Some Aspects of The Thin Film Equation

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Abstract.

1. Introduction

In this lecture I will discuss some aspects of nonnegative solutions $h(x,t)$ of the Thin Film Equation

$$h_t + (h^n h_{xxx})_x = 0\,. \tag{TFE}$$

Here x is the (one-dimensional) space variable, t is time and $n > 0$ is a parameter. The emphasis will be on self-similar solutions and dynamical systems methods for the resulting ordinary differential equations (ODE's).

Equation (TFE) arises in numerous applications of which I mention the spreading of a liquid film along a solid surface ($n = 3$) and Hele-Shaw flows ($n = 1$), see e.g. [21, 13, 20, 49, 5, 28, 30, 34, 52]. The most common derivation of (TFE) is as a lubrication approximation of the Navier-Stokes equations for thin film viscous flows, $h(x,t)$ being the film height, driven by surface tension and occurs with various values of $1 \le n \le 3$.

From a mathematical point of view it is natural to consider (TFE) with n as a parameter. At first sight it looks like a nonlinear degenerate version of the linear parabolic fourth order equation $h_t + h_{xxxx} = 0$. Another perspective is to see (TFE) as a fourth order version of the classical Porous Medium Equation, see e.g. [6],

$$h_t = (h^n h_x)_x\,, \tag{PME}$$

which may be considered as THE prototype second order nonlinear degenerate diffusion equation. It derives its name from applications to flows in porous media and goes back as far as [48]. In applications $h(x,t)$ is typically a density but it also occurs as the height of a groundwater mound, see [9].

Due to their homogeneous nature, both (TFE) and (PME) allow self-similar solutions of the form

$$h(x,t) = t^{-\alpha} f(\eta), \quad \eta = xt^{-\beta}\,, \tag{1}$$

where the similarity exponents respectively have to satisfy $n\alpha + 4\beta = 1$ (TFE) and $n\alpha + 2\beta = 1$ (PME). Such self-similar solutions are the guideline for the general PDE theory of the full partial differential equation (PDE). I will discuss some of these self-similar solutions of (TFE), taking (PME) as a, as we shall see, somewhat deceptive perspective.

2. Source-Type Solutions

Let us first consider (PME) and (TFE) on the whole line. Both equations can be written as the conservation law $h_t + q_x = 0$. Assuming the flux q to tend to zero as x approaches the boundary of the support of h, one expects conservation of mass: the spatial integral $\int h(x,t)dx$ (the mass) is constant in time. For (1) this implies $\alpha = \beta$. For (PME) this gives rise to the famous Zel'dovich-Kompaneetz-Barenblatt (ZKB) solutions given by

$$\alpha = \beta = \frac{1}{n+2}, \quad f(\eta) = \left\{C - \frac{n}{2(n+2)}\eta^2\right\}_+^{\frac{1}{n}}, \tag{2}$$

which feature an important property of (PME). The ZKB-solutions are nonclassical weak solutions with a propagating compact support. As $t \downarrow 0$, they converge, in the sense of distributions, to a multiple $M\delta$ of the Dirac measure δ, where $M = \int f(\eta)d\eta$. Any compactly supported nonnegative weak solution has intermediate asymptotics described by a ZKB-solution: transforming to self-similar variables the ZKB-solutions become equilibria attracting all nonnegative solutions with the same mass. In establishing this behaviour the comparison principle for second order diffusion equations, which says that a pointwise ordering of two initial data is carried over to the corresponding solutions of the initial value problem, is of great help. It allows to control the solutions and their supports using the ZKB-solutions as comparison functions.

The boundary of the support, usually called the interface, has been shown to have interesting properties. It may be stagnant for a while, a phenomenon observed in [47] from an analysis of self-similar solutions, but then, after a finite waiting time [7] it starts to move with a positive speed and is real analytic in time [1].

This discussion of (PME) in a nuttshell applies to all values of $n > 0$. Note that as $n \to 0$, the ZKB-solutions converge to multiples of the heat kernel, i.e. the fundamental solution of the linear heat equation $h_t = h_{xx}$. However, if (PME) is written in terms of the (pressure) variable $v = \frac{1}{n}h^n$ as $v_t = nvv_{xx} + v_x^2$, a very different limiting behaviour occurs, see again [6].

Looking at source-type solutions of (TFE), one arrives at

$$\alpha = \beta = \frac{1}{n+4}, \quad f^n f''' = \frac{\eta f}{n+4}, \tag{3}$$

which is to be compared with

$$\alpha = \beta = \frac{1}{n+2}, \quad f^n f' + \frac{\eta f}{n+2} = 0,$$

leading to (2) for (PME). In other words, where the source-type solutions of (PME) appear as solutions of a simple first order ordinary differential equation (ODE), source-type solutions have to be found from a much more difficult third order ODE.

In view of the physical background the solution f has to be nonnegative and the analogy with (PME) suggest that f should have compact support. At the free boundary of the support obviously $f = 0$. A second physically appropriate condition is that also $f' = 0$ (zero contact angle).

This problem may be solved using Green's functions and a regularisation, see [32], but it was first studied in [17] for $n > 0$ using a shooting argument. The third order ODE in (3) is solved with initial data

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = -\mu < 0\,. \tag{4}$$

Varying μ one expects to find a solution f touching zero (i.e. $f = f' = 0$) at some $\eta = a > 0$ as a borderline case between solutions that have a positive minimum before they can reach zero and solutions that hit zero with a negative slope. Having found this borderline case one then scales the solution f and obtains a source-type solution for any given mass $M > 0$. Surprisingly this method fails for $n \geq 3$ because the solutions with (4) always turn around and never hit zero. Thus source-type solutions can exist only for $n < 3$ and their existence is established along the lines indicated: there is one unique μ for which $f(\eta)$ decreases to touch zero.

Thus $n = 3$ is a critical value. One can show [33] that as $n \to 3$ from below, the source-type solutions with fixed mass M, as functions of x and t, converge to a Dirac mass which is just sitting there in $x = 0$. Apparently, (TFE) does not have moving interfaces and does not behave at all like (PME) when $n \geq 3$. Since $n = 3$ is also the parameter value most common in physical situations, there have been efforts to modify the derivation of the corresponding mathematical models, see [23] and [10].

For $0 < n < 3$ one may still have some validity of the analogy between (TFE) and (PME) but in fact this is only so for $\frac{3}{2} < n < 3$ as the local behaviour of the source-type profile f near the interface indicates. As $\eta \to a$ one has that $f(\eta) \sim A(a-\eta)^b$ with $b = 2$ for $0 < n < \frac{3}{2}$ and $b = \frac{3}{n}$ for $\frac{3}{2} < n < 3$. Note that (2) implies $b = \frac{1}{n}$ for (PME) and there is no way to see $0 < n < \frac{3}{2}$ for (TFE) as being reminiscent of that. I will come back to this in the discussion of self-similar solutions with general α and β.

Another surprise is that, as n goes to and crosses zero, there is no real change as far as source-type solutions are concerned. Source-type solution with compact support and quadratic behaviour at the interface exist throughout the range $-4 < n < \frac{3}{2}$ [19], in sharp contrast to the PME case, where a sharp transition from compact support to algebraic decay occurs as n crosses zero.

3. Strong Solutions

The discussion in section 2 calls for some comments concerning the PDE theory for (TFE). Source-type solutions of (TFE) cannot converge to source-type solutions of the linear equation $h_t + h_{xxxx} = 0$ as $n \to 0$. This linear fourth order parabolic equation has a fundamental solution of the form

$$E(x,t) = t^{-\frac{1}{4}} f(xt^{-\frac{1}{4}})\,,$$

where f is the Fourier inverse of $\exp(-k^4)$ and has oscillating tails. Thus the source-type solutions of (TFE) cannot be obtained by perturbing from $n = 0$ and the linear equation and therefore one cannot expect a PDE theory for (TFE) which starts from the theory for the linear equation. In this context we mention the equation $h_t + (|h|^n h_x)_{xxx} = 0$, see [16], which has compactly supported source-type solutions with infinitely many oscillations near the free boundary. This equation is in fact a natural nonlinear version of $h_t + h_{xxxx} = 0$ but is much less relevant for applications.

So what kind of PDE theory can one expect for (TFE)? The first results have been obtained in [15] and [11]. Since the equation is degenerate at $h = 0$ a weak solution approach is necessary. Multiplying by a testfunction, one can integrate by parts once, but a second integration by parts runs into complications because the third order derivatives are on the inside and not on the outside of the expression to be dealt with. Therefore the commonly used definition uses only one integration, putting only one space derivative on the testfunction, and restricts this integral to the positivity set. As a consequence, this concept of solution is too weak to guarantee uniqueness of solutions with prescribed sufficiently smooth nonnegative initial data. Moreover, it does not imply a weak form of the zero-contact angle condition on the boundary of the support. Zero-contact angle weak solutions are called strong solutions in most of the papers on (TFE) but there is no natural PDE formulation of this concept.

For $n = 0$ the source-type solutions in section 2 are positive on the interior of their support and satisfy $u = u_x = u_{xxx} = 0$ at the interface. This strongly suggests that strong solutions of (TFE) converge to solutions of a free boundary problem for the linear equation with these three conditions at the free boundary. This limit free boundary problem is discussed in [19] and remains to be solved in a PDE setting. It also suggests that strong solutions of (TFE) have to be considered as solutions to a free boundary problem for (TFE) with free boundary conditions $u = u_x = u^n u_{xxx} = 0$.

It is hard to obtain strong nonnegative solutions. For $0 < n < 3$ they are constructed as limits of solutions of approximating problems with very carefully modified initial data and h^n replaced by $D(h)$ with $D(h) \sim h^4$ as $h \to 0$. The approximating solutions in turn are constructed using a regularisation of the equation $h_t + (D(h)h_{xxx})_x = 0$. The nonnegativity of the approximating solutions follows from a nonnegativity property for solutions of (TFE) established in [15] for $n \geq 4$. Precise information about interface behaviour of solutions obtained through different regularisations of (TFE) has also been obtained using formal asymptotics in [25].

There is no comparison principle for strong solutions so it is not possible to use the source-type solutions as comparison functions and thereby control the solution and its free boundary. Nevertheless, some statements derived through such incorrect comparison arguments are essentially correct for strong solutions obtained through some approximation. For instance, as a first step towards a decription of the intermediate asymptotics, the support of a strong solution obtained in a suitable manner ($0 < n < 3$) is indeed controled by $C(1+t)^{\frac{1}{n+4}}$ where C is a constant depending only on the mass of the initial data, see [14] and [43]. These are a posteriori results valid for the constructed strong solutions only. Unfortunately uniqueness of strong solutions is not known.

4. Dipole Solutions for $n = 1$

Equation (PME) has a second class of explicit self-similar solutions, given by

$$\alpha = 2\beta = \frac{1}{n+1}, \quad f(\eta) = \eta^{\frac{1}{n+1}} \{C - \frac{n}{2(n+2)}\eta^{\frac{n+2}{n+1}}\}_+^{\frac{1}{n}} . \tag{5}$$

These are the dipole profiles. The corresponding dipole similarity solutions of (PME) are nonnegative on the half-line $\{x > 0\}$, zero on the lateral boundary $x = 0$ and conserve the first moment. These dipole solutions describe the intermediate asymptotics of general compactly supported solutions with zero boundary values in $x = 0$ in exactly the same fashion as the ZKB-solutions describe the intermediate asymptotics of compactly supported solutions living on the whole line. In the afterword of his book [9] Barenblatt poses the question as to whether such dipole solutions exist for (TFE). If dipoles are understood to be of fixed first moment ($\alpha = 2\beta$) it follows that $n = 1$ because only then there is the appropriate conservation law, reading

$$(xh)_t = -(xhh_{xxx})_x + (hh_{xx})_x - \frac{1}{2}(h_x^2)_x . \tag{6}$$

Thus

$$n = 1, \quad \alpha = \frac{1}{3}, \quad \beta = \frac{1}{6} , \tag{7}$$

and the equation for the profile f comes out as

$$f''' = \frac{f''}{\eta} - \frac{f'^2}{2\eta f} + \frac{\eta}{6} . \tag{8}$$

The ODE (8) looks more difficult to analyse than the ODE in (3). Also it is not clear a priori what the boundary conditions at $\eta = 0$ should be but as we shall see (8) there is a natural choice which follows from the analysis sketched below, see [18] for details. Because of its scaling invariance, the third order non-autonomous ODE may be reduced to a third order autonomous system. Introducing

$$x = \frac{\eta f'}{f}; \quad y = \frac{\eta^2 f''}{f}; \quad z = \frac{\eta^4}{6f}; \quad s = \log \eta , \tag{9}$$

the new unknowns x, y, z must solve

$$\dot{x} = x(1-x) + y; \quad \dot{y} = y(3-x) - \frac{x^2}{2} + z; \quad \dot{z} = z(4-x)\,, \tag{10}$$

where dots denote differentiation with respect to s. The dipole profile will now be identified as corresponding to a unique intersection of a two-dimensional unstable manifold of a finite critical point and the two-dimensional stable manifold of a curve of critical points at infinity.

There are four finite critical points. The dimensions of their unstable manifolds and the characterisation of solutions $f(\eta)$ near $\eta = 0$ corresponding to orbits in these manifolds are listed below. Note that the x-value of the critical point corresponds to the exponent of $f(\eta)$ and $s \to -\infty$ to $\eta = 0$.

$$\dim W^u(0,0,0) = 3\ (f(0) > 0); \qquad \dim W^u(\frac{3}{2}, \frac{3}{4}, 0) = 2 \quad (f(\eta) \sim A\eta^{\frac{3}{2}})\,; \tag{11}$$

$$\dim W^u(2,2,0) = 1\ (f(\eta) \sim A\eta^2); \quad \dim W^u(4,12,20) = 0\,. \tag{12}$$

Since the dipole solution is expected to have compact support, its corresponding orbit does not exist globally in forward s-time so it must escape to infinity where the flow may be analysed using a suitable version of the classical Poincaré transformation, see also [12]. The appropriate transformation reads

$$x = \frac{X}{V}, \quad y = \frac{Y}{V^2}, \quad z = \frac{Z}{V^2}; \quad X^2 + Y^2 + 2Z = 1; \quad Z \geq 0\,, \tag{13}$$

$$(X^2 + 2Y^2 + 2Z)V\frac{d}{ds} = (1 + Y^2)V\frac{d}{ds} = \frac{d}{d\tau}\,. \tag{14}$$

Thus $\{(x, y, z) : z \geq 0\}$ corresponds to $\{(X, Y, V) : X^2 + Y^2 \leq 1,\, V > 0\}$ where the flow is given by

$$\frac{dX}{d\tau} = \frac{(X^2 - 2Y)(X^2 - Y^2 - 1)}{2} + X(X^2 - 1 + \frac{Y}{2}(2X^2 + Y^2 - 1))V\,; \tag{15}$$

$$\frac{dY}{d\tau} = XY(X^2 - 2Y) + \frac{(Y^2 + 2Y - 1)(2X^2 + Y^2 - 1)V}{2}\,; \tag{16}$$

$$\frac{dV}{d\tau} = \frac{X(X^2 + (Y-1)^2)V}{2} + (X^2 - Y^2 - 2 + Y(X^2 + \frac{Y^2 - 1}{2})V^2\,. \tag{17}$$

The critical points at infinity ($V = 0$) are

$$(X, Y) = (\pm 1, 0) \text{ and the parabola } X^2 = 2Y\,, \tag{18}$$

while in $V = 0$ the system is integrable with orbits

$$X^2 + (Y - C)^2 = C^2 + 1\,. \tag{19}$$

Dimensions of stable manifolds contained in $\{V > 0\}$ with corresponding local behaviour of $f(\eta)$ near the boundary $\eta = a > 0$ of the support of f are

$$\dim W^s(X = -1, Y = V = 0) = 3: \quad f(\eta) \sim A(a - \eta)\,; \tag{20}$$

$$\dim W^s(X < 0, X^2 = 2Y) = 2: \qquad f(\eta) \sim A(a - \eta)^2\,. \tag{21}$$

One can show that

$$W^u(\frac{3}{2},\frac{3}{4},0) \cap W^s(X<0, X^2=2Y) = \{\text{one single orbit}\}\,, \tag{22}$$

and this gives the only possible dipole profile, which turns out to satisfy

$$f(0) = f'(0) = f(a) = f'(a) = (ff''')(a) = 0\,. \tag{23}$$

This means zero contact angle on both sides and zero flux at $\eta = 0$. The corresponding dipole similarity solutions looses mass in $x = 0$. Translations in s correspond to scaling a and the first moment of f.

5. General Self-Similar Solutions

In the light of the outcome of the analysis for $n = 1$ it seems natural to expect (for other values of n) dipole-type self-similar solutions with zero contact angle in $x = 0$ and zero contact angle, zero flux at the free boundary to be characteristic for solutions of (TFE) on the half-line. Since there is no conservation law for the first momentum if $n \neq 1$ the values of α and β are unknown a priori and have to be found from the analysis of the full fourth order ODE

$$(f^n f''')' = \beta\eta f' + \alpha f\,. \tag{24}$$

Such exponents are called anomalous and the corresponding solutions are called similarity solutions of the second kind, see again [9], but also [44] and [8].

Equation (24) is the fourth order version of a similar second order ODE for (PME), which, together with its N-dimensional version has been completely analysed in [40] and [42] for all $n > -1$. In particular [42] contains all the relevant forwards and backwards self-similar solutions of the equation $h_t = \nabla\cdot(|h|^n\nabla h)$. The philosophy in the analysis is basically to find similarity exponents for which two one-dimension manifolds (separatrices) of a two-dimensional system coincide. For instance, the ZKB-solutions are contained in an orbit which is the one-dimensional fast unstable manifold of a finite critical point and coincides with the one-dimensional stable manifold of a critical point at infinity. The following transformation mimics the reduction of the ODE for self-similar solutions of (PME) and the reduction of (8) to (10). Let

$$x = \frac{\eta f'}{f};\quad y = \frac{\eta^2 f''}{f};\quad z = \frac{\eta^3 f'''}{f};\quad u = \eta^4 f^{-n};\quad t = \log\eta\,. \tag{25}$$

Then (24) transforms to

$$\begin{aligned}
\dot{x} &= x(1-x) + y;\\
\dot{y} &= y(2-x) + z;\\
\dot{z} &= z(3-(n+1)x) + u(\alpha+\beta x);\\
\dot{u} &= u(4-nx)\,,
\end{aligned}$$

and in this system all (forwards and backwards) self-similar solutions can be found as intersections of two-dimensional unstable manifolds of finite critical points and two-dimensional stable manifolds of critical points at infinity. This is a fairly extensive programme because even just the local analysis is involved. Besides five finite critical points there are also five critical points at infinity, as the transformation

$$x = \frac{X}{V}; \quad y = \frac{Y}{V^2}; \quad z = \frac{Z}{V^3}; \quad u = \frac{U}{V^3}; \quad X^2 + Y^2 + Z^2 + 2U = C\,,$$

see [27] reveals. This transformation gives a much clearer picture than the one used in [41]. There is no space here to list all the relevant dimensions of stable and unstable manifolds but I will briefly mention some results.

Dipole-type solutions with zero-contact angle in $x = 0$ can exist only for $n < 2$ because as n crosses $n = 2$ the dimension of the unstable manifold of the relevant finite critical point drops. In [27] these dipole-type solutions are found using a numerical shooting technique. The results are consistent with a formal perturbation argument giving the derivatives of α and β with respect to n in $n = 1$. As $n \to 2$ the ratio $\frac{\alpha}{\beta} \to 1$, which means conservation of mass. For all $0 < n < 3$ the ODE in (2) allows solutions supported on $[0, a]$ with zero flux in both 0 and a but zero contact-angle only in $x = a$. As $n \uparrow 2$, the dipole-type solution merge with these fixed mass solutions. A formal analysis of the boundary behaviour of solutions of (TFE) confirms that a zero-contact angle at $x = 0$ may be imposed only if $n < 2$.

The jump from exponent 2 to $\frac{3}{n}$ as n crosses $n = \frac{3}{2}$ is due to an exchange of dimensions of stable manifolds of two critical points at infinity, the relevant critical point for exponent $\frac{3}{n}$ disappearing into $\{u < 0\}$ as n is taken outside the range $\frac{3}{2} < n < 3$ ($\beta > 0$). Moreover, when n drops below $\frac{1}{2}$ the two-dimensional stable manifold of another critical point at infinity giving profiles f with exponent $\frac{3}{n+1}$ becomes two-dimensional. Such solutions are flatter at the free boundary than the zero flux, zero contact-angle solutions and taking the limit $n \to 0$ one has $f = f' = f'' = 0$ in $\eta = a > 0$. This suggests that there is another free boundary formulation for (TFE) for $n < \frac{1}{2}$ which imposes optimal flatness at the free boundary and leads to the variational obstacle problem for the linear equation in the limit $n \to 0$.

Other relevant second kind self-similar solutions to be found are solutions which describe rupture (i.e. the initially positive solution developes an interior zero and possibly a dead core. This is of course also and even more relevant in dimensions $N = 2$ and $N = 3$, just as the question of how dry spots disappear. This last issue has been extensively studied for (PME) in which case there is a radial family of second kind self-similar solution (the Aronson-Graveleau solutions) which describe this phenomenon for the radial solutions, see [2]. Recently [3] however symmetry breaking results have been obtained for this class of solutions using n as parameter. Moreover the linearised system obtained in [3] after an ingenious linearisation procedure may be analysed using transformations of the same nature as the one sketched in this lecture. In particular the two-dimensional system in which the Aronson-Graveleau solutions can be found has to be appended with only one first order equation. In [4] we use this to prove that the Aronson-Graveleau profiles are unstable for all $n > 0$. Obviously, the same problem for (TFE) poses a formidable task.

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