A critical case of stability in a free boundary problem

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Abstract

We study the stability of the planar travelling wave solutions to the free boundary heat equation in the whole $\mathbb{R}^2$. We turn the problem into a fully nonlinear parabolic system and establish a stability result which is the proper generalization of the one-dimensional case. The curvature terms contribute a gradient squared corresponding to critical growth. The latter is eliminated by means of the Hopf-Cole transformation.

Keywords: Free boundary problems, stability of travelling waves, fully nonlinear parabolic problems.

1 Introduction

In this paper we consider the one-phase free boundary problem

$$\begin{cases} u_t = u_{xx} + u_{yy} \text{ in } \Omega_t = \{(x,y) \in \mathbb{R}^2 : x \leq s(t,y)\}, \;\; t \geq 0, \\ \\ u = u_n = 1 \text{ on } \partial\Omega_t = \{(x,y) \in \mathbb{R}^2 : x = s(t,y)\}, \;\; t \geq 0, \end{cases} \tag{1.1}$$

with prescribed initial data (s_0, u_0) for s and u. Here subscripts denote partial derivatives, n being the normal vector to the free boundary $\partial\Omega_t = \{(x,y) \in \mathbb{R}^2 : x = s(t,y)\}$ with positive x-component. The unknowns of the problem are the unbounded domains Ω_t (or equivalently the curves $x = s(t,y)$) and the function $u : \{(t,x,y) \in \mathbb{R}^3 : x < s(t,y), t > 0\} \to \mathbb{R}$.

The restriction to space dimension two (with variables x and y) is only for convenience but not essential. We emphasise that Problem (1.1) is set on the whole plane.

In combustion theory, Problem (1.1) is the high activation limit as $\epsilon \to 0$ of a scalar reaction-diffusion equation, which is of the form

$$u_t = \Delta u + f_\epsilon(u), \tag{1.2}$$

with f_ϵ smooth and nonnegative, supported in $[1-\epsilon, 1]$ with integral $\int_{1-\epsilon}^1 f_\epsilon(u)du = \frac{1}{2}$. The convergence of solutions of (1.2) with suitable initial data to solutions of (1.1) was proved in [10]. For travelling wave solutions on the real line this was already done in [2].

Problem (1.1) is a particular case of the model for premixed Near-Equidiffusional Flames (NEF's) which goes back to [19], see also [8, 9], and reads

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u, \;\; t \geq 0, \; x < s(t,y), \; y \in \mathbb{R}, \\ \\ u = 1, \;\; t \geq 0, \; x \geq s(t,y), \; y \in \mathbb{R}, \\ \\ \dfrac{\partial v}{\partial t} = \Delta v - \lambda \Delta u, \;\; x \neq s(t,y), \\ \\ \left[\dfrac{\partial u}{\partial n}\right] = -\exp(v), \quad \left[\dfrac{\partial v}{\partial n}\right] = \lambda \left[\dfrac{\partial u}{\partial n}\right], \; t \geq 0, \; x = s(t,y), \; y \in \mathbb{R}, \end{cases} \tag{1.3}$$

where $[\cdot]$ denotes the jump across the free boundary. Here u represents the temperature, v the enthalpy and λ is a parameter ($\lambda = -\frac{\ell}{2}$ where ℓ is the reduced Lewis number). For the special value $\lambda = 0$, which corresponds to the equidiffusional case, we arrive at (1.1) by setting $v \equiv 0$.

Problem (1.3) allows planar flame fronts, namely one-dimensional travelling wave solutions $(s, u, v) = (-t, U(x+t), V(x+t))$ with $U = e^x$, $V = \lambda x e^x$, if $x < 0$, $U = 1$, $V = 0$, if $x > 0$. Our main interest is to understand their stability properties under two-dimensional perturbations. Instability has recently been proved in [5] for $\lambda > 1$. According to the physical literature, see [21] and [9], the fronts are stable for $\lambda_c < \lambda < 1$ where $\lambda_c < 0$. In this paper we give a rigorous formulation and proof of stability for the reduced problem (1.1). This is a first step. A next project will be to deal with the full problem for the whole range $0 \leq \lambda \leq 1$.

Thus we consider Problem (1.1) with initial data close to the travelling wave solution $u(t,x,y) = U(x+t)$, $U(x) = e^x$ with free boundary $s(t) = -t$. Transforming to travelling wave variables by setting $x' = x+t$, $s' = s+t$ and omitting the primes we look for solutions of

$$\begin{cases} u_t = u_{xx} - u_x + u_{yy}, \;\; t \geq 0, \; x < s(t,y), \; y \in \mathbb{R}, \\ \\ u = u_n = 1, \;\; t \geq 0, \; x = s(t,y), \; y \in \mathbb{R}, \end{cases} \tag{1.4}$$

with initial data near the equilibrium solution $(s(t,y) = 0, u(t,x,y) = U(x))$.

We know that in one dimension, i.e. when there is no y in (1.4), the travelling wave is stable but not asymptotically stable: solutions converge to a translate of the travelling wave, at least when they start from initial data satisfying some decay condition as $x \to -\infty$, see [13, 7]. This is called orbital stability with an asymptotic phase and it is similar to what happens in the case of the bistable reaction-diffusion equation

$$u_t = \Delta u + u(1-u)(u-a), \tag{1.5}$$

see e.g. [12, 1, 11]. The multi-dimensional stability of planar travelling waves of (1.5) and similar bistable semilinear equations in the whole space was studied in [22] and [14]: the travelling wave is asymptotically stable under perturbations localized in both x and y (because of a Sobolev space setting) with an algebraic rate of convergence. In this setting convergence to a translate cannot occur and therefore this result does not contain the full one-dimensional case. Therefore the assumption that the perturbation is localized in y is too strong and should eventually be avoided.

Coming back to problem (1.4), we follow [4, 5] and the references included there, and set

$$\tilde{x} = x - s(t,y), \quad u(t,x,y) = U(\tilde{x}) + s(t,y)U'(\tilde{x}) + w(t,\tilde{x},y) = (1 + s(t,y))e^{\tilde{x}} + w(t,\tilde{x},y). \tag{1.6}$$

Eliminating s from the equations as usual, we arrive at a fully nonlinear parabolic problem for the auxiliary unknown w,

$$\begin{cases} w_t = w_{\tilde{x}\tilde{x}} - w_x + w_{yy} + F(w, Dw, D^2w), & t \geq 0,\ \tilde{x} \leq 0,\ y \in \mathbb{R}, \\ w_{\tilde{x}} - w = G(w_y), & t \geq 0,\ \tilde{x} = 0,\ y \in \mathbb{R}, \end{cases} \tag{1.7}$$

where F and G are smoothly defined near $w = 0$ and both F and G as well as their derivatives with respect to w vanish at $w = 0$. The relation between s and w is given by

$$s(t, y) + w(t, 0, y) = 0. \tag{1.8}$$

The stationary solution $(s(t, y) = 0, u(t, x, y) = e^x)$ corresponds to the null solution of (1.7) and stability is accordingly reformulated.

Local existence and uniqueness of the solution to the initial value problem $w(0, \tilde{x}, y) = w_0(\tilde{x}, y)$ for (1.7) is well understood: the usual techniques of fully nonlinear parabolic problems (see e.g. [16]) may be used to get a unique regular solution in both weighted and nonweighted Hölder spaces for small initial data satisfying the natural compatibility conditions at $\tilde{x} = 0$.

As far as stability is concerned, the principle of linearized stability holds also in the fully nonlinear case, see [16, 6, 4]. The linear part of the right hand side of (1.7) is the elliptic operator $\mathcal{L}w = w_{\tilde{x}\tilde{x}} - w_{\tilde{x}} + w_{yy}$, with homogeneous boundary condition $w_{\tilde{x}} - w = 0$ at $\tilde{x} = 0$. However, its realization in the space of the uniformly continuous and bounded functions $C^0(\mathbb{R}^2_-) = C^0((-\infty, 0) \times \mathbb{R})$ has bad spectral properties: the spectrum consists of the region $\{\lambda \in \mathbb{C} :\ \mathrm{Re}\,\lambda \leq -(\mathrm{Im}\,\lambda)^2\}$. At a first glance, the situation is not much better in the weighted space $X = \{w :\ (\tilde{x}, y) \mapsto e^{-\tilde{x}/2} w(\tilde{x}, y) \in C^0(\mathbb{R}^2_-)\}$, where the spectrum is the whole halfline $(-\infty, 0]$. In the one-dimensional case the weight $e^{-\tilde{x}/2}$ changes the spectrum into $(-\infty, -1/4] \cup \{0\}$ and provides a spectral gap, see [7] and the pioneering work [20]. The lack of a spectral gap is the main difference between the present situation and the situations discussed in [7, 6], and it cannot be resolved using a weight depending also on y. Thus we are always in the critical case of stability. This is similar to the situation in [22] and [14].

However, both in the weighted and in the nonweighted space, the realization L of $\mathcal{L}$ may be seen as $\overline{L_1 + L_2}$, where L_1 is the realization of the operator (acting only on the variable $\tilde{x}$) $w \mapsto w_{\tilde{x}\tilde{x}} - w_{\tilde{x}}$, with boundary condition $w_{\tilde{x}} - w = 0$ at $\tilde{x} = 0$, and L_2 is the realization of the Laplacian with respect to y. In the weighted space X, the spectrum of L_1 is equal to $(-\infty, -1/4] \cup \{0\}$, as in the one-dimensional case. The spectral projection corresponding to the eigenvalue 0 of L_1 is given by

$$(Pv)(\tilde{x}, y) = \int_{-\infty}^{0} v(x', y)\,dx'\, e^{\tilde{x}} \tag{1.9}$$

and the norm of $(I - P)e^{tL_1}$ decays exponentially as $t \to +\infty$. Since the resolvents of L_1 and L_2 commute, then P commutes with both e^{tL_1} and e^{tL_2}. Moreover we have $e^{tL} = e^{tL_1}e^{tL_2} = e^{tL_2}e^{tL_1}$, and hence the norm of $(I - P)e^{tL}$ decays exponentially as $t \to +\infty$ (the norm of e^{tL_2} being 1).

Therefore, even if P and $Q = I - P$ are spectral projections only for L_1 but not for L, they enjoy some properties of the usual spectral projections which are enough to employ a sort of Lyapunov-Schmidt procedure. Indeed we set, like in [22] and [14],

$$w = Pw + Qw = p(t, y)e^{\tilde{x}} + z(t, \tilde{x}, y), \tag{1.10}$$

where we think of the range of P as an "infinite dimensional center subspace". After some computations we arrive at a system for (p, z),

$$p_t - p_{yy} = (1 + (p_y + z_y(t, 0, y))^2)^{\frac{1}{2}} - 1,\ \ t \geq 0,\ y \in \mathbb{R}, \tag{1.11}$$

$$\begin{cases} z_t = Q(z_{\tilde{x}\tilde{x}} + z_{yy} - z_{\tilde{x}}) + QF(pe^{\tilde{x}} + z, D(pe^{\tilde{x}} + z), D^2(pe^{\tilde{x}} + z)), & t \geq 0,\ \tilde{x} \leq 0,\ y \in \mathbb{R}, \\ z_x(t,0,y) - z(t,0,y) = (1 + (p_y + z_y(t,0,y))^2)^{-\frac{1}{2}} - 1, & t \geq 0,\ y \in \mathbb{R}, \end{cases}$$

where the equation for p may be considered as a "center part" of the problem, and the equation for z as a "stable part". Unlike in classical Center Manifold Theory, the equation for p is not an ODE but a PDE. Its right hand side contains the quadratic term $p_y^2/2$, which is critical as far as stability of $p = 0$ is concerned, see the discussion in Section 3.1 of [17]. We circumvent this complication using the Hopf-Cole transformation. Setting

$$q = e^{p/2} - 1,$$

we get

$$q_t - q_{yy} = q_y z_y(t,0,y) + \cdots,$$

The quadratic term $q_y z_y(t,0,y)$ is much better than $p_y^2/2$ in the p-equation because z is to belong to the "stable subspace" $Q(X)$.

Eventually (Theorem 2.12), the initial value problem for the (q,z) system is solved for small initial data q_0 and z_0 in suitable Hölder spaces. In particular we obtain the estimates

$$\sup_{t\geq 0}(\|q\|_\infty + (1+t^{1/2})\|q_y\|_\infty + (1+t)\|q_{yy}\|_\infty + (1+t^{1+\theta/2})[q_{yy}]_{C^\theta})$$

$$\leq C(\|q_0\|_{C^{2+\theta}} + \|e^{-\tilde{x}/2}z_0\|_{C^{2+\theta}}),$$

and

$$\sup_{t\geq 0}(1+t)\|e^{-\tilde{x}/2}z\|_{C^{2+\theta}} \leq C(\|q_0\|_{C^{2+\theta}} + \|e^{-\tilde{x}/2}z_0\|_{C^{2+\theta}}).$$

Here the norms and semi-norms are the usual ones on $\mathbb{R}^2_- = \{(\tilde{x}, y) \in \mathbb{R}^2,\ \tilde{x} \leq 0\}$ for z and on $\mathbb{R}$ for q, see Section 2.1.

These estimates imply that q stays bounded and its derivatives go to zero, with the same algebraic decay as the solutions of the linear heat equation. In particular $\|q_y\|_\infty$ decays at least as fast as $t^{-1/2}$. The same holds for $p = 2\log(q+1)$. The decay rates for z are better. For $z_y(t,0,y)$, which is the only form of z in the PDE for p, we have a uniform decay at least as fast as t^{-1}.

When we translate these results back to the original problem (1.4), we have to invert (1.6). Using (1.8) and (1.10) we get

$$\begin{cases} s(t,y) = -(p(t,y) + z(t,0,y)), & t \geq 0,\ y \in \mathbb{R}, \\ u(t,x,y) = (1 - z(t,0,y))e^{x - s(t,y)} + z(t, x - s(t,y), y), & t \geq 0,\ x \leq s(t,y),\ y \in \mathbb{R}, \end{cases}$$

and conversely

$$\begin{cases} p(t,y) = \displaystyle\int_{-\infty}^0 u(t, x' + s(t,y), y)dx' - (1 + s(t,y)), & t \geq 0,\ y \in \mathbb{R}, \\ z(t,\tilde{x},y) = u(t, \tilde{x} + s(t,y), y) - \displaystyle\int_{-\infty}^0 u(t, x' + s(t,y), y)dx' e^{\tilde{x}}, & t \geq 0,\ \tilde{x} \leq 0,\ y \in \mathbb{R}. \end{cases}$$

Therefore, (p_0, z_0) close to $(0,0)$ means (s_0, u_0) close to $(0, U) = (0, e^x)$.

Theorem 2.12 implies that if $s_0 : \mathbb{R} \to \mathbb{R}$, $u_0 : \Omega_0 = \{(x,y) : x \leq s_0(y)\} \to \mathbb{R}$ satisfy

$$u_0(s_0(y), y) = 1, \ \frac{\partial u_0}{\partial n}(s_0(y), y) = 1,$$

(n being the unit exterior normal vector to $\partial\Omega_0$), and the norms

$$\|s_0\|_{C^{2+\theta}(\mathbb{R})}, \ \|e^{-x/2}(u_0 - e^x)\|_{C^{2+\theta}(\Omega_0)}$$

are small enough, then problem (1.4) has a global classical solution (s, u). The convergence of z to zero as $t \to \infty$ now implies that

$$p(t,y) + s(t,y) \to 0, \quad u(t,x,y) - e^{x-s(t,y)} \to 0, \quad x \le s(t,y),\ y \in \mathbb{R}.$$

Since in general q (and hence also p) cannot be expected to converge as $t \to \infty$, there is no convergence of s and u, despite the convergence of s_y and p_y to zero. In fact, if we assume that the perturbations are localized in (x, y) we can prove convergence back to the travelling wave, as in [22] and [14], working in weighted Hölder spaces with exponential weight $e^{-x/2}$ with respect to x and suitable polynomial weight with respect to y. However, we emphasize once again that our result is the appropriate generalisation of the one-dimensional case, in which one does have convergence to a translate of the wave and all the translates form a one-dimensional center manifold with trivial dynamics. This is because the p-equation (1.11) is simply $p_t = 0$.

The two-dimensional case is much more complicated. All we can say is that asymptotically the behaviour is governed by $p = 2\log(q + 1)$. Our method of proof, in which the Hopf-Cole transformation is crucial to get global existence, suggests us to expect that eventually q behaves like a solution of the heat equation $q_t = q_{yy}$. Mutatis mutandis we expect p to behave for large time like a solution of the integrated Burgers' equation

$$p_t = p_{yy} + \frac{p_y^2}{2}.$$

This explains why it is not possible to prove convergence: it is not true for the heat equation itself and will not be true here either.

It remains to be seen to what extent our expectation concerning the role of Burgers' equation is true.

Let us describe briefly the contents of the paper. After deriving problem (1.7) for the auxiliary unknown w, in Section 2.1 we study the spectral properties of the linear operator $\mathcal{L}$ and we introduce the projections P and Q. The systems for $(Pw, Qw) = (pe^{\tilde{x}}, z)$ and for the final unknowns $(q = e^{p/2} - 1, z)$ are written down in Section 2.2. The (fully nonlinear) system for (q, z) is solved in Section 2.4 by a fixed point argument, which works thanks to optimal regularity and asymptotic behavior results for linear systems which are proved in Section 2.3. Section 2.3 is the technical part of the paper and uses as main tools the theory of analytic semigroups in Banach spaces and interpolation theory.

2 The fully nonlinear problem

We invoke the methods developed in [3, 4, 5, 6, 7] to transform problem (1.4) to a fully nonlinear problem on the fixed domain $\{(x, y) \in \mathbb{R}^2 : x < 0\}$.

Setting

$$\tilde{x} = x - s(t,y),\ \tilde{y} = y,\ \tilde{t} = t; \quad u(x,y,t) = U(\tilde{x}) + s(\tilde{y},\tilde{t})U'(\tilde{x}) + w(\tilde{x},\tilde{y},\tilde{t}), \tag{2.1}$$

we arrive at a system for the new couple of unknowns (s, w), which reads as, omitting the tildes,

$$w_t = w_{xx} - w_x + w_{yy} + (s_t - s_{yy})w_x + s_y^2 w_{xx} - 2s_y w_{xy} + \left(ss_t - ss_{yy} + ss_y^2 - s_y^2\right)e^x, \tag{2.2}$$

for $x < 0$, $y \in \mathbb{R}$, with the boundary conditions

$$w + s = 0, \quad w_x + s = (1 + s_y^2)^{-1/2} - 1, \quad x = 0,\ y \in \mathbb{R}. \tag{2.3}$$

The first equality in (2.3) allows us to decouple the system. An equation for s_t is derived as usual by taking the trace of (2.2) at $x = 0$. Combining again with both equations in (2.3) this gives

$$s_t - \frac{1 - s_y^2}{1 + s_y^2} s_{yy} = -s_y^2 - (1 + s_y^2)^{3/2}(w_{xx}(0) - w_x(0)), \tag{2.4}$$

where we use the notation $f(0)$ for $f(t, 0, y)$. Eliminating s_t then gives

$$\begin{aligned} w_t &= w_{xx} - w_x + w_{yy} + 2w_y(0)w_{xy} + w_{yy}(0)w_x + w_y(0)^2 w_{xx} \\ &- \Big(\frac{2w_y(0)^2}{1 + w_y(0)^2} w(0)w_{yy}(0) + w_y(0)^2 + w(0)(1 + w_y(0)^2)^{\frac{3}{2}}(w_{xx}(0) - w_x(0))\Big)e^x \\ &- \Big(\frac{1 - w_y(0)^2}{1 + w_y(0)^2} w_{yy}(0) + w_y(0)^2 + (1 + w_y(0)^2)^{3/2}(w_{xx}(0) - w_x(0))\Big)w_x, \end{aligned} \tag{2.5}$$

with boundary condition

$$w_x(0) - w(0) = (1 + w_y(0)^2)^{-1/2} - 1. \tag{2.6}$$

Setting

$$\mathcal{L}w = w_{xx} - w_x + w_{yy},$$

the problem for w is then of the form

$$\begin{cases} w_t = \mathcal{L}w + F(w, Dw, D^2 w), \;\; t \geq 0, \; x \leq 0, \; y \in \mathbb{R}, \\ \\ \mathcal{B}w = w_x(0) - w(0) = \varphi(w_y(0)), \;\; t \geq 0, \; x = 0, \; y \in \mathbb{R}, \end{cases} \tag{2.7}$$

where F and φ are defined by the right hand sides of (2.5), (2.6), respectively. Since both F and φ vanish at $w = 0$ with all their first order derivatives, problem (2.7) is parabolic near $w = 0$. The usual techniques of fully nonlinear parabolic equations can be used to get wellposedness of the initial value problem for (2.7) in Hölder spaces, for small initial data. However, the linear part of the problem has not good spectral properties in the usual Hölder spaces, and for this reason we will consider weighted spaces.

2.1 The realization of $\mathcal{L}$ in weighted Hölder spaces

Let us define the (weighted and nonweighted) spaces we need.

Let Ω be an open set in $\mathbb{R}^n$ with regular boundary (in what follows Ω will be either $\mathbb{R}$ or $\mathbb{R}^2_- = (-\infty, 0) \times \mathbb{R}$). For $k \in \mathbb{N} \cup \{0\}$ we denote by $C^k(\Omega)$ the space of all uniformly continuous bounded functions in $\overline{\Omega}$ with derivatives up to the k-th order uniformly continuous and bounded in $\overline{\Omega}$, endowed with its natural norm

$$\|\varphi\|_{C^k(\Omega)} = \sum_{|\gamma|=0}^{k} \|D^\gamma \varphi\|_\infty.$$

For $\theta \in (0, 1)$, $C^\theta(\Omega)$ is, as usual, the space of the bounded functions in Ω such that $[\varphi]_{C^\theta(\Omega)} = \sup_{x,y \in \Omega,\, x \neq y} |\varphi(x) - \varphi(y)|/|x - y|^\theta < \infty$, endowed with the norm

$$\|\varphi\|_{C^\theta(\Omega)} = \|\varphi\|_\infty + [\varphi]_{C^\theta(\Omega)}.$$

If $k \in \mathbb{N}$ and $\theta \in (0, 1)$, $C^{k+\theta}(\Omega)$ is the set of the k times differentiable functions in Ω such that $D^\gamma \varphi \in C^\theta(\Omega)$ for every multi-index γ with $|\gamma| \leq k$. The norm is, as usual,

$$\|\varphi\|_{C^{k+\theta}(\Omega)} = \sum_{|\gamma|=0}^{k} \|D^\gamma \varphi\|_\infty + \sum_{|\gamma|=k} [D^\gamma \varphi]_{C^\theta(\Omega)}.$$

Let $\alpha = k + \theta \geq 0$. We set

$$C^\alpha_\omega(\mathbb{R}^{2_\omega}_-) = \{f \in C(\mathbb{R}^2_-) : (x, y) \mapsto e^{-x/2} f(x, y) \in C^\alpha(\mathbb{R}^2_-)\},$$

$$\|f\|_{C^\alpha_\omega(\mathbb{R}^2_-)} = \|(x, y) \mapsto e^{-x/2} f(x, y)\|_{C^\alpha(\mathbb{R}^2_-)}.$$

The realization of the operator $\mathcal{L}$ with homogeneous boundary condition in $C^0_\omega(\mathbb{R}^2_-)$ will be seen as (the closure of) the sum of the realizations of the operators $\partial^2/\partial x^2 - \partial/\partial x$ and $\partial^2/\partial y^2$ in the space $C^0_\omega(\mathbb{R}^2_-)$.

More precisely, we define

$$D(L_1) = \{f \in C^0_\omega(\mathbb{R}^2_-) : \ f_{xx} \in C^0_\omega(\mathbb{R}^2_-), \ \mathcal{B}f(0, y) = f_x(0, y) - f(0, y) = 0\}, \ \ L_1 f = f_{xx} - f_x,$$

$$D(\Delta_y) = \{f \in C^0_\omega(\mathbb{R}^2_-) : \ f_{yy} \in C^0_\omega(\mathbb{R}^2_-)\}, \ \ \Delta_y f = f_{yy}.$$

The main properties of L_1 and Δ_y are stated in the next lemma.

Lemma 2.1 *Both L_1 and Δ_y generate analytic bounded semigroups in $C^0_\omega(\mathbb{R}^2_-)$. The spectrum of L_1 is $(-\infty, -1/4] \cup \{0\}$. The spectral projection corresponding to the spectral set $\{0\}$ is the projection P defined by (1.9).*

Proof — The proof of the first statement is standard and it is left to the reader. To prove the statement about the spectral properties of L_1 it is sufficient to follow closely [7, Sect. 3], who considered a purely one-dimensional version of the operator L_1, i.e.

$$X = \{u \in C^0((-\infty, 0]) : x \mapsto e^{-x/2} u(x) \in L^\infty(-\infty, 0)\},$$

$$A : D(A) = \{u \in C^2((-\infty, 0]) : u, \, u'' \in X, \, u'(0) = u(0)\} \mapsto X; \ \ Au = u'' - u'.$$

The variable y acts only as a parameter in our operator L_1. Following step by step the arguments of [7] the statement follows. ∎

The following proposition relates the operators L_1 and Δ_y to the operator $\mathcal{L}$, and it is crucial in our analysis.

Proposition 2.2 *The operator $L_1 + \Delta_y : D(L_1) \cap D(\Delta_y) \mapsto C^0_\omega(\mathbb{R}^2_-)$ is closable, and its closure L generates the analytic semigroup $e^{tL} = e^{tL_1} e^{t\Delta_y}$. If P is the projection defined in (1.9) and $Q = I - P$, then e^{tL} commutes with P and Q, and $\|Qe^{tL}\|$ decays exponentially as $t \to \infty$. More precisely, for each $\varepsilon > 0$ there exists $M > 0$ such that*

$$\|Qe^{tL} f\|_{C^0_\omega(\mathbb{R}^2_-)} \leq M e^{(\varepsilon - 1/4)t} \|f\|_{C^0_\omega(\mathbb{R}^2_-)}, \ \ t > 0, \ f \in C^0_\omega(\mathbb{R}^2_-). \tag{2.8}$$

Proof — Since L_1 and Δ_y commute (in the resolvent sense), their sum is closable and the closure $L = \overline{L_1 + \Delta_y}$ still generates an analytic semigroup. Moreover,

$$e^{tL} = e^{tL_1} e^{t\Delta_y} = e^{t\Delta_y} e^{tL_1}, \ \ t > 0.$$

By lemma 2.1, the spectrum of L_1 consists of the halfline $(\infty, -1/4]$ plus the isolated eigenvalue 0, and the projection P defined in (1.9) is nothing else but the spectral projection associated to the eigenvalue 0 of the operator L_1. Therefore, $\|e^{tL_1}(I - P)\| = \|e^{tL_1} Q\| \leq M e^{(\varepsilon - 1/4)t}$, and since $\|e^{t\Delta_y}\| = 1$ and Q commutes with $e^{t\Delta_y}$, then

$$\|Qe^{tL}\| = \|e^{t\Delta_y} e^{tL_1} Q\| \leq M e^{(\varepsilon - 1/4)t}, \ \ t > 0. \ ∎$$

Remark 2.3 Note that

$$\sigma(L) = \sigma(L_1) + \sigma(\Delta_y) = ((-\infty, -1/4] \cup \{0\}) + (-\infty, 0] = (-\infty, 0],$$

while the spectrum of the part of L in $Q(X)$ consists of the halfline $(-\infty, -1/4]$. Therefore Q kills the whole interval $(-1/4, 0]$ of the spectrum of L.

Another lemma which will be used later is about the characterization of weighted Hölder spaces as real interpolation spaces relevant to the operator L. If Y is a Banach space and $L : D(L) \mapsto Y$ is the generator of a bounded semigroup e^{tL}, for $0 < \theta < 1$ we denote as usual by $D_L(\theta, \infty)$ the real interpolation space $(Y, D(L))_{\theta,\infty}$, and by $D_L(\theta + 1, \infty) = \{u \in D(L) : Lu \in D_L(\theta, \infty)\}$ the domain of the part of L in $D_L(\theta, \infty)$. It is well known that the norms

$$f \mapsto \|f\|_Y + \sup_{\lambda>0} \|\lambda^\theta L R(\lambda, L) f\|_Y, \quad f \mapsto \|f\|_Y + \sup_{t>0} \|t^{1-\theta} L e^{tL} f\|_Y$$

are equivalent to the norm of $D_L(\theta, \infty)$. See e.g. [16, Prop. 2.2.6], where we may take $a = 0$.

Lemma 2.4 *For every $\theta \in (0, 1)$ we have*

$$C^\theta_\omega(\mathbb{R}^2_-) = D_L(\theta/2, \infty),$$

with equivalence of the respective norms.

Moreover,

$$C^1_\omega(\mathbb{R}^2_-) \subset D_L(1/2, \infty),$$

with continuous embedding.

Finally, for every $\theta \in (0, 1)$ we have

$$\{f \in C^{2+\theta}_\omega(\mathbb{R}^2_-) : \mathcal{B}f = 0\} = D_L(\theta/2 + 1, \infty),$$

with equivalence of the respective norms.

Proof — For $\lambda > 0$ and $f \in C^0_\omega(\mathbb{R}^2_-)$ the equation $\lambda u - Lu = f$ in $C^0_\omega(\mathbb{R}^2_-)$ is equivalent to (setting $v(x, y) = u(x, y)e^{-x/2}$)

$$\begin{cases} \lambda v - \Delta v + \dfrac{1}{4} v = e^{x/2} f, & x \leq 0, \; y \in \mathbb{R}, \\ \\ v_x - \dfrac{1}{2} v = 0, & x = 0, \; y \in \mathbb{R}. \end{cases}$$

Therefore, $f \in D_L(\theta/2, \infty)$ iff $e^{x/2} f \in D_A(\theta/2, \infty)$, where A is the realization of the Laplace operator with boundary condition $v_x(0, y) - v(0, y)/2 = 0$ in the space of the uniformly continuous and bounded functions in $(-\infty, 0] \times \mathbb{R}$. The equivalence $C^\theta_\omega(\mathbb{R}^2_-) = D_L(\theta/2, \infty)$ and the embedding $C^1_\omega(\mathbb{R}^2_-) \subset D_L(1/2, \infty)$ follow now from [16, Prop. 3.1.27, Thm. 3.1.30]. The equivalence $\{f \in C^{2+\theta}_\omega(\mathbb{R}^2_-) : \mathcal{B}f = 0\} = D_L(\theta + 1, \infty)$ follows from Schauder's regularity theory. ■

Note that the last statement of lemma 2.4 is nothing but a Schauder regularity result for the equation

$$\mathcal{L}f = \psi$$

in the weighted space $C^\theta_\omega(\mathbb{R}^2_-)$. Indeed, it means that a function $f \in D(L)$ is such that $Lf = \psi \in D_L(\theta/2, \infty)$ if and only if $f \in C^{2+\theta}_\omega(\mathbb{R}^2_-)$. On the other hand, by the first statement of the lemma, $D_L(\theta/2, \infty) = C^\theta_\omega(\mathbb{R}^2_-)$. Therefore, $\psi \in C^\theta_\omega(\mathbb{R}^2_-)$ iff $f \in C^{2+\theta}_\omega(\mathbb{R}^2_-)$.

2.2 A Lyapunov-Schmidt type procedure

We emphasize that the projections P and Q which we used in the previous subsection are not spectral projections for the operator L. However, they share some important features with spectral projections: they commute with e^{tL} and moreover $\|Qe^{tL}\|$ decays exponentially as $t \to \infty$. This is enough to perform a sort of Lyapunov-Schmidt procedure, splitting w as

$$w = Pw + Qw = p(t,y)e^x + z(t,x,y) \tag{2.9}$$

and applying P and Q to (2.2). We have

$$(Pw_x)(t,x,y) = w(t,0,y)e^x; \quad (Pw_{xx})(t,x,y) = w_x(t,0,y)e^x;$$

$$(Pw_{xy})(t,x,y) = w_y(t,0,y)e^x; \quad (Pw_{yy})(t,x,y) = \int_{-\infty}^{0} w_{yy}(t,x',y)dx'e^x,$$

so that

$$(Qz_x)(t,x,y) = z_x(t,x,y) - z(t,0,y)e^x \quad (Qz_{xx})(t,x,y) = z_{xx}(t,x,y) - z_x(t,0,y)e^x;$$

$$(Qz_{xy})(t,x,y) = z_{xy}(t,x,y) - z_y(t,0,y)e^x;$$

$$(Qz_y)(t,x,y) = z_y(t,x,y); \ (Qz_{yy})(t,x,y) = z_{yy}(t,x,y).$$

Using the above equalities we derive, applying P and Q to (2.2),

$$p_t - p_{yy} = (1 + w_y(0)^2)^{\frac{1}{2}} - 1, \ \ t \geq 0, \ y \in \mathbb{R}, \tag{2.10}$$

$$z_t = Q(z_{xx} + z_{yy} - z_x) + QF = \mathcal{L}z - (z_x(0) - z(0))e^x + QF, \tag{2.11}$$

with boundary condition

$$z_x(0) - z(0) = (1 + w_y(0)^2)^{-\frac{1}{2}} - 1. \tag{2.12}$$

Replacing $w_y(0) = p_y + z_y(0)$ in (2.10) we see that the equation for p is of the type

$$p_t - p_{yy} = \frac{1}{2}(p_y + z_y(0))^2 + O(p_y + z_y(0))^4,$$

which gives technical problems because of the term $\frac{1}{2}p_y^2$, corresponding to quadratic (critical) growth near $p = 0$. As explained in the introduction we avoid this complication using the Hopf-Cole transformation, which transforms the integrated Burgers equation $p_t = p_{yy} + \frac{1}{2}p_y^2$ into the heat equation $q_t = q_{yy}$, setting

$$q = e^{p/2} - 1, \tag{2.13}$$

and we get

$$q_t - q_{yy} = q_y z_y(0) + \frac{1}{4}(q+1)(z_y(0))^2 + \frac{q+1}{2}\,\omega\Big(\frac{2q_y}{q+1} + z_y(0)\Big), \tag{2.14}$$

where

$$\omega(\xi) = (1+\xi^2)^{1/2} - 1 - \xi^2/2 = O(\xi^4)$$

as $\xi \to 0$. The quadratic term $q_y z_y(0)$ here is much better than $p_y^2/2$ in the p-equation because z is sought in the stable subspace and z_y will be allowed to decay to 0 sufficiently fast as $t \to +\infty$.

Replacing $w_y(0) = p_y + z_y(0)$ and $w_{yy}(0) = p_{yy}(0) + z_{yy}(0)$ also in (2.11) and in (2.12) we get

$$\begin{cases} z_t = \mathcal{L}z + f(q,z), \ \ t \geq 0, \ x \leq 0, \ y \in \mathbb{R}, \\ \\ \mathcal{B}z = g(q,z), \ \ t \geq 0, \ x = 0, \ y \in \mathbb{R}, \end{cases} \tag{2.15}$$

where

$$f(q,z) = -g(q,z)e^x$$

$$+\Big(\frac{2q_y}{q+1} + z_y(0))^2 Q(z_{xx} - z_x) + 2\Big(\frac{2q_y}{q+1} + z_y(0)\Big) Q z_{xy} + Q z_x \cdot$$

$$\cdot\Big(\frac{2\Big(\frac{2q_y}{q+1} + z_y(0)\Big)^2 \Big(\frac{2q_{yy}}{q+1} - \frac{2q_y^2}{(q+1)^2} + z_{yy}(0)\Big)}{1 + \Big(\frac{2q_y}{q+1} + z_y(0)\Big)^2} - \Big(1 + \Big(\frac{2q_y}{q+1} + z_y(0)\Big)^2\Big)^{3/2} Q z_{xx}(0)\Big), \tag{2.16}$$

$$g(q,z) = \Big(1 + \Big(\frac{q_y}{q+1} + z_y(0)\Big)\Big)^{-1/2} - 1. \tag{2.17}$$

We shall solve the initial value problem for (2.14,2.15) by a fixed point theorem in a ball centered at $(0,0)$ of a suitable product of function spaces.

We recall that the unknown q depends on (t,y), while the unknown z depends on (t,x,y). The functional spaces where q and z will live are weighted C^k and Hölder spaces, with powerlike weight with respect to t and exponential weight with respect to x, no weight with respect to y. Precisely, we shall set our problem in the space $\mathcal{X} = \mathcal{X}_q \times \mathcal{X}_z$, endowed with the product norm $\|(q,z)\|_{\mathcal{X}} = \|q\|_{\mathcal{X}_q} + \|z\|_{\mathcal{X}_z}$. Here $\mathcal{X}_q$ consists of the functions q such that $t \mapsto q(t,\cdot)$ is continuous in $[0,+\infty)$ with values in $C^2(\mathbb{R})$, continuously differentiable with values in $C^0(\mathbb{R})$, bounded with values in $C^{2+\theta}(\mathbb{R})$, and moreover

$$\|q\|_{\mathcal{X}_q} = \sup_{t\geq 0} \|q(t,\cdot)\|_{C^0(\mathbb{R})} + \sup_{t\geq 0}(1+t)(\|q_t(t,\cdot)\|_{C^\theta(\mathbb{R})} + \|q_{yy}(t,\cdot)\|_{C^\theta(\mathbb{R})}) < \infty.$$

$\mathcal{X}_z$ consists of the functions z such that $t \mapsto z(t,\cdot)$ is continuous in $[0,+\infty)$ with values in $C^2_\omega(\mathbb{R}^2_-)$, continuously differentiable with values in $C^0_\omega(\mathbb{R}^2_-)$, bounded with values in $C^{2+\theta}_\omega(\mathbb{R}^2_-)$, and moreover

$$\|z\|_{\mathcal{X}_z} = \sup_{t\geq 0}(1+t)(\|z(t,\cdot)\|_{C^{2+\theta}_\omega(\mathbb{R}^2_-)} + \|z_t(t,\cdot)\|_{C^\theta_\omega(\mathbb{R}^2_-)}) < \infty.$$

The following interpolation inequalities will be thoroughly used in the sequel.

Lemma 2.5 *If $q \in \mathcal{X}_q$ then*

$$\begin{cases} (a) \quad \sup_{t\geq 0}(1+t^{1/2})\|q_y(t,\cdot)\|_{C^0(\mathbb{R})} \leq C\|q\|_{\mathcal{X}_q}, \\[2ex] (b) \quad \sup_{0\leq s\leq t}(1+s)(t-s)^{-1/2-\theta/2}\|q_y(t,\cdot) - q_y(s,\cdot)\|_{C^0(\mathbb{R})} \leq C\|q\|_{\mathcal{X}_q}. \end{cases} \tag{2.18}$$

If $z \in \mathcal{X}_z$ then

$$(c) \quad \sup_{0\leq s\leq t}(1+s)(t-s)^{-1/2-\theta/2}\|Dz(t,\cdot) - Dz(s,\cdot)\|_{C^0_\omega(\mathbb{R}^2_-)} \leq C\|z\|_{\mathcal{X}_z}. \tag{2.19}$$

Proof — Estimate (2.18)(a) is standard. To prove (2.18)(b) and (2.19)(c) it is sufficient to use the inequality

$$\|Df\|_{L^\infty(\Omega)} \leq C(\|f\|_{C^\theta(\Omega)})^{\frac{1+\theta}{2}}(\|D^2 f\|_{C^\theta(\Omega)})^{\frac{1-\theta}{2}},$$

applied to the function $y \mapsto q(t,y) - q(s,y)$ with $\Omega = \mathbb{R}$ and to the function $y \mapsto e^{-x/2}(z(t,x,y) - z(s,x,y))$ with $\Omega = \mathbb{R}^2_-$, respectively. We get, for every $y \in \mathbb{R}$ and

$t \geq s \geq 0$,

$$|q_y(t,y) - q_y(s,y)| \leq C \left\| \int_s^t q_t(\sigma,\cdot)d\sigma \right\|_{C^\theta(\mathbb{R})}^{\frac{1+\theta}{2}} \Big(\|q_{yy}(t,\cdot)\|_{C^\theta(\mathbb{R})} + \|q_{yy}(s,\cdot)\|_{C^\theta(\mathbb{R})} \Big)^{\frac{1-\theta}{2}}$$

$$\leq C(t-s)^{\frac{1+\theta}{2}} \Big(\sup_{\sigma \geq s} \|q_t(\sigma,\cdot)\|_{C^\theta(\mathbb{R})} \Big)^{\frac{1+\theta}{2}} \Big(2 \sup_{\sigma \geq s} \|q_{yy}(\sigma,\cdot)\|_{C^\theta(\mathbb{R})} \Big)^{\frac{1-\theta}{2}}$$

$$\leq C' \frac{(t-s)^{\frac{1+\theta}{2}}}{1+s} \|q\|_{\mathcal{X}_q},$$

and (2.18)(b) follows. (2.19)(c) is proved similarly. ∎

The main tool for the fixed point argument to work are the next theorems 2.6 and 2.11, which provide optimal regularity results and estimates for linear problems in the above weighted spaces. Such results will be obtained in the next subsection, using analytic semigroups and interpolation techniques, and constitute the technical part of the paper.

2.3 Optimal regularity and estimates for linear problems

Let us begin with regularity results and estimates which will be used for the linearized q-equation,

$$\begin{cases} q_t(t,y) = q_{yy}(t,y) + \varphi(t,y), \quad t \geq 0, \ y \in \mathbb{R}, \\ \\ q(0,y) = q_0(y), \quad y \in \mathbb{R}, \end{cases}$$

Since the linear part is the heat equation, we shall use some well known estimates about the heat semigroup $T(t)$ given by $(T(t)f)(y) = \frac{1}{(4\pi t)^{1/2}} \int_{-\infty}^{+\infty} \exp(-(y-s)^2/4t) f(s)ds$. Precisely,

$$\|T(t)f\|_\infty \leq \|f\|_\infty, \quad \|\partial^k/\partial t^k T(t)f\|_\infty \leq \frac{C}{t^{k/2}} \|f\|_\infty, \quad k \geq 1, \ t > 0, \tag{2.20}$$

and

$$[D^k T(t) f]_{C^\theta} \leq \frac{C}{t^{(k+\theta)/2}} \|f\|_\infty, \quad t > 0, \tag{2.21}$$

for every $f \in C^0(\mathbb{R})$.

Theorem 2.6 *Let $\varphi : [0,\infty) \times \mathbb{R} \mapsto \mathbb{R}$ be such that both φ and φ_y are continuous, and for some $\alpha > 0$*

$$[[\varphi]] = \sup_{t \geq 0} (1 + t^{1+\alpha}) \|\varphi(t,\cdot)\|_\infty + \sup_{t \geq 0} (1 + t^{3/2}) \|\varphi_y(t,\cdot)\|_\infty < \infty.$$

Let in addition $q_0 \in C^{2+\theta}(\mathbb{R})$, $0 < \theta < 1$. Then the problem

$$\begin{cases} q_t(t,y) = q_{yy}(t,y) + \varphi(t,y), \quad t \geq 0, \ y \in \mathbb{R}, \\ \\ q(0,y) = q_0(y), \quad y \in \mathbb{R}, \end{cases}$$

has a unique bounded classical solution q, which belongs to $\mathcal{X}_q$ and satisfies

$$\|q\|_{\mathcal{X}_q} \leq C(\|q_0\|_{C^{2+\theta}(\mathbb{R})} + [[\varphi]]). \tag{2.22}$$

In addition,

$$\sup_{t \geq 0} (1 + t^{\theta/2}) [q_{yy}(t,\cdot)]_{C^\theta(\mathbb{R})} \leq C(\|q_0\|_{C^{2+\theta}(\mathbb{R})} + [[\varphi]]).$$

Proof — The unique bounded classical solution is

$$q(t,\cdot) = T(t)q_0 + \int_0^t T(t-s)\varphi(s,\cdot)ds = T(t)q_0 + q_1(t,\cdot), \ \ t \geq 0,$$

where $T(t)$ is the heat semigroup. Concerning $T(t)q_0$, from estimates (2.20) and (2.21) we get easily

$$\|T(\cdot)q_0\|_{\mathcal{X}_q} + \sup_{t\geq 0}(1+t^{\theta/2})\left[\frac{\partial^2}{\partial y^2}T(t)q_0\right]_{C^\theta(\mathbb{R})} \leq C(\|q_0\|_{C^{2+\theta}(\mathbb{R})} + [[\varphi]]).$$

Concerning q_1 we get easily for $0 \leq t \leq 1$

$$\|q_1(t,\cdot)\|_{C^{2+\theta}(\mathbb{R})} \leq C(\sup_{0\leq t\leq 1}\|\varphi(t,\cdot)\|_\infty + \sup_{0\leq t\leq 1}\|\varphi_y(t,\cdot)\|_\infty).$$

For $t \geq 1$, using estimates (2.20) we get

$$\|q_1(t,\cdot)\|_\infty \leq \int_0^{+\infty}\frac{1}{1+s^{1+\alpha}}\,ds\ \sup_{s\geq 0}(1+s^{1+\alpha})\|\varphi(s,\cdot)\|_\infty$$

$$\|\tfrac{\partial}{\partial y}\,q_1(t,\cdot)\|_\infty \leq \left(\int_0^{t/2} + \int_{t/2}^t\right)\frac{1}{1+s^{1+\alpha}}\frac{C}{(t-s)^{1/2}}\,ds\ \sup_{s\geq 0}(1+s^{1+\alpha})\|\varphi(s,\cdot)\|_\infty$$

$$\leq \left(\frac{C}{(t/2)^{1/2}}\int_0^{+\infty}\frac{1}{1+s^{1+\alpha}}\,ds + \frac{C}{1+(t/2)^{1+\alpha}}\int_{t/2}^t\frac{1}{s^{1/2}}\,ds\right)\sup_{s\geq 0}(1+s^{1+\alpha})\|\varphi(s,\cdot)\|_\infty,$$

$$\leq \frac{C}{t^{1/2}}\sup_{s\geq 0}(1+s^{1+\alpha})\|\varphi(s,\cdot)\|_\infty,$$

and

$$\|\tfrac{\partial^2}{\partial y^2}\,q_1(t,\cdot)\|_\infty \leq \Bigg(\int_0^{t/2}\frac{1}{1+s^{1+\alpha}}\frac{C}{t-s}\,ds\ \sup_{s\geq 0}(1+s^{1+\alpha})\|\varphi(s,\cdot)\|_\infty$$

$$+\int_{t/2}^t\frac{1}{1+s^{3/2}}\frac{C}{(t-s)^{1/2}}\,ds\ \sup_{s\geq 0}(1+s^{3/2})\|\varphi_y(s,\cdot)\|_\infty\Bigg)$$

$$\leq \frac{C}{t/2}\int_0^{+\infty}\frac{1}{1+s^{1+\alpha}}\,ds\ \sup_{s\geq 0}(1+s^{1+\alpha})\|\varphi(s,\cdot)\|_\infty$$

$$+\frac{C}{1+(t/2)^{3/2}}\int_{t/2}^t\frac{1}{(t-s)^{1/2}}\,ds\ \sup_{s\geq 0}(1+s^{3/2})\|\varphi_y(s,\cdot)\|_\infty$$

$$\leq \frac{C}{t}(\sup_{s\geq 0}(1+s^{1+\alpha})\|\varphi(s,\cdot)\|_\infty + \sup_{s\geq 0}(1+s^{3/2})\|\varphi_y(s,\cdot)\|_\infty) \leq \frac{C}{t^{1/2}}\,[[\varphi]].$$

Concerning the C^θ-seminorm of $\frac{\partial^2}{\partial y^2}\,q_1$ we get, using estimates (2.21)

$$\left[\tfrac{\partial^2}{\partial y^2}\,q_1(t,\cdot)\right]_{C^\theta(\mathbb{R})} \leq \Bigg(\int_0^{t/2}\frac{1}{1+s^{1+\alpha}}\frac{C}{(t-s)^{1+\theta/2}}\,ds\ \sup_{s\geq 0}(1+s^{1+\alpha})\|\varphi(s,\cdot)\|_\infty$$

$$+\int_{t/2}^t\frac{1}{1+s^{3/2}}\frac{C}{(t-s)^{(1+\theta)/2}}\,ds\ \sup_{s\geq 0}(1+s^{3/2})\|\varphi_y(s,\cdot)\|_\infty\Bigg)$$

$$\leq \frac{C}{(t/2)^{1+\theta/2}}\int_0^{+\infty}\frac{1}{1+s^{1+\alpha}}\,ds\ \sup_{s\geq 0}(1+s^{1+\alpha})\|\varphi(s,\cdot)\|_\infty$$

$$+\frac{C}{1+(t/2)^{3/2}}\int_{t/2}^t\frac{1}{(t-s)^{(1+\theta)/2}}\,ds\ \sup_{s\geq 0}(1+s^{3/2})\|\varphi_y(s,\cdot)\|_\infty \leq \frac{C}{t^{1+\theta/2}}\,[[\varphi]],$$

and (2.22) follows. ■

The regularity results and the estimates we need for the linearized z-equation,

$$\begin{cases} z_t = \mathcal{L}z + Q\eta - \zeta e^x, \;\; t \geq 0, \; x \leq 0, \; y \in \mathbb{R}, \\ \\ \mathcal{B}z = \zeta, \;\; t \geq 0, \; x = 0, \; y \in \mathbb{R}, \\ \\ z(0,\cdot) = z_0, \;\; x \leq 0, \; y \in \mathbb{R}, \end{cases}$$

are stated in the next propositions. To use analytic semigroups and interpolation techniques we shall use a variant of the Balakrishnan formula, representing z as

$$e^{tL}z_0 + \int_0^t e^{(t-s)L}(Q\eta(s,\cdot) + \mathcal{L}\mathcal{N}\zeta(s,\cdot) - \zeta(s,\cdot)e^x)ds - L\int_0^t e^{(t-s)L}\mathcal{N}\zeta(s,\cdot)ds, \;\; t \geq 0, \quad (2.23)$$

where $\mathcal{N}$ is a suitable lifting operator such that $\mathcal{B}\mathcal{N}\zeta = \zeta$ for each sufficiently regular ζ. The construction of $\mathcal{N}$ is given in the next lemma.

Lemma 2.7 *There exists a linear operator $\mathcal{N} \in L(C^{k+\theta}(\mathbb{R}); C^{k+\theta+1}_\omega(\mathbb{R}^2_-))$, for each $k \in \mathbb{N} \cup \{0\}$, $\theta \in [0,1)$, such that*

$$(a)\; \mathcal{B}\mathcal{N}\zeta = \zeta, \quad (b)\; P\mathcal{N}\zeta = 0, \;\; \forall \zeta \in C^0(\mathbb{R}),$$

$$(c)\; P\mathcal{L}\mathcal{N}\zeta = \zeta e^x, \;\; \forall \zeta \in C^1(\mathbb{R}).$$

Proof — Fix a function $\varphi \in C^\infty(\mathbb{R})$, with compact support, and such that $\int_{\mathbb{R}} \varphi(\xi)d\xi = 1$. Fix moreover $\eta \in C^\infty((-\infty,0])$ such that $\eta \equiv 0$ for $x \leq -2$, $\eta \equiv 1$ for $-1 \leq x \leq 0$.

For each $k \in \mathbb{N}$ and $\zeta \in C^0(\mathbb{R})$ set

$$(N_1\zeta)(x,y) = x\eta(x)\int_{\mathbb{R}} \varphi(\xi)\zeta(y+\xi x)d\xi, \;\; x \leq 0, \; y \in \mathbb{R}. \quad (2.24)$$

Then $N_1 \in L(C^{k+\theta}(\mathbb{R}); C^{k+\theta+1}_\omega(\mathbb{R}^2_-))$, for each $k \in \mathbb{N} \cup \{0\}$, $\theta \in [0,1)$, and $N_1\zeta(0,y) = 0$, $(N_1\zeta)_x(0,y) = \zeta(y)$, so that $\mathcal{B}N_1\zeta = \zeta$ for each $\zeta \in C^0(\mathbb{R})$. Let us define

$$\mathcal{N} = N_1 - PN_1.$$

Then $\mathcal{N}$ satisfies (b), and since $\mathcal{B}P = 0$, it satisfies also (a). If $\zeta \in C^1(\mathbb{R})$, then $\varphi = \mathcal{N}\zeta \in Q(C^2_\omega(\mathbb{R}^2_-))$ so that $P(\varphi_{yy}) = 0$ and consequently $P\mathcal{L}\varphi = \mathcal{B}\varphi e^x = \zeta e^x$, and statement (c) follows. ■

The next proposition is concerned with the first two addenda in (2.23). Note that the function $Q\eta(s,\cdot) + \mathcal{L}\mathcal{N}\zeta(s,\cdot) - \zeta(s,\cdot)e^x$ in the first integral belongs to $Q(C^\theta_\omega(\mathbb{R}^2_-))$, thanks to statement (b) of lemma 2.7.

Proposition 2.8 *Let $\psi : [0,\infty) \times \mathbb{R}^2_- \mapsto \mathbb{R}$ be a continuous function such that for each t $\psi(t,\cdot) \in Q(C^\theta_\omega(\mathbb{R}^2_-))$, with $0 < \theta < 1$, and*

$$\sup_{t \geq 0}(1+t)\|\psi(t,\cdot)\|_{C^\theta_\omega(\mathbb{R}^2_-)} < \infty.$$

Let in addition $v_0 \in Q(C^{2+\theta}_\omega(\mathbb{R}^2_-))$. Then the problem

$$\begin{cases} v_t(t,x,y) = \mathcal{L}v(t,x,y) + \psi(t,x,y), \;\; t \geq 0, \; x \leq 0, \; y \in \mathbb{R}, \\ \\ \mathcal{B}v(t,0,y) = 0, \;\; t \geq 0, \; y \in \mathbb{R}, \\ \\ v(0,y) = v_0(x,y), \;\; x \leq 0, \; y \in \mathbb{R}, \end{cases}$$

has a unique bounded classical solution v, given by

$$v(t,\cdot) = e^{tL}v_0 + \int_0^t e^{(t-s)L}\psi(s,\cdot)ds = Qe^{tL}v_0 + \int_0^t e^{(t-s)L}Q\psi(s,\cdot)ds, \ \ t \geq 0.$$

For every $t > 0$, $v(t,\cdot)$, $Dv(t,\cdot)$, $D^2(t,\cdot)$, $v_t(t,\cdot)$ belong to $C^{\theta}_{\omega}(\mathbb{R}^2_-)$, and

$$\begin{array}{l} \sup_{t\geq 1} t(\|v(t,\cdot)\|_{C^{2+\theta}_{\omega}(\mathbb{R}^2_-)} + \|v_t(t,\cdot)\|_{C^{\theta}_{\omega}(\mathbb{R}^2_-)}) \\ \\ \leq C(\|v_0\|_{C^0_{\omega}(\mathbb{R}^2_-)} + \sup_{t\geq 1} t\|\psi(t,\cdot)\|_{C^{\theta}_{\omega}(\mathbb{R}^2_-)}). \end{array} \tag{2.25}$$

If in addition $\mathcal{B}v_0(0,y) \equiv 0$, then we have also

$$\begin{array}{l} \sup_{0\leq t\leq 1}(\|v(t,\cdot)\|_{C^{2+\theta}_{\omega}(\mathbb{R}^2_-)} + \|v_t(t,\cdot)\|_{C^{\theta}_{\omega}(\mathbb{R}^2_-)}) \\ \\ \leq C(\|v_0\|_{C^{2+\theta}_{\omega}(\mathbb{R}^2_-)} + \sup_{0\leq t\leq 1} \|\psi(t,\cdot)\|_{C^{\theta}_{\omega}(\mathbb{R}^2_-)}), \end{array} \tag{2.26}$$

so that $v \in \mathcal{X}_z$, and

$$\|v\|_{\mathcal{X}_z} \leq C(\|v_0\|_{C^{2+\theta}_{\omega}(\mathbb{R}^2_-)} + \sup_{t\geq 0}(1+t)\|\psi(t,\cdot)\|_{C^{\theta}_{\omega}(\mathbb{R}^2_-)}). \tag{2.27}$$

Proof — Since $t \mapsto \psi(t,\cdot)$ is continuous with values in $C^0_{\omega}(\mathbb{R}^2_-)$ and bounded with values in $C^{\theta}_{\omega}(\mathbb{R}^2_-) = D_L(\theta/2,\infty)$, then the initial value problem $v'(t) = Lv(t) + \psi(t,\cdot)$, $v(0) = v_0$ has a unique classical solution in the space $C^0_{\omega}(\mathbb{R}^2_-)$, given by the variation of constants formula. Due to [18], both $v_t(t,\cdot)$ and $Lv(t,\cdot)$ have values in $D_L(\theta/2,\infty) = C^{\theta}_{\omega}(\mathbb{R}^2_-)$ for $t > 0$, and lemma 2.4 implies that for each $t > 0$ $v(t,\cdot)$ has values in $C^{2+\theta}_{\omega}(\mathbb{R}^2_-)$. If in addition $\mathcal{B}v_0(0,y) \equiv 0$ then $v_0 \in D_L(\theta/2+1,\infty)$ by lemma 2.4, and estimate (2.26) follows from [18].

To prove (2.25) we use the estimates

$$\|e^{tL}Qf\|_{C^0_{\omega}(\mathbb{R}^2_-)} \leq Me^{-\omega t}\|f\|_{C^0_{\omega}(\mathbb{R}^2_-)},$$

$$\|Le^{tL}Qf\|_{C^0_{\omega}(\mathbb{R}^2_-)} \leq \frac{Me^{-\omega t}}{t^{1-\theta/2}}\|f\|_{C^{\theta}_{\omega}(\mathbb{R}^2_-)},$$

$$\|L^2e^{tL}Qf\|_{C^0_{\omega}(\mathbb{R}^2_-)} \leq \frac{Me^{-\omega t}}{t^{2-\theta/2}}\|f\|_{C^{\theta}_{\omega}(\mathbb{R}^2_-)},$$

with $\omega > 0$, which follow from the usual properties of analytic semigroups of negative type and from the characterization of $C^{\theta}_{\omega}(\mathbb{R}^2_-)$ as the interpolation space $D_L(\theta/2,\infty)$. We get immediately $\|e^{tL}v_0\|_{C^{2+\theta}_{\omega}(\mathbb{R}^2_-)} \leq Ce^{-\omega t}\|v_0\|_{C^{2+\theta}_{\omega}(\mathbb{R}^2_-)}$, and for $t \geq 1$

$$\left\|\int_0^t e^{(t-s)L}\psi(s,\cdot)ds\right\|_{C^0_{\omega}(\mathbb{R}^2_-)} \leq \left(\int_0^{t/2} + \int_{t/2}^t\right)\frac{Me^{-\omega(t-s)}}{1+s}\,ds\,\sup_{t\geq 0}(1+t)\|\psi(t,\cdot)\|_{C^0_{\omega}(\mathbb{R}^2_-)}$$

$$\leq M\left(e^{-\omega t/2}\log(t/2+1) + \frac{1}{\omega(1+t/2)}\right)\sup_{t\geq 0}(1+t)\|\psi(t,\cdot)\|_{C^0_{\omega}(\mathbb{R}^2_-)},$$

$$\left\|L\int_0^t e^{(t-s)L}\psi(s,\cdot)ds\right\|_{C^0_{\omega}(\mathbb{R}^2_-)} \leq \left(\int_0^{t/2} + \int_{t/2}^t\right)\frac{Me^{-\omega(t-s)}}{(1+s)(t-s)^{1-\theta/2}}\,ds\cdot$$

$$\cdot\sup_{t\geq 0}(1+t)\|\psi(t,\cdot)\|_{C^{\theta}_{\omega}(\mathbb{R}^2_-)}$$

$$\leq M\left(\frac{e^{-\omega t/2}\log(t/2+1)}{(t/2)^{1-\theta/2}} + \frac{1}{1+t/2}\int_0^{\infty}\frac{e^{-\omega s}}{s^{1-\theta/2}}\,ds\right)\sup_{t\geq 0}(1+t)\|\psi(t,\cdot)\|_{C^{\theta}_{\omega}(\mathbb{R}^2_-)},$$

and for each $\xi > 0$

$$
\begin{aligned}
&\left\| \xi^{1-\theta/2} L^2 e^{\xi L} \int_0^t e^{(t-s)L} \psi(s,\cdot) ds \right\|_{C^0_\omega(\mathbb{R}^2_-)} \\
&\leq \xi^{1-\theta/2} \Big(\int_0^{t/2} + \int_{t/2}^t \Big) \frac{M e^{-\omega(\xi+t-s)}}{(1+s)(\xi+t-s)^{2-\theta/2}}\, ds \ \sup_{t\geq 0} (1+t) \|\psi(t,\cdot)\|_{C^\theta_\omega(\mathbb{R}^2_-)} \\
&\leq M \Big(\frac{\xi^{1-\theta/2} e^{-\omega(\xi+t/2)} \log(t/2+1)}{(t/2)^{2-\theta/2}} \\
&+ \frac{\xi^{1-\theta/2}}{1+t/2} \frac{1}{1-\theta/2} \Big(\frac{1}{\xi^{1-\theta/2}} - \frac{1}{(\xi+t/2)^{1-\theta/2}} \Big) \Big) \sup_{t\geq 0} (1+t) \|\psi(t,\cdot)\|_{C^\theta_\omega(\mathbb{R}^2_-)}, \\
&\leq \frac{C}{1+t} \sup_{t\geq 0} (1+t) \|\psi(t,\cdot)\|_{C^\theta_\omega(\mathbb{R}^2_-)},
\end{aligned}
$$

so that

$$
\sup_{t\geq 0} (1+t) \| Lv(t,\cdot)]_{D_L(\theta/2,\infty)} \leq C \sup_{t\geq 0} (1+t) \|\psi(t,\cdot)\|_{C^\theta_\omega(\mathbb{R}^2_-)},
$$

and the statement follows through the characterization of $D_L(\theta/2+1,\infty)$ provided by lemma 2.4. ∎

The investigation on the term $L\int_0^t e^{(t-s)L} \mathcal{N}\zeta(s,\cdot)ds$ in formula (2.23) is performed using a general result about abstract parabolic equations in Banach spaces, which we state below.

Proposition 2.9 *Let Y be a Banach space, let $A : D(A) \subset Y \mapsto Y$ be the generator of an analytic semigroup of negative type in X, and let $f : [0,+\infty) \mapsto X$ be a γ-Hölder continuous function such that*

$$
\sup_{t\geq 0} \|f(t)\|_Y + \sup_{t>s\geq 0} (1+s) \frac{\|f(t)-f(s)\|_Y}{(t-s)^\gamma} < \infty,
$$

with $0 < \gamma < 1$. Then the function

$$
u(t) = \int_0^t e^{(t-s)A} f(s) ds, \quad t \geq 0
$$

is the unique classical solution of $u'(t) = Au(t) + f(t)$, $t > 0$; $u(0) = 0$, u' has values in $D_A(\gamma,\infty)$ for $t > 0$ and

$$
\sup_{t\geq 1} t \|u'(t)\|_{D_A(\gamma,\infty)} \leq C \Big(\sup_{t\geq 1} \|f(t)\|_Y + \sup_{t>s\geq 1} (1+s) \frac{\|f(t)-f(s)\|_Y}{(t-s)^\gamma} \Big).
$$

If in addition $f(0) = 0$ then u' is bounded with values in $D_A(\gamma,\infty)$ also for $0 \leq t \leq 1$, and we have

$$
\sup_{0<t\leq 1} \|u'(t)\|_{D_A(\gamma,\infty)} \leq C \|f\|_{C^\gamma([0,1];Y)}.
$$

Proof — The regularity of u and the estimate $\sup_{0\leq t\leq 1} \|u'(t)\|_{D_A(\gamma,\infty)} \leq C\|f\|_{C^\gamma([0,1];Y)}$ (if $f(0) = 0$) follow from well known old results, which go back to Kato [15], see [18] for a detailed proof. Here we check the statement about asymptotic behavior. Let M_k, $\omega > 0$ be such that

$$
t^k \|A^k e^{tA}\|_{L(Y)} \leq M_k e^{-\omega t}, \quad t > 0, \ k = 0,1,2,
$$

and set

$$[f] = \sup_{t>s\geq 0} (1+s)\frac{\|f(t)-f(s)\|_Y}{(t-s)^\gamma}.$$

Then

$$u'(t) = \int_0^t Ae^{(t-s)A}(f(s)-f(t))ds + e^{tA}f(t), \;\; t\geq 0,$$

so that for $t \geq 1$

$$\begin{aligned}
\|u'(t)\| &\leq M_1\Big(\int_0^{t/2} + \int_{t/2}^t\Big)\frac{e^{-\omega(t-s)}}{(t-s)^{1-\gamma}(1+s)}[f]ds + M_0e^{-\omega t}\|f(t)\|_Y \\
&\leq M_1\Big(\frac{e^{-\omega t/2}}{(t/2)^{1-\gamma}}\int_0^{t/2}\frac{ds}{1+s} + \frac{1}{1+t/2}\int_0^{t/2}\frac{e^{-\omega\sigma}}{\sigma^{1-\gamma}}d\sigma\Big)[f] + M_0e^{-\omega t}\|f(t)\|_Y \\
&\leq Ct^{-1}([f]+\|f(t)\|_Y),
\end{aligned}$$

and for $\xi > 0$, $t \geq 1$ we have

$$\|\xi^{1-\gamma}Ae^{\xi A}u'(t)\| \leq \xi^{1-\gamma}M_2\Big(\int_0^{t/2} + \int_{t/2}^t\Big)\frac{e^{-\omega(\xi+t-s)}(t-s)^\gamma}{(\xi+t-s)^2(1+s)}[f]ds$$

$$+M_1\xi^{1-\gamma}\frac{e^{-\omega(\xi+t)}}{\xi+t}\|f(t)\|_Y$$

$$\leq M_2\xi^{1-\gamma}\Big(\frac{(t/2)^\gamma e^{-\omega(\xi+t/2)}}{(\xi+t/2)^2}\int_0^{t/2}\frac{ds}{1+s} + \frac{1}{1+t/2}\int_{t/2}^t\frac{1}{(\xi+t-s)^{2-\gamma}}d\sigma\Big)[f]$$

$$+M_1\frac{e^{-\omega t}}{t}\|f(t)\|_Y \leq \frac{C'}{t}([f]+\|f(t)\|_Y),$$

and the statement follows. ∎

We shall use proposition 2.9 to study the function $-L\int_0^t e^{(t-s)L}\mathcal{N}\zeta(s,\cdot)ds$. The procedure is the same of [16, Ch. 5], [4], adapted to the present situation.

Proposition 2.10 *Let $\zeta : [0,\infty)\times\mathbb{R} \mapsto \mathbb{R}$ belong to $C^{(1+\theta)/2,1+\theta}([0,\infty)\times\mathbb{R})$, be such that $\zeta(0) = 0$ and*

$$[[\zeta]] = \sup_{t\geq 0}(1+t)\|\zeta(t,\cdot)\|_{C^{1+\theta}(\mathbb{R})} + \sup_{t>s\geq 0}(1+s)\frac{\|\zeta(t,\cdot)-\zeta(s,\cdot)\|_{C^0(\mathbb{R})}}{(t-s)^{(1+\theta)/2}} < \infty.$$

Then the function

$$w(t,\cdot) = -L\int_0^t e^{(t-s)L}\mathcal{N}\zeta(s,\cdot)ds, \;\; t\geq 0,$$

is twice continuously differentiable with respect to (x,y), once continuously differentiable with respect to t, and

$$\begin{aligned}
&\sup\nolimits_{t\geq 1} t(\|w(t,\cdot)\|_{C^{2+\theta}_\omega(\mathbb{R}^2_-)} + \|w_t(t,\cdot)\|_{C^\theta_\omega(\mathbb{R}^2_-)}) \\
&\leq C\Big(\sup_{t\geq 1} t\|\zeta(t,\cdot)\|_{C^{1+\theta}(\mathbb{R})} + \sup_{t>s\geq 0}(1+s)\frac{\|\zeta(t,\cdot)-\zeta(s,\cdot))\|_{C^0(\mathbb{R})}}{(t-s)^{(1+\theta)/2}}\Big).
\end{aligned} \tag{2.28}$$

Moreover,

$$w_t = \mathcal{L}w - \mathcal{L}\mathcal{N}\zeta, \;\; t>0, \; x\leq 0, \; y\in\mathbb{R}, \tag{2.29}$$

$$\mathcal{B}w(t,0,y) = \zeta(y), \;\; t>0, \; y\in\mathbb{R}. \tag{2.30}$$

If in addition $\zeta(0,\cdot)=0$, then we have also

$$\begin{aligned} &\sup\nolimits_{0\le t\le 1}(\|w(t,\cdot)\|_{C^{2+\theta}_\omega(\mathbb{R}^2_-)} + \|w_t(t,\cdot)\|_{C^{\theta}_\omega(\mathbb{R}^2_-)}) \\ &\le C\bigg(\sup_{0\le t\le 1}\|\zeta(t,\cdot)\|_{C^{1+\theta}(\mathbb{R})} + \sup_{0\le s<t\le 1}\frac{\|\zeta(t,\cdot)-\zeta(s,\cdot))\|_{C^0(\mathbb{R})}}{(t-s)^{(1+\theta)/2}}\bigg). \end{aligned} \tag{2.31}$$

Proof — The function $t\mapsto \mathcal{N}\zeta(t,\cdot)$ belongs to $C^{(1+\theta)/2}([0,\infty);C^1_\omega(\mathbb{R}^2_-))$ and it is bounded with values in $Q(C^{2+\theta}_\omega(\mathbb{R}^2_-))$. Moreover,

$$\sup_{t>s\ge 0}(1+s)\frac{\|\mathcal{N}\zeta(t,\cdot)-\mathcal{N}\zeta(s,\cdot))\|_{C^1_\omega(\mathbb{R}^2_-)}}{(t-s)^{(1+\theta)/2}} \le C[[\zeta]],$$

$$\sup_{t\ge 0}(1+t)\|\mathcal{N}\zeta(t,\cdot)\|_{C^{2+\theta}_\omega(\mathbb{R}^2_-)} \le C[[\zeta]].$$

Since $C^1_\omega(\mathbb{R}^2_-)$ is continuously embedded in $D_L(1/2,\infty)$, we may apply proposition 2.9 with $Y=Q(D_L(1/2,\infty))$, $A=QL_{|D_L(1/2,\infty)}$, $\gamma=1/2+\theta/2$, $u=\int_0^t e^{(t-s)L}\mathcal{N}\zeta(s,\cdot)ds$, getting

$$\left\{\begin{array}{l} -u' = w(t,\cdot)-\mathcal{N}\zeta(t,\cdot)\in D_A(1/2+\theta/2,\infty), \;\; t>0, \\ \\ \sup_{t\ge 1}\, t\|(w(t,\cdot)-\mathcal{N}\zeta(t,\cdot)\|_{D_A(1/2+\theta/2,\infty)} \le C[[\zeta]], \end{array}\right. \tag{2.32}$$

and, if $\zeta(0,\cdot)=0$,

$$\sup_{0\le t\le 1}\|w(t,\cdot)-\mathcal{N}\zeta(t,\cdot)\|_{D_A(1/2+\theta/2,\infty)} \le C[[\zeta]]. \tag{2.33}$$

By reiteration, $D_A(1/2+\theta/2,\infty) = QD_L(1+\theta/2,\infty)$, and by lemma 2.4, $QD_L(1+\theta/2,\infty) = \{g\in QC^{2+\theta}_\omega(\mathbb{R}^2_-) : \mathcal{B}g(0,y)=0\}$. It follows that $\mathcal{B}(w-\mathcal{N}\zeta)=0$, i.e. (2.30) holds. Since $\|\mathcal{N}\zeta(t,\cdot)\|_{C^{2+\theta}_\omega(\mathbb{R}^2_-)} \le C\|\zeta(t,\cdot)\|_{C^{1+\theta}_\omega(\mathbb{R})}$, from (2.32) and (2.33) we get

$$\sup_{t\ge 1} t\|w(t,\cdot)\|_{C^{2+\theta}_\omega(\mathbb{R}^2_-)} \le C[[\zeta]],$$

and if $\zeta(0,\cdot)=0$,

$$\sup_{0\le t\le 1}\|w(t,\cdot)\|_{C^{2+\theta}_\omega(\mathbb{R}^2_-)} \le C[[\zeta]].$$

Since u' is continuous with values in $Y=QD_L(1/2,\infty)$ and bounded with values in $D_A(1/2+\theta/2,\infty) = QD_L(1+\theta/2,\infty)\subset D(L)$, then u' is continuous with values in $D(L)$, so that Lu is time differentiable and $(Lu)'=Lu'$. It follows that $w=-Lu(t,\cdot)$ is differentiable with respect to time and $w_t = -Lu'(t,\cdot) = \mathcal{L}w - \mathcal{L}\mathcal{N}\zeta$, i.e. (2.29) holds. The statement follows. ∎

Taking into account the results of propositions 2.8 and 2.10 we may state the regularity and asymptotic behavior results we need for the z-problem.

Theorem 2.11 *Let $\psi:[0,\infty)\times\mathbb{R}^2_-\mapsto\mathbb{R}$, $\zeta:[0,\infty)\times\mathbb{R}\mapsto\mathbb{R}$ satisfy the assumptions of propositions 2.8, 2.10, and let $z_0\in QC^{2+\theta}_\omega(\mathbb{R}^2_-)$ be such that $\mathcal{B}z_0=\zeta(0,\cdot)$. Then the function z defined by*

$$z(t,\cdot) = e^{tL}z_0 + \int_0^t e^{(t-s)L}(\psi(s,\cdot)+\mathcal{L}\mathcal{N}\zeta(s,\cdot)-e^x\zeta(s,\cdot))ds - L\int_0^t e^{(t-s)L}\mathcal{N}\zeta(s,\cdot)ds, \tag{2.34}$$

is the unique bounded solution of

$$\begin{cases} z_t(t,x,y) = \mathcal{L}z(t,x,y) + \psi(t,x,y) - \zeta(t,y)e^x, & t \geq 0,\ x \leq 0,\ y \in \mathbb{R}, \\ \\ \mathcal{B}z(t,0,y) = \zeta(t,y), & t \geq 0,\ y \in \mathbb{R}, \\ \\ z(0,x,y) = z_0(x,y), & x \leq 0,\ y \in \mathbb{R}. \end{cases} \tag{2.35}$$

Moreover $z \in \mathcal{X}_z$ *and*

$$\begin{aligned} \|z\|_{\mathcal{X}_z} \leq C \Big(&\|z_0\|_{C^{2+\theta}_\omega(\mathbb{R}^2_-)} + \sup\nolimits_{t \geq 0}(1+t)\|\psi(t,\cdot)\|_{C^\theta_\omega(\mathbb{R}^2_-)} \\ &+ \sup_{t\geq 0}(1+t)\|\zeta(t,\cdot)\|_{C^{1+\theta}(\mathbb{R})} + \sup_{t > s \geq 0}(1+s)\frac{\|\zeta(t,\cdot) - \zeta(s,\cdot))\|_{C^0(\mathbb{R})}}{(t-s)^{(1+\theta)/2}} \Big). \end{aligned} \tag{2.36}$$

Proof — To study the regularity of z for $t > 0$ and to get estimates for t near 0, say $0 \leq t \leq 1$, write it as $z = z_1 + z_2$, where

$$z_1(t,\cdot) = e^{tL}(z_0 - \mathcal{N}\zeta(0,\cdot)) + \int_0^t e^{(t-s)L}(\psi(s,\cdot) + \mathcal{L}\mathcal{N}\zeta(s,\cdot) - \zeta(s,\cdot)e^x)ds,$$

$$z_2(t,\cdot) = -L\int_0^t Qe^{(t-s)L}\mathcal{N}(\zeta(s,\cdot) - \zeta(0,\cdot))ds + \mathcal{N}\zeta(0,\cdot).$$

Let us consider z_1: the initial datum $v_0 = z_0 - \mathcal{N}\zeta(0,\cdot)$ belongs to $C^{2+\theta}_\omega(\mathbb{R}^2_-)$, and $\mathcal{B}v_0(0,y) \equiv 0$. Moreover the function $s \mapsto \psi(s,\cdot) + \mathcal{L}\mathcal{N}\zeta(s,\cdot) - \zeta(s,\cdot)e^x$ is continuous with values in $Q(C^0_\omega(\mathbb{R}^2_-))$ and bounded with values in $Q(C^\theta_\omega(\mathbb{R}^2_-))$. By proposition 2.8, z_1 solves

$$\begin{cases} D_t z_1 = \mathcal{L}z_1 + \psi + \mathcal{L}\mathcal{N}\zeta - \zeta e^x, & t > 0,\ x \leq 0,\ y \in \mathbb{R}, \\ \\ \mathcal{B}z_1 = 0, & t > 0,\ y \in \mathbb{R}, \\ \\ z_1(0,x,y) = z_0(x,y) - (\mathcal{N}\zeta(0,\cdot))(x,y), & x \leq 0,\ y \in \mathbb{R}, \end{cases}$$

and it satisfies

$$\begin{aligned} &\sup\nolimits_{0\leq t\leq 1}(\|z_1(t,\cdot)\|_{C^{2+\theta}_\omega(\mathbb{R}^2_-)} + \|D_t z_1(t,\cdot)\|_{C^\theta_\omega(\mathbb{R}^2_-)}) \\ &\leq C(\|z_0\|_{C^{2+\theta}_\omega(\mathbb{R}^2_-)} + \sup\nolimits_{0\leq t\leq 1}\|\psi(t,\cdot)\|_{C^\theta_\omega(\mathbb{R}^2_-)} + \sup\nolimits_{0\leq t\leq 1}\|\zeta(t,\cdot)\|_{C^{1+\theta}(\mathbb{R})}). \end{aligned}$$

Let us consider now z_2. The function $s \mapsto \zeta(s,\cdot) - \zeta(0,\cdot)$ is $(\theta+1)/2$-Hölder continuous with values in $C^0(\mathbb{R})$, and bounded with values in $C^{1+\theta}(\mathbb{R})$; moreover it vanishes at $s = 0$. By proposition 2.10, z_2 solves

$$\begin{cases} D_t z_2 = \mathcal{L}z_2 - \mathcal{L}\mathcal{N}\zeta(t,\cdot), & t > 0,\ x \leq 0,\ y \in \mathbb{R}, \\ \\ \mathcal{B}z_2(t,0,y) = \zeta(t,y), & t > 0,\ y \in \mathbb{R}, \\ \\ z_2(0,x,y) = (\mathcal{N}\zeta(0,\cdot))(x,y), & x \leq 0,\ y \in \mathbb{R}, \end{cases}$$

and it satisfies

$$\begin{aligned} &\sup\nolimits_{0\leq t\leq 1}(\|z_2(t,\cdot)\|_{C^{2+\theta}_\omega(\mathbb{R}^2_-)} + \|D_t z_2(t,\cdot)\|_{C^\theta_\omega(\mathbb{R}^2_-)}) \\ &\leq C(\|z_0\|_{C^{2+\theta}_\omega(\mathbb{R}^2_-)} + \sup\nolimits_{0\leq s<t\leq 1}(t-s)^{-(1+\theta)/2}(\|\zeta(t,\cdot) - \zeta(s,\cdot)\|_{C^0_\omega(\mathbb{R}^2_-)} \\ &+ \sup\nolimits_{0\leq t\leq 1}\|\zeta(t,\cdot)\|_{C^{1+\theta}(\mathbb{R})}). \end{aligned}$$

Summing up, we get that z solves (2.35) and

$$\sup_{0\le t\le 1}(\|z(t,\cdot)\|_{C^{2+\theta}_\omega(\mathbb{R}^2_-)} + \|D_t z(t,\cdot)\|_{C^\theta_\omega(\mathbb{R}^2_-)})$$

$$\le C(\|z_0\|_{C^{2+\theta}_\omega(\mathbb{R}^2_-)} + \sup_{0\le t\le 1}\|\psi(t,\cdot)\|_{C^\theta_\omega(\mathbb{R}^2_-)} + \sup_{0\le t\le 1}\|\zeta(t,\cdot)\|_{C^{1+\theta}(\mathbb{R})}).$$

To get estimates for $t\ge 1$ we split $z = z_1 + z_2$, where now

$$z_1 = e^{tL}z_0 + \int_0^t e^{(t-s)L}(\psi(s,\cdot) + \mathcal{L}\mathcal{N}\zeta(s,\cdot) - \zeta(s,\cdot)e^x)ds,$$

$$z_2(t,\cdot) = -L\int_0^t e^{(t-s)L}\mathcal{N}\zeta(s,\cdot)ds.$$

Using respectively estimates (2.25) and (2.28) we get

$$\sup_{t\ge 1} t(\|z_1(t,\cdot)\|_{C^{2+\theta}_\omega(\mathbb{R}^2_-)} + \|D_t z_1(t,\cdot)\|_{C^\theta_\omega(\mathbb{R}^2_-)})$$

$$\le C(\|z_0\|_{C^{2+\theta}_\omega(\mathbb{R}^2_-)} + \sup_{t\ge 0}(1+t)(\|\psi(t,\cdot)\|_{C^\theta_\omega(\mathbb{R}^2_-)} + \|\zeta(t,\cdot)\|_{C^{1+\theta}(\mathbb{R})})),$$

and

$$\sup_{t\ge 1} t(\|z_2(t,\cdot)\|_{C^{2+\theta}_\omega(\mathbb{R}^2_-)} + \|D_t z_2(t,\cdot)\|_{C^\theta_\omega(\mathbb{R}^2_-)})$$

$$\le C\Big(\sup_{t\ge 0}(1+t)\|\zeta(t,\cdot)\|_{C^{1+\theta}(\mathbb{R})} + \sup_{t>s\ge 0}(1+s)\frac{\|\zeta(t,\cdot)-\zeta(s,\cdot))\|_{C^0(\mathbb{R})}}{(t-s)^{(1+\theta)/2}}\Big),$$

and the proof is complete. ∎

2.4 The solution of the nonlinear system

The (q,z)-system will be solved in the obvious way, looking for a fixed point of the operator Γ defined in a small ball $B(0,r)\subset\mathcal{X}$, $\Gamma(q,z)$ being the couple $(\widetilde{q},\widetilde{z})$ defined by

$$\widetilde{q}(t,y) = (T(t)q_0)(y) + \int_0^t T(t-s)h(q,z)ds\,(y),$$

$$\widetilde{z}(t,x,y) = (e^{tL}z_0)(x,y) + \int_0^t Qe^{(t-s)L}(f(q,z)(s,\cdot) + \mathcal{L}\mathcal{N}g(q,z)(s,\cdot))ds\,(x,y)$$

$$-L\int_0^t e^{(t-s)L}\mathcal{N}g(q,z)(s,\cdot)ds(x,y).$$

Here $T(t)$ is the heat semigroup in dimension $n=1$, e^{tL} is the semigroup generated by L, $h(q,z)$ is defined by the right hand side of (2.14),

$$h(q,z) = q_y z_y(0) + \frac{1}{4}(q+1)(z_y(0))^2 + \frac{q+1}{2}\,\omega\Big(\frac{2q_y}{q+1} + z_y(0)\Big),$$

$g(q,z)$ is defined by (2.17), and $f(q,z)$ is defined by (2.16).

Theorem 2.12 *Let $q_0\in C^{2+\theta}(\mathbb{R})$ and $z_0\in Q(C^{2+\theta}_\omega(\mathbb{R}^2_-))$ be such that the compatibility condition*

$$\mathcal{B}z_0(y) = \left(1 + \left(\frac{2q_0'(y)}{q_0(y)+1} + \frac{\partial}{\partial y}z_0(0,y)\right)^2\right)^{-1/2} - 1 \tag{2.37}$$

holds.

There are r, $\rho > 0$ such that if

$$\|q_0\|_{C^{2+\theta}(\mathbb{R})} + \|z_0\|_{C^{2+\theta}_\omega(\mathbb{R}^2_-)} \le r$$

then problem (2.14,2.15) with initial data (q_0, z_0) has a unique solution in the ball $B(0,\rho) \subset \mathcal{X}$. There is $K > 0$, independent of q_0 and z_0, such that the solution (q,z) satisfies

$$\|(q,z)\|_{\mathcal{X}} \le K(\|q_0\|_{C^{2+\theta}(\mathbb{R})} + \|z_0\|_{C^{2+\theta}_\omega(\mathbb{R}^2_-)}).$$

Proof — The initial value problem for (2.14,2.15) is rewritten in the equivalent form $(q,z) = \Gamma(q,z)$, Γ being the nonlinear operator defined above.

It is important to understand why Γ maps (a small ball of) $\mathcal{X}$ into itself. Due to the behavior of g near $q = 0$, $z = 0$, we get

$$|h(q,z)(t,y)| \le C(|q_y(t,y) z_y(t,0,y)| + |z_y(t,0,y)|^2 + |q_y(t,y)|^4) \le \frac{C}{1+t^{3/2}} \|(q,z)\|^2_{\mathcal{X}},$$

$$|\frac{\partial}{\partial y} h(q,z)(t,y)| \le \frac{C}{1+t^2} \|(q,z)\|^2_{\mathcal{X}}.$$

Therefore the function $\varphi(t,y) = g(q,z)(t,y)$ satisfies the assumptions of theorem 2.6, with $\alpha = 1/2$. Theorem 2.6 guarantees that $\tilde{q}$ is in the right space $\mathcal{X}_q$, provided $q_0 \in C^{2+\theta}(\mathbb{R})$. In addition,

$$\|\tilde{q}\|_{\mathcal{X}_q} \le C(\|q_0\|_{C^{2+\theta}(\mathbb{R})} + \|(q,z)\|^2_{\mathcal{X}}). \tag{2.38}$$

Moreover,

$$|g(q,z)(t,y)| \le C(|q_y(t,y)|^2 + |z_y(t,0,y)|^2) \le \frac{C}{1+t} \|(q,z)\|^2_{\mathcal{X}},$$

and using the inequality

$$\left| \frac{1}{(1+a^2)^{1/2}} - \frac{1}{(1+b^2)^{1/2}} \right| \le |a-b|\,|a+b|$$

we get

$$|g(q,z)(t,y) - g(q,z)(s,y)| \le \left(\left| \frac{q_y(t,y)}{2(q(t,y)+1)} - \frac{q_y(s,y)}{2(q(s,y)+1)} \right| + |z_y(t,0,y) - z_y(s,0,y)| \right)$$

$$\cdot \left| \frac{q_y(t,y)}{2(q(t,y)+1)} + \frac{q_y(s,y)}{2(q(s,y)+1)} + z_y(t,0,y) + z_y(s,0,y) \right|$$

so that

$$\sup_{t>s\ge 0} (1+s) \frac{\|g(q,z)(t,\cdot) - g(q,z)(s,\cdot))\|_{C^0(\mathbb{R})}}{(t-s)^{(1+\theta)/2}} \le C\|(q,z)\|^2_{\mathcal{X}}.$$

Therefore, the function

$$\zeta(t,y) = g(q,z)(t,y), \quad t \ge 0,\ y \in \mathbb{R},$$

satisfies the assumptions of proposition 2.10.

Concerning the nonlinear function f, we get easily

$$\sup_{t\ge 0} (1+t) \|f(q,z)(t,\cdot)\|_{C^\theta_\omega(\mathbb{R}^2_-)} \le C\|(q,z)\|^2_{\mathcal{X}}.$$

Therefore the function

$$\psi(t,x,y) = f(q,z)(t,x,y) + e^x g(q,z)(t,y), \quad t \ge 0,\ x \le 0,\ y \in \mathbb{R},$$

satisfies the assumptions of proposition 2.8. The initial datum z_0 is in $Q(C^{2+\theta}_\omega(\mathbb{R}^2_-))$ and the compatibility condition

$$\mathcal{B}z_0 = \zeta(0,y), \quad y \in \mathbb{R}$$

is satisfied thanks to (2.37). We may apply theorem 2.11 which gives $\tilde{z} \in \mathcal{X}_z$ and

$$\|\tilde{z}\|_{\mathcal{X}_z} \leq C(\|z_0\|_{C^{2+\theta}_\omega(\mathbb{R}^2_-)} + \|(q,z)\|^2_{\mathcal{X}}). \tag{2.39}$$

Using (2.38) and (2.39) it follows that if

$$C\rho \leq 1/2$$

and

$$\|q_0\|_{C^{2+\theta}(\mathbb{R})} + \|z_0\|_{C^{2+\theta}_\omega(\mathbb{R}^2_-)} \leq r = \frac{\rho}{2C}$$

then Γ maps the ball $B(0,\rho) \subset \mathcal{X}$ into itself. Similar arguments show that if $C\rho \leq 1/2$ then Γ is a 1/2-contraction. Therefore it has a unique fixed point (q,z) in the ball, which satisfies

$$\|(q,z)\|_{\mathcal{X}} \leq 2C(\|q_0\|_{C^{2+\theta}(\mathbb{R})} + \|z_0\|_{C^{2+\theta}_\omega(\mathbb{R}^2_-)}). \ \blacksquare$$

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