue. The stable manifold is contained in $z = 0$ and has no meaning for the f-equation here but the unstable manifold, which is tangent to the plane spanned by $(7-\sqrt{13}, 9, 0)$ and $(4, 18, 27)$, contains solutions with

$$x(t) = \frac{3}{2} + O(e^{\lambda t}) \text{ as } t \to -\infty \text{ with } \lambda > 0 \quad \Rightarrow$$

$$\frac{f'(\eta)}{f(\eta)} = \frac{3}{2\eta} + O(\eta^{\lambda-1}) \text{ as } \eta \downarrow 0,$$

so that

$$f(\eta) \sim c\eta^{3/2} \text{ as } \eta \downarrow 0 \text{ with } c > 0, \qquad \text{and } f'(0) = 0. \tag{5.6}$$

The eigenvalues of $A(2,2,0)$ are -1 (algebraic multiplicity 2, geometric multiplicity 1) and 2. Thus $\dim W^s(2,2,0) = 2$ and $\dim W^u(2,2,0) = 1$. Again the stable manifold is contained in $z = 0$ and has no meaning for the f-equation. The unstable manifold, which is tangent to the line spanned by $(1,5,9)$, contains solutions with $x(t) \to 2$ exponentially, implying $f(\eta) \sim c\eta^2$ as $\eta \downarrow 0$ with $c > 0$.

The eigenvalues of $A(4,12,20)$ are -1.52 and $-3.24 \pm 1.63i$. Thus $(4,12,20)$ is a sink. Orbits going into it contain solutions with $f(\eta) \sim \eta^4/20$ as $\eta \uparrow +\infty$.

5.3 Critical points at infinity

Next we examine the behaviour of unbounded orbits using a Poincaré type transformation. Introduce new dependent variables X, Y, Z and V by

$$x = \frac{X}{V}, \quad y = \frac{Y}{V^2}, \quad z = \frac{Z}{V^2} \text{ and } X^2 + Y^2 + 2Z = 1 \text{ with } Z \geq 0, \tag{5.7}$$

and a new independent variable τ by

$$(X^2 + 2Y^2 + 2Z)V\frac{d}{dt} = (1 + Y^2)V\frac{d}{dt} = \frac{d}{d\tau}. \tag{5.8}$$

Then $\{(x, y, z) : z \geq 0\}$ corresponds to $\{(X, Y, V) : X^2 + Y^2 \leq 1, V > 0\}$ and unbounded (x, y, z)-orbits correspond to (X, Y, V)-orbits with V approaching zero, i.e. (X, Y, V) approaches the invariant set $\{V = 0\}$. Also the invariant set $\{z = 0\}$ corresponds to $\{X^2 + Y^2 = 1\}$.

Eliminating Z this gives the following system for X, Y and V:

$$\begin{cases} \frac{dX}{d\tau} = \frac{1}{2}(X^2 - 2Y)(X^2 - Y^2 - 1) + X(X^2 - 1 + \frac{Y}{2}(2X^2 + Y^2 - 1))V; \\ \frac{dY}{d\tau} = XY(X^2 - 2Y) + \frac{1}{2}(Y^2 + 2Y - 1)(2X^2 + Y^2 - 1)V; \\ \frac{dV}{d\tau} = \frac{1}{2}X(X^2 + (Y-1)^2)V + (X^2 - Y^2 - 2 + Y(X^2 + \frac{1}{2}Y^2 - \frac{1}{2}))V^2. \end{cases} \tag{5.9}$$

Modified Poincaré transformation like the one above were introduced in [10]. The critical points of the (X, Y, V)-system at $\{X^2 + Y^2 \leq 1, V = 0\}$ are

$$(X, Y) = (\pm 1, 0) \text{ and the points of the parabola } X^2 = 2Y, \tag{5.10}$$

the orbits (at $V = 0$) being given by the family of circles

$$X^2 + (Y - C)^2 = C^2 + 1. \tag{5.11}$$

(In fact each circle contains two orbits and two critical points on the parabola).

The coefficient of V in the V-equation in (5.9) determines which parts of infinity (i.e. $\{V = 0\}$) are repelling and attracting. This implies that $(X, Y, V) = (-1, 0, 0)$ is a sink and $(X, Y, V) = (1, 0, 0)$ is a source. Orbits going into the sink have

$$\frac{dV}{dt} \sim \frac{1}{V}\frac{dV}{d\tau} \sim -1 \quad \Rightarrow \quad V \sim \bar{t} - t \quad \Rightarrow \quad \frac{d\tau}{dt} \sim \frac{1}{\bar{t} - t} \quad \Rightarrow \quad \tau \sim -\log(\bar{t} - t),$$

where $\bar{t}$ is the time at which the orbit reaches infinity and corresponds to $\tau = +\infty$. (Of course $\bar{t}$ and its corresponding $\bar{\eta}$ are finite). Since the eigenvalues of the linearised equations near $(X, Y, V) = (-1, 0, 0)$ are all negative, we have for some $\lambda > 0$ that

$$Y = O(e^{-\lambda\tau}) \quad \Rightarrow \quad Y = O((\bar{t} - t)^\lambda) = O((\bar{\eta} - \eta)^\lambda),$$

where possibly the second λ has to be chosen slightly smaller. We rewrite the x-equation in (5.4) as

$$\dot{w} + w = 1 - \frac{Y}{X^2} \quad \text{for } w = \frac{1}{x} \quad \Rightarrow \quad w(t) = 1 - e^{\bar{t}-t} + e^{-t}\int_t^{\bar{t}} e^{t'}\,\frac{Y}{X^2}\,dt'.$$

This gives

$$\frac{f(\eta)}{f'(\eta)} = (\eta - \bar{\eta})\left(1 - \frac{1}{\bar{\eta}-\eta}\int_\eta^{\bar{\eta}} \frac{Y}{X^2}\,d\eta'\right) = (\eta - \bar{\eta})(1 - R(\eta)) \quad \Rightarrow$$

$$f(\eta) = f(a)\,\frac{\bar{\eta}-\eta}{\bar{\eta}-a}\,\exp\left(\int_a^\eta \frac{r(\eta')}{\eta'-\bar{\eta}}\,d\eta'\right),$$

where a is some η-value less than $\bar{\eta} = \exp \bar{t}$, and we have set $1 - R = 1/(1+r)$, hence $r = R/(1-R)$ and $r = R + R^2 + R^3 + \cdots = O(R) = O((\bar{\eta}-\eta)^\lambda)$. Thus orbits going into the sink contain solutions with $f(\eta)$ going to zero linearly as $\eta \uparrow \bar{\eta}$. A similar analysis shows that orbits coming out of the source have $f(\eta)$ going to zero linearly as $\eta \downarrow \bar{\eta} > 0$.

Points on the parabolic arc $P_- = \{X^2 = 2Y,\ V = 0,\ X < 0\}$ have two orbits coming out, both given by one of the circles above in $\{V = 0\}$, corresponding to a positive eigenvalue of their linearisation, which also has a zero eigenvalue with eigenvector tangent to the parabola. The third eigenvalue is the coefficient of V in the V-equation in (5.9) which is negative on P_-. Thus every point on it has a one-dimensional stable manifold which together form the 2-dimensional stable manifold of P_-. Orbits in this stable manifold have

$$\frac{dV}{dt} \sim \frac{1}{(1+Y^2)V}\frac{dV}{d\tau} \sim \frac{X}{2} \quad \Rightarrow \quad \frac{d\tau}{dt} \sim \frac{2}{-X(1+Y^2)(\bar{t}-t)} \quad \Rightarrow$$

$$\tau \sim -\frac{2}{-X(1+Y^2)}\log(\bar{t}-t).$$

We rewrite the x-equation in (5.4) as

$$\dot{w} + w = \frac{1}{2} + \frac{X^2 - 2Y}{2X^2} \quad \text{for } w = \frac{1}{x} \quad \Rightarrow$$

$$w(t) = \frac{1}{2}(1 - e^{\bar{t}-t}) + e^{-t}\int_t^{\bar{t}} e^{t'}\,\frac{2Y - X^2}{2X^2}\,dt'.$$

This gives

$$\frac{f(\eta)}{f'(\eta)} = (\eta - \bar{\eta})\left(\frac{1}{2} - \frac{1}{\bar{\eta}-\eta}\int_\eta^{\bar{\eta}} \frac{2Y - X^2}{2X^2}\,d\eta'\right) = (\eta - \bar{\eta})\left(\frac{1}{2} - R(\eta)\right) \quad \Rightarrow$$

$$f(\eta) = f(a)\left(\frac{\bar{\eta}-\eta}{\bar{\eta}-a}\right)^2 \exp\left(\int_a^\eta \frac{r(\eta')}{\eta'-\bar{\eta}}\,d\eta'\right),$$

where a is some η-value less than $\bar\eta = \exp\bar t$, and we have set $1/2 - R = 1/(2+r)$, hence $r = 4R/(1-2R)$ and $r = 4R + 8R^2 + 16R^3 + \cdots = O(R) = O((\bar\eta - \eta)^\lambda)$. Here we have used that

$$2Y - X^2 = O((\bar t - t)^\lambda) = O((\bar\eta - \eta)^\lambda),$$

for some $\lambda > 0$. This follows as in the previous case because orbits in the stable manifold approach P_- exponentially fast in the τ-variable. We conclude that the stable manifold of P_- contains the solutions $f(\eta)$ which go to zero quadratically as $\eta \uparrow \bar\eta$. Again a similar statement with $\eta \downarrow \bar\eta$ holds for solutions in the unstable manifold of $P_+ = \{X^2 = 2Y,\ V = 0,\ X > 0\}$.

The only critical point not discussed yet is the degenerate point $(X, Y, V) = (0, 0, 0)$. In the existence proof below we will show that the desired connection cannot converge to $(X, Y, V) = (0, 0, 0)$.

5.4 Existence of the connection

In view of the analysis above we have to show that the intersection of $W^u(3/2, 3/4, 0)$ and $W^s(P_-)$ is nonempty. Both manifolds are 2-dimensional so generically we expect them to intersect. A priori we might also have that $W^u(2, 2, 0)$ is contained in $W^s(P_-)$ but in the course of the analysis below this will be excluded.

We parametrise $W^u(3/2, 3/4, 0) \cap \{z \ge 0\}$ as a smooth family of orbits γ_μ with parameter μ such that γ_0 is the orbit contained in $\{z = 0\}$ coming out of $(3/2, 3/4, 0)$ along $(-7+\sqrt{13}, -9, 0)$, γ_1 the (fast) orbit coming out along $(4, 18, 27)$ and γ_2 the orbit contained in $\{z = 0\}$ coming out of $(3/2, 3/4, 0)$ along $(7-\sqrt{13}, 9, 0)$. For $\mu \in [0, 1)$ the orbit γ_μ starts off in the set $\{x < 3/2,\ x(1-x) + y < 0\}$ and for $\mu \in [1, 2]$ it starts off in the set $\{x > 3/2,\ x(1-x) + y > 0\}$. The latter orbits may be ignored in view of the differential equations

$$\dot w = (3 - 2x)w + z \quad \text{for} \quad w = y - \frac{x^2}{2} \tag{5.12}$$

and

$$\dot v = (4 - 3x)v + x(x - \tfrac{3}{2})(2 - x) + z \quad \text{for } v = \dot x = x(1-x) + y. \tag{5.13}$$

For $\mu \in [1, 2]$ we see from (5.12)-(5.13) and the x-equation in (5.4) that γ_μ is completely contained in the positively invariant region

$$\{x > \frac{3}{2},\ y > \min(x(1-x),\ \frac{1}{2}x^2)\},$$

so that in particular $f'(\eta) > 0$ as long as the solution $f(\eta)$ exists. The same holds for $W^u(2, 2, 0) \cap \{z \ge 0\}$.

In view of these observations we can restrict our attention to $\mu \in [0, 1)$. For $\mu = 0$ we have that γ_0 is contained in the invariant plane $\{z = 0\}$. It is easy to see from the phase plane for the reduced system that for this orbit $\dot x$ and $\dot y$

remain negative, that γ_0 enters $\{x < 0, y < 0\}$ and γ_0 escapes to infinity. In terms of X, Y and V it has no other choice than to go to $(X, Y, V) = (-1, 0, 0)$.

We introduce the maximal interval I contained in $[0, 1)$ with $0 \in I$, such that for every $\mu \in I$ the orbit γ_μ has the property that $x(t)$ decreases to zero in finite time, say, after a time shift, in $t = 0$, and then becomes negative. In view of (5.13), which says that $\dot{v} > 0$ in points where $x = \dot{x} = v = 0$, we must have $\dot{x}(0) < 0$. Thus I is open on the right and of the form $I = [0, \mu^*)$ with $0 < \mu^* < 1$.

For $\mu = \mu^*$ we have that $x(t)$ decreases until it hits $x = 0$. This may happen for $t \to \infty$ or for $t \to t^* < \infty$. In the latter case we may shift again to have $t^* = 0$. We claim that the first case is impossible. Indeed, we would then have from (5.4) that $\dot{x} \sim y$, $\dot{y} \sim z$ and $\dot{z} \sim 4z > 0$, contradicting $x \to 0$.

Next we introduce the maximal interval J contained in I with $0 \in J$, such that for all $\mu \in J$ the orbit γ_μ connects to $(X, Y, V) = (-1, 0, 0)$ and has $x(t)$ decreasing on the maximal t-interval on which it exists. Again by (5.13) we must have $\dot{x} < 0$ if $x \le 0$, so that again J is open in I and of the form $J = [0, \mu_*)$. Since $x(t)$ has a nondegenerate minimum for $\mu = \mu^*$ we cannot have that $\mu_* = \mu^*$ so $\mu_* < \mu^*$. We conclude that $x(t)$ is decreasing for $\mu = \mu_*$ and thus, in view of the maximality of J, the orbit γ_{μ_*} cannot go to $(X, Y, V) = (-1, 0, 0)$. We also cannot have that $x(t)$ remains bounded because then the exponentially growing z-term in the y-equation in (5.4) would force $y(t)$ and hence also $\dot{x}(t)$ to become large. Thus γ_{μ_*} approaches $V = 0$ where it must follow the flow of the reduced system for X and Y. Thus it can only converge to one of the critical points at $V = 0$. Since $(X, Y, V) = (-1, 0, 0)$ is already excluded, it remains to exclude $(X, Y, V) = (0, 0, 0)$ for then it must converge to a point on P_-.

So suppose that $x(t) \downarrow -\infty$ in finite or infinite time and that $(X, Y, V) \to (0, 0, 0)$. Then $Z \to 1/2$ so that also $z(t) \to \infty$. ¿From the y- and the z-equations in (5.4) we then have

$$\frac{dy}{dz} - \frac{y}{z} = \frac{z - y - \frac{1}{2}x^2}{z(4 - x)} = \frac{Z - Y - \frac{1}{2}X^2}{Z(4 - x)} \sim -\frac{1}{x} \downarrow 0.$$

This means that eventually y cannot go from positive to negative so it must remain one-signed. If it is eventually positive we have from this equation that y grows at least at the same order as z contradicting $y/z = Y/Z \to 0$. Thus eventually y must be negative. But then the x- and the z-equations in (5.4) imply that

$$\frac{dx}{dz} = \frac{x(1 - x)}{z(4 - x)} + \frac{y}{z(4 - x)} < \frac{x(1 - x)}{z(4 - x)} \sim \frac{x}{z}.$$

This implies that for any $a < 1$ a lower estimate of the form $-x \ge Cz^a$ holds whence $X^2 \ge CZ^{2a}$, again a contradiction with $X \to 0$ and $Z \to 1/2$.

As a result of the previous arguments we have that γ_{μ_*} connects $(x, y, z) = (3/2, 3/4, 0)$ to a point on P_-. This completes the proof of the existence of a dipole solution.

Remark 5.1 As byproduct we have proven that $\dot{x} < 0$, hence $\eta f'/f$ is decreasing and $ff' + \eta(ff'' - (f')^2) < 0$. This concavity like property implies also that f has a unique maximum. Recall (cf. Lemma 4.4) that we also know that $\sqrt{f}$ is concave, $f'/\sqrt{f}$ decreasing and $ff'' - (f')^2/2$ negative.

5.5 Uniqueness of the connection

To show that there are no other connections we first derive a monotonicity property. Suppose we have two orbits with $x(t)$ decreasing from $x = b$ to $x = a$ and $\dot{x} < 0$. Then we can write them as $y = y_1(x), z = z_1(x)$ and $y = y_2(x), z = z_2(x)$ respectively. Suppose also that $z_2(b) \geq z_1(b)$ and that $y_2(x) > y_1(x)$ for all $a < x < b$. We claim that then also $y_2(a) > y_1(a)$. This can be proved as follows. If the claim is false we must have $y_2(a) = y_1(a)$ and $y_2'(a) \geq y_1'(a)$. In view of the x-and y-equations in (5.4) this implies $z_1(a) \geq z_2(a)$. But on the x-interval (a, b) we have

$$\frac{1}{z_1}\frac{dz_1}{dx} = \frac{4 - x}{x - x^2 + y_1(x)} > \frac{4 - x}{x - x^2 + y_2(x)} = \frac{1}{z_2}\frac{dz_2}{dx},$$

so that $z_1(b) > z_2(b)$, a contradiction. This argument also shows that $z_2(x) > z_1(x)$ for all $a \leq x < b$.

Let $\mu_1 = \mu_*$ be the μ-value obtained in the existence proof. We have seen that the corresponding orbit, denoted now by γ_1, connects to P_- and has $\dot{x} < 0$. Suppose that for some $\mu_2 > \mu_1$ the same is true. We claim that, in the notation of the monotonicity argument above,

$$y_2(x) > y_1(x) \quad \text{and} \quad z_2(x) > z_1(x) \quad \text{for all} \quad x < \frac{3}{2}. \tag{5.14}$$

We first establish (5.14) in a neighbourhood of the critical point $(x, y, z) = (3/2, 3/4, 0)$. The two orbits under consideration are ordered in $W^u(3/2, 3/4, 0)$ for the simple reason that this is a 2-dimensional (smooth) manifold. Since $W^u(3/2, 3/4, 0)$ is tangent to the eigenvectors $(7 - \sqrt{13}, 9, 0)$ and $(4, 18, 27)$ in $(x, y, z) = (3/2, 3/4, 0)$, the projections of $W^u(3/2, 3/4, 0)$ on the (x, y)-plane and the (x, z)-plane are local diffeomorphisms. Clearly these projections preserve the ordering of orbits. Since $\mu_2 > \mu_1$, the ordering is locally as in (5.14) because of the direction of the eigenvectors and the parametrisation chosen at the beginning of Section 5.4. Once (5.14) holds locally, the monotonicity property implies that it holds for all $x < 3/2$.

By the local analysis of P_-, see Section 5.3, as $x \to -\infty$ we have that $Y_i \sim X^2/2$, and by (5.7) $y_i \sim x^2/2$. Hence

$$y_2(x) - y_1(x) = o(x^2) \quad \text{as} \quad x \to -\infty. \tag{5.15}$$

We proceed considering

$$\frac{d(y_2 - y_1)}{dx} = (y_2 - y_1)\frac{3 - x}{x(1 - x) + y_1} - \frac{\frac{1}{2}x^2}{x(1 - x) + y_2} + \frac{\frac{1}{2}x^2}{x(1 - x) + y_1}$$

$$+\frac{y_2(3-x)}{x(1-x)+y_2}-\frac{y_2(3-x)}{x(1-x)+y_1}+\frac{z_2}{x(1-x)+y_2}-\frac{z_1}{x(1-x)+y_1}\,.$$

The last line is negative for $|x|$ large enough because $y_i>0$, $z_i>0$, $x(1-x)+y_i<0$ and (5.14) holds. Dropping this line and rewriting the other terms we obtain

$$\frac{d(y_2-y_1)}{dx}<\left(\frac{2}{x}+R(x)\right)(y_1-y_2)\quad\text{as } x\to-\infty,$$

where

$$R(x)=R_1(x)+R_2(x)=$$

$$\frac{x+x^2-2y_1(x)}{x(x(1-x)+y_1(x))}+\frac{x^2}{2}\int_0^1\frac{ds}{(sy_2(x)+(1-s)y_1(x)+x(1-x))^2}\,.$$

Integrating this differential inequality and taking any $\hat{x}<0$ with $|\hat{x}|$ large enough it follows that

$$y_2(x)-y_1(x)>(y_2(\hat{x})-y_1(\hat{x}))\left(\frac{x}{\hat{x}}\right)^2\exp\left(-\int_x^{\hat{x}}R(x')\,dx'\right)$$

$$\sim(y_2(\hat{x})-y_1(\hat{x}))\left(\frac{x}{\hat{x}}\right)^2\exp\left(-\int_{-\infty}^{\hat{x}}R(x')\,dx'\right)\,,$$

which contradicts (5.15), provided that the integral

$$\int_{-\infty}^{\hat{x}}R(x)\,dx=\int_{-\infty}^{\hat{x}}R_1(x)\,dx+\int_{-\infty}^{\hat{x}}R_2(x)\,dx$$

converges. For the convergence of the first integral we rewrite, omitting the subscripts,

$$-\int_{-\infty}^{\hat{x}}R_1(x)\,dx=\int_{\hat{t}}^{\bar{t}}\frac{x(t)+x(t)^2-2y(t)}{x(t)}\,dt=\int_{\hat{t}}^{\bar{t}}r(t)\,dt,$$

with

$$r(t)=\frac{X(\tau)V(\tau)+X(\tau)^2-2Y(\tau)}{X(\tau)V(\tau)}\,,$$

where we intentionally use both variables t and τ. The numerator converges to zero exponentially fast in the τ-variable and is therefore, see the local analysis in Section 5.3, of order $O((\bar{t}-t)^\lambda)$ for some $\lambda>0$ as $t\uparrow\bar{t}$. Since, again by the local analysis (cf. $dV/d\tau\sim X/2$), the denominator satisfies $XV\sim-\frac{1}{2}X^2(\bar{t}-t)$, the convergence of the first integral follows. The convergence of the second integral is immediate from

$$R_2(x)=\frac{x^2}{2}\int_0^1\frac{ds}{(sy_2(x)+(1-s)y_1(x)+x(1-x))^2}\sim\frac{2}{x^2}\,.$$

We conclude that for $\mu>\mu_*$ there cannot be another connection between $(3/2,3/4,0)$ and P_- with $\dot{x}<0$ besides the one obtained in the existence proof for $\mu = \mu_*$. It remains to show that a sign change of $\dot{x}$ makes it impossible for γ to connect to P_-. So suppose we have an orbit γ_μ with $\mu_* < \mu < 1$ for which $\dot{x}$ changes sign. This means that $\gamma = \gamma_\mu$ has to enter the region $\{y > x(x-1)\}$. We observe that, in view of (5.12)-(5.13), besides $\{x > 3/2,\ y > \min(x(1-x),\ \frac{1}{2}x^2)\}$, also

$$\{(x,y,z) : z > 0,\ y > x(1-x) \text{ for } x \le 0,\ y > \frac{1}{2}x^2 \text{ for } x \ge 0\}$$

is positively invariant and cannot contain orbits in the stable manifold of P_-. Thus we only have to deal with the case that γ enters and subsequently exits the set

$$\{(x,y,z) : z > 0,\ x(1-x) < y < \frac{1}{2}x^2\}$$

through points (x_0, y_0, z_0) and (x_1, y_1, z_1) on $y = x(x-1)$ with $0 < x < 3/2$ and $z > 0$, after which $x(t)$ decreases to some $x_2 \ge -\infty$. Let $x \to (y_i(x), z_i(x))$ for $i = 1, 2, 3$ represent γ before entrance ($i = 1$, $x_0 < x < 3/2$), between entrance and exit ($i = 2$, $x_0 < x < x_1$) and after exit ($i = 3$, $x_2 < x < x_1$). Clearly $y_3(x_1) > y_1(x_1)$ and, by the z-equation in (5.4), also $z_3(x_1) > z_1(x_1)$. Applying the monotonicity argument, we conclude that for $\max(x_0, x_2) < x < x_1$ the same ordering persists, i.e. $y_3(x) > y_1(x)$ and $z_3(x) > z_1(x)$. This implies that $x_2 \ge x_0$. In particular the orbit enters again, upon which the above reasoning may be repeated so that γ cannot connect to P_-. This completes the proof of uniqueness.

6 Self-similar solutions of the second kind

In this section we present briefly some qualitative properties of the self-similar problem of the second kind (1.6). The analysis of Section 2 suggests that Problem (1.6) has a solution only if $n < 2$. We prove below that (1.6) has no solution if $n \ge 2$, and that any solution for $n < 2$ has positive draining, i.e. decreasing mass (which in turn is equivalent to $\alpha > \beta$).

To begin with, we integrate once the differential equation of (1.6):

$$f^n(\eta) f'''(\eta) = -(\alpha - \beta) \int_\eta^a f(s)\, ds + \beta \eta f(\eta). \tag{6.1}$$

Recalling (1.3) we observe that

$$\text{If } n > 0 \text{ then} \quad \alpha \le \beta \Longleftrightarrow \alpha \le \frac{1}{n+4} \Longleftrightarrow \beta \ge \frac{1}{n+4}. \tag{6.2}$$

Theorem 6.1 *Let $n > 0$ and $\alpha \le \beta$. Then Problem (1.6) has no solution.*

Proof. By (6.1)-(6.2) $f''' > 0$ in $(0, a)$ and f'' is increasing. Since $f''(0^+) \ge 0$, we have that $f'' > 0$ in $(0, a)$, which contradicts $f'(0) = f'(a) = 0$. $\square$

Theorem 6.2 *Given any $\alpha \in \mathbb{R}$, if $n \geq 2$ then Problem (1.6) has no solution.*

Proof. In the sequel C stands for a positive constant which may be different in different occurrences. By (6.1)

$$f^n(\eta)f'''(\eta) \sim -(\alpha - \beta)C \quad \text{as} \quad \eta \to 0. \tag{6.3}$$

If $\alpha = \beta$ there is no solution by the former theorem. So assume that $\alpha \neq \beta$. From $f(0) = f'(0) = 0$ we deduce that, near $\eta = 0$, $f(\eta) \leq C\eta$ and by (6.3) $|f'''(\eta)| \geq C\eta^{-n}$. Since $n \geq 2$ this implies that f' is unbounded, a contradiction which completes the proof. □

Finally, we consider the asymptotic behaviour at $\eta = 0$. Let $n > 0$ and let f be a solution of Problem (1.6). According to the above theorems we must have $\alpha > \beta$ and $n < 2$. Using Taliaferro's theorem [46] as in [15], it follows from (6.3) that as $\eta \to 0$:

$$\begin{array}{ll} \text{For } 0 < n < 1/2 & f(\eta) \sim C_1\eta^2\,. \\ \text{For } n = 1/2 & f(\eta) \sim C_2\eta^2|\log\eta|^{2/3}\,. \\ \text{For } 1/2 < n < 2 & f(\eta) \sim C_3\eta^{3/(n+1)}\,. \end{array} \tag{6.4}$$

Remark 6.3 With regard to the behaviour at $\eta = a$ we conjecture that it is given by (1.8), excepting of course that $3/2 < n < 3$ is to be replaced by $3/2 < n < 2$. A rigorous proof needs to exclude that f' is oscillatory near $\eta = a$. If it is proved that $f' \leq 0$ near $\eta = a$, then (1.8) follows from [46] and [15].

Remark 6.4 As pointed out in Section 2.2, another interesting possibility is to prescribe zero pressure at $x = 0$, $B(t) = 0$ in (2.8), i.e. to consider the boundary conditions $h = h_{xx} = 0$ at $x = 0$. This leads to a self-similar problem of the second kind identical to Problem (1.6), excepting that $f'(0) = 0$ is replaced by $f''(0^+) = 0$. For this problem $f'(0) > 0$, Theorem 6.1 still holds and (6.3) implies that no solution exists if $n \geq 1$, because now f'' must be bounded near $\eta = 0$. This agrees with the formal analysis of Section 2.2: the expression (2.8) with $B(t) = 0$ requires $3 - n > 2$, i.e. $n < 1$.

7 Conclusions and discussion

One of the most interesting features of the current work is the emergence of $n = 2$ as a key critical exponent, in the sense that for $2 \leq n < 3$ the conditions $h = h^n h_{xxx} = 0$, which imply that fluid extraction is impossible (by mass conservation), are the only relevant boundary conditions at the draining line $x = 0$. This is supported by the formal asymptotic analysis of Section 2.2 and by the results on existence, uniqueness and non-existence for similarity solutions obtained throughout the paper. A reliable heuristic principle attributes a fundamental significance to these similarity solutions. At the same time, this suggests new lines of research and, in particular, deserves a further study from the point of view of mathematical analysis.

This value of $n = 2$ is of particular interest because two different slips laws proposed in the modelling of moving contact lines involve $n = 1$ and $n = 2$ as dominant exponents when the droplet height h tends to zero. For example, Greenspan [28], Greenspan & McCay [29] and Neogi & Miller [43] adopt a slip model with $n = 1$, while Hocking [31]-[33], Lacey [41] and Ehrhard & Davis [25] take $n = 2$. These values lead to very different behaviour in the current context. In particular, fluid extraction is impossible for $n = 2$, which provides a possible way of distinguishing between the two slip laws. Some authors explain that the choice of slip law depends on conditions at the solid surface and on the type of fluid: see the overview in [17]. In other papers the two slip laws seem to be competing choices.

Another feature of the critical value $n = 2$ suggested by the current work is that, while one can impose a zero contact angle at a moving contact line or free boundary for any $n < 3$, cf. Section 2, one can do so at a fixed contact line through which drainage occurs only if $n < 2$. This is again supported by the asymptotic analysis of Section 2.2 and by the non-existence results of Section 6, and also deserves a further mathematical analysis.

With regard to the range of exponents $0 < n < 2$, the formal analysis of Section 2.2, cf. (2.8), indicates that there is an infinite number of possibilities for prescribing the second boundary condition at the draining line $x = 0$. We have concentrated on the two outstanding choices (I) $h = h^n h_{xxx} = 0$ and (II) $h = h_x = 0$. And there are other possibilities, in particular the zero pressure conditions $h = h_{xx} = 0$ briefly considered in Remark 6.4. This again suggests additional physical and mathematical research on the draining line.

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