Existence of travelling wave solutions in a combustion-radiation model *

Claude-Michel Brauner
Université Bordeaux I, Mathématiques Appliquées de Bordeaux
33405 Talence Cedex, France
E-mail: brauner@math.u-bordeaux.fr

Josephus Hulshof
Faculty of Sciences, Mathematics and Computer Science division
Free University of Amsterdam, De Boelelaan 1081
1081 HV Amsterdam, The Netherlands
E-mail: jhulshof@cs.vu.nl

Jean-Francois Ripoll
Université Bordeaux I, Mathématiques Appliquées de Bordeaux
33405 Talence Cedex, France
E-mail: ripoll@math.u-bordeaux.fr
&
CEA-CESTA, BP 2, 33114 Le Barp Cedex, France
E-mail: ripoll@bordeaux.cea.fr

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Abstract

We consider a simple model of premixed flames propagating in a gaseous mixture containing inert dust. The radiation field is modelled by the classical Eddington equation. The main parameters are the dimensionless opacity and the Boltzmann number. We prove the existence of travelling solutions with increased speed w.r.t. the adiabatic case. Three singular limiting cases (including a modification involving an ignition temperature) of the parameter values are discussed.

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1 Introduction

We consider a simplified model of premixed flames, propagating in a gaseous mixture which contains inert dust, see Joulin et al. [10, 11, 2, 3], and Buckmaster & Jackson ([6]). The model is based on classical hypotheses in combustion with simple chemistry and high activation energy asymptotics.

The presence of particles in the mixture generates a radiation field approximated by the well known Eddington equation of radiative transfer [14, 17, 13, 12, 15, 9]. Depending on the opacity of the medium, radiations may significantly influence the flame profile.

To begin with, we write the unsteady system of equations in two space dimensions for the mass fraction Y, the temperature T and the radiative flux $\mathbf{q} = (q_1, q_2)$, the flame front being located at $x = s(y,t)$:

$$\frac{\partial}{\partial t}(\rho Y) - \nabla(\rho D \nabla Y) = 0, \quad x < s(y,t); \quad Y = 0, \quad x \geq s(y,t); \tag{1.1}$$

$$\frac{\partial}{\partial t}(\rho C_p T) - \lambda \Delta T + \nabla \cdot \mathbf{q} = 0, \quad x \neq s(y,t); \tag{1.2}$$

$$-L^2 \nabla(\nabla \cdot \mathbf{q}) + 3\mathbf{q} + 4\sigma L \nabla T^4 = 0, \quad x \neq s(y,t). \tag{1.3}$$

At the front, Y, T, $\mathbf{q}$ and the first order derivatives of $\mathbf{q}$ are continuous, while the following jump conditions for the normal derivatives hold:

$$[\frac{\partial Y}{\partial n}] = \frac{B}{\rho D} \exp(-\frac{E}{2RT}); \quad [\frac{\partial T}{\partial n}] = -\frac{QB}{\lambda} \exp(-\frac{E}{2RT}), \quad x = s(y,t). \tag{1.4}$$

Finally, when $x \to -\infty$, $Y \to Y_f$, $T \to T_f$, $\mathbf{q} \to 0$, and as $x \to \infty$, $T \to T_b$ (unknown), $\mathbf{q} \to 0$. As usual the subscripts f and b stand for "fresh" and "burnt".

The "combustion" constants in the problem are the diffusion constant D, the heat conduction coefficient λ, the specific heat C_p, the reaction heat Q, the pre-exponential factor B, the activation energy E and the perfect gas constant R. Concerning the density ρ, we adopt the constant density model rather than relating it to the temperature by a state law. The constants related to the radiative field are the Stefan-Boltzmann constant σ and the radiative mean free path L [13, 12, 16]. The optically thin limit (transparent medium) corresponds to $L \to \infty$ and the optically thick limit (opaque medium) to $L \to 0$.

In this paper we are interested in planar (one-dimensional) travelling wave solutions moving along the x-axis with constant velocity $-u$. Thus we introduce the travelling wave variable $z = x + ut$ and look at the following steady state system for Y, T and $q = q_1$:

$$\rho u \frac{dY}{dz} - \rho D \frac{d^2Y}{dz^2} = 0, \quad z < 0; \quad Y = 0, \quad z \geq 0; \tag{1.5}$$

$$\rho C_p u \frac{dT}{dz} - \lambda \frac{d^2T}{dz^2} + \frac{dq}{dz} = 0, \quad z \neq 0; \tag{1.6}$$

$$-L^2 \frac{d^2q}{dz^2} + 3q + 4\sigma L \frac{dT^4}{dz} = 0, \quad z \neq 0. \tag{1.7}$$

At the front, which is now located at $z = 0$, we have that Y, T, q and $\frac{dq}{dz}$ are continuous, with jump conditions on the prime derivatives

$$[\frac{dY}{dz}] = \frac{B}{\rho D} \exp(-\frac{E}{2RT}); \quad [\frac{dT}{dz}] = -\frac{QB}{\lambda} \exp(-\frac{E}{2RT}), \quad z = 0. \tag{1.8}$$

Finally

$$T(-\infty) = T_f, \quad Y(-\infty) = Y_f, \quad q(\pm\infty) = 0, \quad T(+\infty) = T_b. \tag{1.9}$$

We note that the condition for q is automatic from (1.7). Physically, it means that radiative equilibrium is achieved at infinity[1].

To determine T_b, which is not a constant we can choose freely, we first solve System (1.5),(1.6),(1.8) in the absence of radiative field. Thus we forget

[1] See also the discussion in [11, p. 1584].

equation (1.7) and drop the q-term in (1.6). This is equivalent to putting $L = 0$: in the optically thick limit, the radiative flux is 0. In such a case, the combustion process is purely *adiabatic*, see [18, p.543]. The temperature is then constant for $z > 0$ (see [7, chap. 2]), and given by elementary calculation. We will denote it by (u_a, Y_a, T_a), and refer herafter to it as the "adiabatic solution" :

$$\rho u_a = \frac{B}{Y_f} \exp(-\frac{E}{2RT_{ad}}) = \frac{QB}{C_p(T_{ad} - T_f)} \exp(-\frac{E}{2RT_{ad}});$$

$$Y_a(z) = Y_f - Y_f \exp(\frac{u_a z}{D}), \quad z < 0; \quad Y_a(z) = 0, \quad z \geq 0;$$

$$T_a(z) = T_f + (T_{ad} - T_f) \exp(\frac{\rho u_a C_p z}{\lambda}), \quad z < 0; \quad T_a(z) = T_{ad}, \quad z \geq 0,$$

where

$$T_{ad} = T_f + \frac{QY_f}{C_p} \tag{1.10}$$

is known as the "adiabatic flame temperature".

Returning to the full system (1.5)–(1.9), integration with respect to z yields, in view of $q(\pm\infty) = 0$,

$$\rho u = \frac{B}{Y_f} \exp(-\frac{E}{2RT(0)}) = \frac{QB}{C_p(T_b - T_f)} \exp(-\frac{E}{2RT(0)}),$$

and therefore we obtain by global energy conservation [3, p.221]

$$T_b = T_f + \frac{QY_f}{C_p}, \tag{1.11}$$

which is nothing but the adiabatic temperature T_{ad}, see (1.10). The flame is called "globally adiabatic" [3].

Next we make the system (1.5)–(1.9) dimensionless. As characteristic quantities we take

$$T_c = T_{ad}, \quad Y_c = Y_f, \quad u_c = u_a, \quad l_c = \frac{\lambda}{\rho u_a C_p}, \quad q_c = 4\sigma T_{ad}^4,$$

and set

$$\tilde{z} = \frac{z}{l_c}, \quad \theta = \frac{T}{T_c}, \quad \tilde{Y} = \frac{Y}{Y_c}, \quad \mu = \frac{u}{u_c}, \quad \tilde{q} = \frac{q}{q_c}.$$

Omitting the tildes we arrive at

$$\mu Y' - \frac{1}{\mathrm{Le}} Y'' = 0, \quad z < 0; \quad Y \equiv 0, \quad z \geq 0; \tag{1.12}$$

$$\mu\theta' - \theta'' + \beta q' = 0, \quad z \neq 0; \tag{1.13}$$

$$-q'' + 3\alpha^2 q + \alpha(\theta^4)' = 0, \quad z \neq 0; \tag{1.14}$$

with jump conditions

$$[Y] = [\theta] = [q] = [q'] = 0, \quad [Y'] = \delta \exp(-\frac{\mathrm{Ze}}{\theta}), \quad [\theta'] = -\gamma \exp(-\frac{\mathrm{Ze}}{\theta}), \tag{1.15}$$

at $z = 0$. The conditions at infinity are

$$\theta(-\infty) = \theta_-, \quad Y(-\infty) = 1, \quad \theta(+\infty) = 1, \quad q(\pm\infty) = 0. \tag{1.16}$$

Here

$$\mathrm{Ze} = \frac{E}{2RT_{ad}}, \quad \mathrm{Le} = \frac{\lambda}{\rho C_p D}, \quad \gamma = \frac{QBl_c}{\lambda T_{ad}}, \quad \delta = \frac{Bl_c}{\rho D Y_f}, \quad \theta_- = \frac{T_f}{T_{ad}}, \tag{1.17}$$

where Ze is a Zel'dovich number (dimensionless activation energy) and Le the Lewis number. Note that $\gamma = (1 - \theta_-)\exp(\mathrm{Ze})$, using (1.11) and (1.17).

Since Y decouples from the system, we are left with a problem for μ, θ and q only. The Y-part of the solution then follows, due to the obviously satisfied compatibility relation

$$1 - \theta_- = \frac{\gamma \mathrm{Le}}{\delta}.$$

The two main parameters are

$$\alpha = \frac{\lambda}{\rho u_a C_p L} \quad \text{and} \quad \beta = \frac{4\sigma T_{ad}^4}{\rho u_a C_p T_{ad}},$$

the dimensionless opacity (the inverse of the dimensionless mean free path) and the Boltzmann number.

In the dimensionless adiabatic case $\alpha = \infty$ we simply have

$$\mu_a = 1; \quad \theta_a(z) = \theta_- + (1 - \theta_-)\exp(z), \quad z < 0; \quad \theta(z) = 1, \quad z \geq 0. \tag{1.18}$$

In Section 2 we show that for every $\alpha, \beta > 0$ there exists a travelling wave profile with the property that $\theta(0) > \theta_a(0) = 1$ and hence $\mu > \mu_a = 1$. This enhancement of the front temperature and thereby the speed is well known in the literature, see e.g. [12] for a radiative shock. In the case of small β, it is referred to as the "Joulin effect" [6].

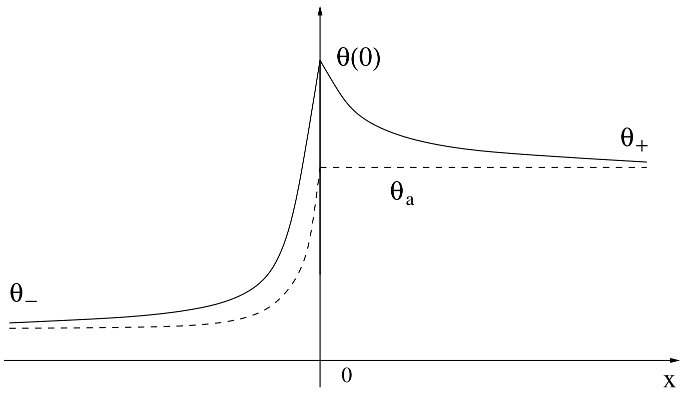

Figure 1: A typical temperature profile with radiative effects (solid line) vs. the adiabatic temperature solution (dotted line).

Our proof is based on using a fixed point argument to construct approximating solutions on a sequence of intervals $[-b, b]$. Taking the limit $b \to \infty$ along a suitable sequence we obtain the travelling wave. This strategy is reminiscent of the one in [1].

In Section 3 we rigorously establish three different physically interesting limit cases, namely (i) $\beta > 0$ fixed, $\alpha \to 0$ (the transparent case); (ii) $\alpha > 0$ fixed, $\beta \to \infty$ (large Boltzmann number); (iii) combined limits $\alpha \to 0$ and $\beta \to \infty$, $\alpha\beta$ fixed. Case (i) is a singular limit where one might expect to recover the adiabatic case but this turns out to be false. Case (ii) is also singular but here we recover the adiabatic speed $\mu = \mu_a$. The limits are rather subtle and inner and outer scales are needed to give a complete picture. The third case yields just one limiting equation for θ.

This paper is only a first step in the mathematical analysis of premixed flames propagating in gaseous mixtures with dust. A next step will be a rigorous stability analysis of the travelling waves along the lines of [5] and [4]. This is of course related to the uniqueness question. There may be travelling waves with different speeds for the same set of parameters. Some partial uniqueness results can be obtained from the limit situations considered in Section 3 but it is not at all clear for instance that the limit problem in case (iii) above has a unique solution. In Section 3.4 we consider case (iii) with a modified reaction rate involving an ignition temperature and exhibit the possibility of both extinction (i.e. no travelling wave solutions) and multiple solutions, see also [11, Fig. 3].

Finally, the Eddington equation itself is only a rough approximation in the modelling of radiative transfer: more realistic models have been suggested in [12, 8, 9, 16]. and they will be subject of further research.

2 The travelling wave system

2.1 Existence

In this section we formulate the results for the travelling wave problem. We write x instead of z, set $q = -\mu w$ and drop Y from the system. Then (1.12)–(1.16) have to be replaced by

$$\mu\theta' = \theta'' + \beta\mu w', \quad x \neq 0; \tag{2.1}$$

$$[\theta] = 0 \text{ and } [\theta'] = -F(\theta) = -\gamma \exp(-\frac{\text{Ze}}{\theta}) \text{ in } x = 0; \tag{2.2}$$

$$\theta(-\infty) = \theta_- \text{ and } \theta(+\infty) = \theta_+; \tag{2.3}$$

$$-w'' + 3\alpha^2 w = \frac{\alpha}{\mu}(\theta^4)'. \tag{2.4}$$

Here

$$\alpha > 0,\ \beta > 0,\ \gamma > 0,\ \text{Ze} > 0,\ 0 < \theta_- < \theta_+ \tag{2.5}$$

are given fixed parameters satisfying

$$\theta_+ = 1, \quad \gamma = (1 - \theta_-)\exp(\text{Ze}). \tag{2.6}$$

The unknowns are the travelling wave speed $\mu > 0$ and the functions θ and w.

The last equation (2.4) is to be understood in the distributional sense. Since θ has to be continuous, it follows that w' is continuous. In particular

$$[w] = 0 \text{ and } [w'] = 0 \text{ in } x = 0.$$

Note that we have only written the limiting conditions at infinity which are not prescribed by the differential equations. Any solution of Problem (2.1)–(2.4) automatically also has $w(\pm\infty) = 0$. Also, the restriction (2.6) follows only from the derivation of the model problem but is otherwise unnecessary and we shall not use it hereafter. This means that we now make the system (1.5)–(1.9) dimensionless w.r.t. general characteristic quantities.

We recall from the introduction that (2.6) implies that the adiabatic speed

$$\mu_a = \frac{F(\theta_+)}{\theta_+ - \theta_-} = \frac{\gamma}{\theta_+ - \theta_-} \exp(-\frac{\mathrm{Ze}}{\theta_+}) \tag{2.7}$$

is normalised to $\mu_a = 1$. Since we drop (2.6) the adiabatic solution is

$$\theta_a(x) = \theta_- + (\theta_+ - \theta_-)\exp(\mu_a x), \quad x < 0; \quad \theta_a(x) = \theta_+, \quad x \geq 0, \tag{2.8}$$

with μ_a given by(2.7).

Before stating the main result it is convenient to introduce the auxillary functions $\Theta(x)$ and $\theta_\pm(x)$ defined by

$$\Theta(x) = \theta_- + (\theta_+ - \theta_-)\exp(\mu x), \quad x < 0; \quad \Theta(x) = \theta_+, \quad x \geq 0, \tag{2.9}$$

and

$$\theta_\pm(x) = \theta_-, \quad x < 0; \quad \theta_\pm(x) = \theta_+, \quad x \geq 0, \tag{2.10}$$

Note that $\Theta(x)$ depends on μ so it is not the adiabatic solution $\theta_a(x)$, unless $\mu = \mu_a$.

Theorem 2.1 *If (2.5) is satisfied, then Problem (2.1)–(2.4) has a solution (μ, θ, w). This solution has the following properties:*

$$\mu(\theta - \theta_\pm) = \theta' + \beta\mu w, \quad x \neq 0; \tag{2.11}$$

$$\mu = \frac{F(\theta(0))}{\theta_+ - \theta_-} = \frac{\gamma}{\theta_+ - \theta_-}\exp(-\frac{\mathrm{Ze}}{\theta(0)}); \tag{2.12}$$

$$\int_{-\infty}^{+\infty} w(s)ds = \frac{1}{3\alpha\mu}(\theta_+^4 - \theta_-^4); \tag{2.13}$$

$$-w'' + (3\alpha^2 + 4\alpha\beta\theta^3)w = 4\alpha\theta^3(\theta - \theta_\pm); \tag{2.14}$$

$$\theta(x) = \Theta(x) + \beta\int_x^\infty \mu\exp(\mu(x-s))w(s)ds, \quad x \in I\!R; \tag{2.15}$$

$$0 < w(x) \leq \frac{\theta_+ - \theta_-}{\beta}, \quad x \in I\!R; \tag{2.16}$$

$$0 \leq \theta(x) - \Theta(x) \leq \theta_+ - \theta_-, \quad x \in I\!R; \tag{2.17}$$

$$\theta(x) - \Theta(x) \leq \frac{\beta}{3\alpha}(\theta_+^4 - \theta_-^4), \quad x \in I\!R; \tag{2.18}$$

$$\mu_a < \mu < \frac{\gamma}{\theta_+ - \theta_-} = \bar{\mu}. \tag{2.19}$$

Both θ and w are decreasing and convex on $[0,\infty)$ and $w(\pm\infty) = 0$. Equation (2.14) holds in the sense of distributions. This implies that it holds pointwise for $x \neq 0$ and that w' is continuous also in $x = 0$.

Theorem 2.1 will be proved in the subsections below: the properties in Section 2.2 and the existence in Section 2.3.

2.2 Proof of a priori properties

We first observe that the properties of the solution listed in Theorem 2.1 are a priori properties for solutions with $w \geq 0$. It is easy to see from (2.4) and (2.3) that $w(\pm\infty) = 0$. The first order ODE in (2.11) follows from integration of (2.1) upon which (2.12) follows from (2.2). The integral in (2.13) is immediate from (2.4). Combining (2.11) and (2.4) gives the second order ODE in (2.14).

Solving (2.11) we find (2.15) which gives θ and in particular the temperature at the front

$$\theta(0) = \theta_+ + \beta \int_0^\infty \mu \exp(-\mu s) w(s) ds, \tag{2.20}$$

in terms of w. Assuming $w \geq 0$ this implies $\theta \geq \Theta \geq \theta_\pm$. Consequently, since $\Theta \not\equiv \theta_\pm$ we have $w > 0$ everywhere from (2.14).

As for (2.16), assume that w has a maximum in x_0. Then integration of (2.4) gives

$$\int_{x_0}^{+\infty} w(s) ds = \frac{1}{3\alpha\mu}(\theta_+^4 - \theta(x_0)^4), \tag{2.21}$$

so that $\theta(x_0) < \theta_+$ whence $x_0 < 0$. Thus w and hence also θ are strictly decreasing on $[0,\infty)$. It is then easy to see from (2.1) and (2.4) that w and θ are convex on $[0,\infty)$. Furthermore (2.14) implies that

$$w(x_0) \leq \frac{4\alpha\theta(x_0)^3}{3\alpha^2 + 4\alpha\beta\theta(x_0)^3}(\theta(x_0) - \theta_\pm(x_0)) < \frac{\theta_+ - \theta_-}{\beta + \frac{3\alpha}{4\theta_+^3}} \leq \frac{\theta_+ - \theta_-}{\beta}. \tag{2.22}$$

Finally (2.17) follows from (2.15) and (2.16), (2.18) from (2.15), (2.13) and $w \geq 0$, and (2.19) from (2.12) and (2.17).

Remark 2.2 *In this paper, we are only interested in solutions such that* $\theta \geq \theta_\pm$*, the latter being equivalent to* $w > 0$ *via (2.14) – which plays a crucial role – and the maximum principle. Here we have taken advantage of the ellipticity of Eddington equation. Physically,* $w > 0$*, i.e.* $q < 0$*, means that only the radiative flux from the burnt region to the fresh one is significant.*

Remark 2.3 *From (2.17) we recover the classical estimate for the dimensionless temperature at the flame front,*

$$\theta(0) \leq 2\theta_+ - \theta_-, \tag{2.23}$$

see [12],[3]. This bound is sharp, see Remark 3.3 and Proposition 3.5.

2.3 The fixed point argument

As a result of Section 2.2, solving (2.1)–(2.4) amounts to solving the system (2.12),(2.14),(2.15) for μ, θ and w. This suggests to look for fixed points of the mapping $T : (\mu, w) \to (M, W)$ defined by the iterative scheme

$$\theta(x) = \Theta(x) + \beta \int_x^\infty \mu \exp(\mu(x-s)) w(s) ds; \tag{2.24}$$

$$M = \frac{F(\theta(0))}{\theta_+ - \theta_-} = \frac{\gamma}{\theta_+ - \theta_-} \exp(-\frac{\mathrm{Ze}}{\theta(0)}); \tag{2.25}$$

$$-W'' + (3\alpha^2 + 4\alpha\beta\theta^3)W = 4\alpha\theta^3(\theta - \theta_\pm). \tag{2.26}$$

Since Schauder's Fixed Point Theorem is not directly applicable to T, we approximate the problem on $I\!R$ by a family of problems on $[-b, b]$ with $b \to \infty$. Thus we will apply the Schauder Fixed Point Theorem to suitably defined mappings T_b defined on convex sets with bounds depending on b. The fixed points thus obtained will satisfy bounds uniform in b and this will allow to take the limit $b \to \infty$ along a subsequence.

We fix $b > 0$ and replace the scheme (2.24),(2.25),(2.26) by

$$\theta(x) = \Theta(x) + \beta \int_x^b \mu \exp(\mu(x-s)) w(s) ds; \tag{2.27}$$

$$M = \frac{F(\theta(0))}{\theta_+ - \theta_-} = \frac{\gamma}{\theta_+ - \theta_-} \exp(-\frac{\mathrm{Ze}}{\theta(0)}); \tag{2.28}$$

$$-W'' + (3\alpha^2 + 4\alpha\beta\theta^3)W = 4\alpha\theta^3(\theta - \theta_\pm) \quad \text{on} \quad [-b, b], \quad W'(\pm b) = 0. \tag{2.29}$$

Thus the only changes are that the upper bound ∞ in the integral is replaced by b and the equation for W is solved on $[-b, b]$ with Neumann boundary conditions.

The new scheme defines a mapping

$$T_b : (\mu, w) \to (M, W). \tag{2.30}$$

Note that (2.27) is equivalent to

$$\mu(\theta - \theta_\pm) = \theta' + \beta\mu w \quad \text{on} \quad [-b, b], \quad \theta(b) = \theta_+, \tag{2.31}$$

with $\theta(-b) \geq \theta_-$, so that for a fixed point of T_b we also have

$$-w'' + 3\alpha^2 w = \frac{\alpha}{\mu}(\theta^4)' \quad \text{on} \quad [-b, b], \quad w'(\pm b) = 0. \tag{2.32}$$

Proposition 2.4 *Let $\mu > 0$ and $w \in X_b = C([-b, b])$, the space of continuous functions on $[-b, b]$, endowed with the maximum norm. Defining θ by (2.27), equation (2.29) has a unique solution W in X_b. If $w \geq 0$, then $\theta \geq \Theta$, $W > 0$,*

$$\begin{aligned} 0 \leq \theta(x) - \Theta(x) &\leq \beta(\max_{x \leq s \leq b} w(s)) \int_x^b \mu \exp(\mu(x - s))ds \\ &\leq (1 - \exp(-2\mu b))\beta||w||_\infty, \end{aligned} \tag{2.33}$$

whence, for some $x_0 \in [-b, b]$,

$$\begin{aligned} ||W||_\infty = W(x_0) &\leq \frac{4\alpha\theta(x_0)^3}{3\alpha^2 + 4\alpha\beta\theta(x_0)^3}(\theta(x_0) - \theta_\pm(x_0)) \\ &\leq (1 - \exp(-2\mu b))||w||_\infty + \frac{\theta_+ - \theta_-}{\beta}. \end{aligned} \tag{2.34}$$

Proof Estimate (2.33) is immediate from (2.27). The estimate (2.34) follows from the classical maximum principle and (2.33), using also that $\theta(x_0) - \theta_\pm(x_0) = \theta(x_0) - \Theta(x_0) + \Theta(x_0) - \theta_\pm(x_0)$. Here we note that if $x_0 = 0$, the estimate may be read with $\theta_\pm(x_0)$ replaced by either θ_- or θ_+. ∎

Corrollary 2.5 *The constant in front of the norm in (2.34) is sufficient to get an invariant set. Clearly M defined by (2.28) satisfies*

$$\mu_a = \frac{\gamma}{\theta_+ - \theta_-} \exp(-\frac{\mathrm{Ze}}{\theta_+}) \leq M \leq \frac{\gamma}{\theta_+ - \theta_-} = \bar{\mu}. \tag{2.35}$$

Let $B = \exp(2\bar{\mu}b)(\theta_+ - \theta_-)/\beta$. Then $w \in X_{b,B} = \{w \in X_b : \ 0 \leq w \leq B\}$ and $\mu \in [\mu_a, \bar{\mu}]$ imply $W \in X_{b,B}$. Moreover,

$$0 \leq \theta(x) - \Theta(x) \leq (\theta_+ - \theta_-)(\exp(2\bar{\mu}b) - 1),$$

so that also $|W''| \leq C$ on $[-b, b]$ and $|W'| \leq 2bC$ for some constant C depending on all the parameters, including b. Thus the mapping T_b defined by (2.30) maps $[\mu_a, \bar{\mu}] \times X_{b,B}$ to a precompact subset of itself. Hence T_b has a fixed point.

Remark 2.6 *The point $(\mu, \theta, w) = (\mu_b, \theta_b, w_b)$, in which (μ_b, w_b) is the fixed point of T_b and $\theta = \theta_b$ is defined by (2.27), has properties similar to the a priori properties listed in Section 2.2. By the same arguments we have $w \geq 0$ and*

$$\int_{-b}^{b} w(s)ds = \frac{1}{3\alpha\mu}(\theta_+^4 - \theta(-b)^4) \leq \frac{1}{3\alpha\mu}(\theta_+^4 - \theta_-^4),$$

$$0 \leq \theta(x) - \Theta(x) \leq \beta\mu \int_x^b w(s)ds \leq \frac{\beta}{3\alpha}(\theta_+^4 - \theta_-^4). \tag{2.36}$$

If w' has a zero in x_0, then

$$\theta(x_0) \leq \theta_+.$$

In particular, if w has its (global) maximum in x_0, then $x_0 < 0$,

$$||w||_\infty = w(x_0) \leq \frac{4\alpha\theta_+^3}{3\alpha^2 + 4\alpha\beta\theta_+^3}(\theta_+ - \theta_\pm(x_0)) \leq \frac{\theta_+ - \theta_-}{\beta + \frac{3\alpha}{4\theta_+^3}} \leq \frac{\theta_+ - \theta_-}{\beta}. \tag{2.37}$$

Also,

$$0 \leq \theta(x) - \Theta(x) \leq \frac{\beta}{\beta + \frac{3\alpha}{4\theta_+^3}}(\theta_+ - \theta_-) \leq \theta_+ - \theta_-. \tag{2.38}$$

It is now easy to construct a solution of (2.15), (2.12) and (2.14) by standard limit arguments. In particular we use that the solutions $(\mu, \theta, w) = (\mu_b, \theta_b, w_b)$ satisfy, in view of (2.38) and (2.31),

$$\Theta(x) \leq \theta_b(x) \leq 2\theta_+; \quad -\mu(\theta_+ - \theta_-) \leq \theta_b'(x) \leq 2\mu\theta_+,$$

for all $x \in [-b, b]$. Because of (2.37) we can now conclude from (2.32) that w_b' is uniformly bounded by a constant depending only on θ_+, α and β. Thus we can extract a subsequence $b \to \infty$ such that, in addition to the convergence of μ_b, the functions θ_b and w_b converge to limits θ and w defined on the whole real line, locally uniform. Taking the limits in (2.27), (2.28) and (2.29) with $M = \mu_b$ and $w = W = w_b$, we obtain (2.24), (2.25) and (2.26) with $M = \mu$ and $W = w$, μ and w being the limits. The convergence of w_b to w is in $W^{2,\infty}_{loc}(I\!R)$.

3 Limit cases

In this section we examine the singular limit cases. However, we begin by a few words about the limits $\alpha \to \infty$ (optically thick limit, opaque medium) and $\beta \to 0$ (radiative flux negligible vs. convective flux) which are straightforward. In both cases we easily see from our a priori properties that $\theta \to \theta_a$ and $\mu \to \mu_a$. However, the limits for w are different. In the first case, the limit for w is zero, as can be seen dividing by α^2 in (2.14) : one fully recovers the adiabatic case. In the second case the limit for w is simply the solution of (2.14) with $\beta = 0$.

3.1 Transparent limit: $\alpha \to 0$

For fixed β we consider a family of solutions $(\mu, \theta, w) = (\mu_\alpha, \theta_\alpha, w_\alpha)$ of (2.1),(2.2),(2.3),(2.4) and examine the limit $\alpha \to 0$. Assuming that the limit (μ_0, θ_0, w_0) exists and setting $\alpha = 0$ in (2.1),(2.2),(2.3),(2.4), it looks like (μ_0, θ_0, w_0) should simply be the adiabatic solutions (μ_a, θ_a) with $w \equiv 0$. This is not true and a more subtle analysis is needed in which we have to consider two scales. The result is

Proposition 3.1 *For $\beta > 0$ fixed there is a constant $C_0 > 0$ such that, along some sequence $\alpha \to 0$, the solutions $(\mu, \theta, w) = (\mu_\alpha, \theta_\alpha, w_\alpha)$ satisfy*

$$\mu \to \mu_0, \quad w \to C_0, \quad \theta \to \Theta + \beta C_0,$$

locally uniform in x, where Θ is defined by (2.9) with

$$\mu = \mu_0 = \frac{F(\theta_+ + \beta C_0)}{\theta_+ - \theta_-} = \frac{\gamma}{\theta_+ - \theta_-} \exp(-\frac{\mathrm{Ze}}{\theta_+ + \beta C_0}).$$

Proof We have that the solutions are controlled by the properties listed in Theorem 2.1, in which the α-dependence is explicit. Clearly, with $\beta > 0$ fixed and $\alpha \to 0$, the functions w, w'', θ and θ' (and also the speed μ) are uniformly bounded. Using the ODE's also w''' and θ'' are uniformly bounded on both sides of $x = 0$. Thus we may pass to the limit along a sequence $\alpha \to 0$ and obtain a limit (μ_0, θ_0, w_0). The convergence of w, w', θ is locally uniform on $(-\infty, \infty)$. The convergence of w'' and θ' is locally uniform on both $(-\infty, 0]$ and $[0, \infty)$. Thus we know for the limit that $w_0' \equiv 0$, so that w_0 is a nonnegative constant. In view of (2.13) we may expect the constant to be nonzero. In any case we have that

$$w_0 \equiv C_0,\ \theta_0 = \Theta + \beta C_0,\ \mu = \mu_0 = \frac{F(\theta_+ + \beta C_0)}{\theta_+ - \theta_-}. \tag{3.1}$$

To determine the constant C_0 we need another spatial scale. We set

$$\hat{x} = \alpha x, \quad \hat{\theta}(\hat{x}) = \theta(x), \quad \hat{w}(\hat{x}) = w(x),$$

and obtain, the primes now denoting differentiation with respect to $\hat{x}$,

$$\mu(\hat{\theta} - \theta_\pm) = \alpha\hat{\theta}' + \beta\mu\hat{w}; \tag{3.2}$$

$$-\hat{w}'' + 3\hat{w} = \frac{1}{\mu}(\hat{\theta}^4)'. \tag{3.3}$$

The difficulties in taking the limit $\alpha \to 0$ come from the nonlinearity on the right hand side of (3.3).

For $\hat{w}$ and $\hat{\theta}$ we have from (2.13) and (3.2) the L^1-estimates

$$\int_{-\infty}^{+\infty} \hat{w}(\hat{x})d\hat{x} = \frac{1}{3\mu}(\theta_+^4 - \theta_-^4); \tag{3.4}$$

$$\mu\int_{-\infty}^{+\infty} (\hat{\theta}(\hat{x}) - \theta_\pm(\hat{x}))d\hat{x} = \alpha(\theta_+ - \theta_-) + \frac{\beta}{3}(\theta_+^4 - \theta_-^4). \tag{3.5}$$

In view of the uniform bounds on $\hat{w}$ and $\hat{\theta}$ the L^2-norms of $\hat{w}$, $\hat{\theta} - \theta_\pm$ and $\hat{\theta}^4 - \theta_\pm^4$ remain also bounded as $\alpha \to 0$.

Next, since $(\hat{\theta}^4)' = (\hat{\theta}^4 - \theta_\pm^4)' + (\theta_\pm^4)'$ in the distributional sense, the right hand side of (3.3) splits up as the sum of an $H^{-1}(I\!R)$ distribution and a (fixed) Dirac mass. Clearly the $H^{-1}(I\!R)$-part remains bounded as $\alpha \to 0$. Thus $\hat{w}$ is bounded in $H^1(I\!R)$. This gives a bound

$$\alpha\int_{-\infty}^{\infty} \hat{\theta}'^2 \le C, \tag{3.6}$$

simply multiplying (3.2) by θ' and integrating by parts on both sides of $x = 0$.

The estimate (3.6) allows us to conclude that in (3.2), possibly along a further subsequence,

$$\alpha\hat{\theta}' \to 0, \quad \hat{w} \to \hat{w}_0,$$

strongly in respectively $L^2(I\!R)$ (because of the above estimate) and $L^2_{loc}(I\!R)$, therefore

$$\hat{\theta} - \theta_\pm \to \hat{\theta}_0 - \theta_\pm,$$

strongly in $L^2_{loc}(I\!R)$.

We now conclude that in the $\hat{x}$-scale the limit solution $(\mu_0, \hat{\theta}_0, \hat{w}_0)$ satisfies

$$\hat{\theta}_0(\hat{x}) = \theta_\pm(\hat{x}) + \beta\hat{w}_0(\hat{x}), \tag{3.7}$$

where $\hat{w}_0$ is the (continuous) weak solution of

$$-\hat{w}_0'' + 3\hat{w}_0 = \frac{1}{\mu_0}(\hat{\theta}_0^4)'. \tag{3.8}$$

It remains to determine $\hat{w}_0$ and μ_0. Matching both scales we clearly have, since $\hat{w}(0) = w(0)$ for every α, that $C_0 = \hat{w}_0(0)$. Hence, dropping hats and subscripts zero in (3.1), (3.7) and (3.8),

$$(w' + \frac{1}{\mu}\theta^4)' = 3w, \quad \theta = \theta_\pm + \beta w, \quad \mu = \frac{F(\theta_+ + \beta w(0))}{\theta_+ - \theta_-}. \tag{3.9}$$

Proposition 3.2 *There is a unique solution $(\mu, w, \theta) = (\mu_0, \hat{w}_0, \hat{\theta}_0)$ of (3.9). The solution has the property that $\hat{w}_0 > 0$.*

Proof Let $z = \mu w' + \theta^4$. Then (w, z) must satisfy

$$\mu w' = z - (\theta_\pm + \beta w)^4; \quad z' = 3\mu w, \tag{3.10}$$

which is a two dimensional Lienard type of system. For $\hat{x} < 0$ we have $\theta_\pm = \theta_-$ and for $\hat{x} > 0$ we have $\theta_\pm = \theta_+$. The θ_--system has a saddle in $(w, z) = (0, \theta_-^4)$ with one unique orbit coming out into the first quadrant. Likewise, the θ_+-system has a saddle in $(w, z) = (0, \theta_+^4)$ with one unique orbit going in from the first quadrant. For each $\mu > 0$ there is a unique intersection point P_μ of these two orbits. Setting $\hat{x} = 0$ in P_μ, it is easy to see that w must have $w(0)$ equal to the positive w-coordinate of P_μ, which is decreasing as a function of μ and varies from zero to $(\theta_+ - \theta_-)/\beta$ as we vary μ from ∞ to 0. This implies that there is a unique $\mu = \mu_0$ for which the last equality in (3.9) holds. ∎

Remark 3.3 *It is interesting to note that solutions of the limit problem satisfy $\theta_0(0) \to 2\theta_+ - \theta_-$ (the upperbound in (2.23)) as $\beta \to \infty$, consistent with [3, Fig. 1(c)]. This can be seen by setting $v = \beta w$ in (3.10). It is easy to see that the two orbits of (3.10) used in the proof of Proposition 3.2 converge to the curves $z = (\theta_- + v)^4$ and $z = \theta_+^4$ respectively as $\beta \to 0$ whence $v_0(0) = \beta w_0(0) = \beta w_0(0) \to \theta_+ - \theta_-$.*

The limits in Proposition 3.1 hold for $\alpha \to 0$ without restriction to a (sub)sequence because of the uniqueness in Proposition 3.2.

To conclude this subsection, we point out that $\hat{\theta}_0$ has a jump $\theta_+ - \theta_-$ in $\hat{x} = 0$, which is in agreement with [11, Fig. 1], [3, Fig. 1(b)] and [6, Fig. 3(a)]. For the convenience of the reader, a typical profile is shown in Figure 2.

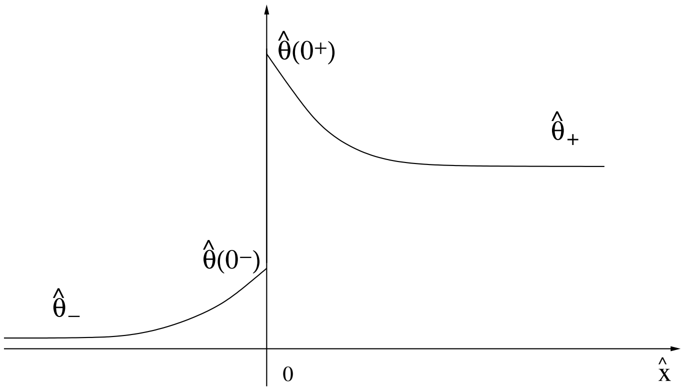

Figure 2: $\alpha \to 0$, β fixed: profile of the limit $\hat{\theta}_0$ in the $\hat{x}$ variable. $\hat{\theta}_0$ has a jump $\theta_+ - \theta_-$ at $\hat{x} = 0$, $\hat{\theta}(0^\pm) = \theta_\pm + \beta C_0$.

3.2 Large Boltzmann numbers: $\beta \to \infty$

The limit $\beta \to \infty$ with α positive fixed is a little easier and is in agreement with Fig. 4 in [6], as far as the behaviour of the speed μ is concerned.

Proposition 3.4 *For $\alpha > 0$ fixed the solutions $(\mu, \theta, w) = (\mu_\beta, \theta_\beta, w_\beta)$ satisfy*

$$\mu \to \mu_a, \quad w \to 0, \quad \theta \to \theta_+,$$

locally uniformly in x as $\beta \to \infty$.

Proof It is not difficult to show, using Theorem 2.1 and in particular (2.16) and (2.4), that

$$w \to 0, \quad \theta \to \theta_\infty \equiv C_\infty, \quad \mu \to \mu_\infty = \frac{F(C_\infty)}{\theta_+ - \theta_-} = \frac{\gamma}{\theta_+ - \theta_-} \exp(-\frac{\mathrm{Ze}}{C_\infty}),$$

locally uniformly. As in the previous subsection it remains to identify μ_∞ and C_∞ and this is again done using another scale. We set

$$x = \beta\hat{x}, \quad \hat{v}(\hat{x}) = \beta w(x), \quad \hat{\theta}(\hat{x}) = \theta(x).$$

Then

$$\mu(\hat{\theta} - \theta_\pm) = \frac{1}{\beta}\hat{\theta}' + \mu\hat{v}; \quad -\frac{1}{\beta^2}\hat{v}'' + 3\alpha^2\hat{v} = \frac{\alpha}{\mu}(\hat{\theta}^4)'.$$

Clearly $\hat{v}$ is bounded in L^∞ after (2.16). We look only on the right where we control the L^1-norm of $\hat{\theta}'$ since it has a sign. Therefore we can pass to the limit for $\hat{x} > 0$ and conclude for the limits $\hat{v}_\infty$ and $\hat{\theta}_\infty$ that on $[0, \infty)$,

$$\hat{v}_\infty = \hat{\theta}_\infty - \theta_+ \geq 0, \quad 3\alpha^2\hat{v}_\infty = \frac{\alpha}{\mu_\infty}(\hat{\theta}_\infty^4)' \leq 0,$$

hence $\hat{v}_\infty \equiv 0$ and $\hat{\theta}_\infty \equiv \theta^+$ on $[0, \infty)$. By matching (as in the previous subsection $\alpha \to 0$), $C_\infty = \hat{\theta}_\infty(0^+) = \theta_+$ and thus we conclude that $\mu_\infty = \mu_a$. ∎

3.3 Transparent limit combined with large Boltzmann numbers: $\alpha \to 0$ with $\alpha\beta$ constant

Here we assume that $\alpha\beta = \chi$ is positive and fixed. As in Remark 3.3 we set $v = \beta w$, so

$$-v'' + 3\alpha^2 v = \frac{\chi}{\mu}(\theta^4)'; \quad \mu(\theta - \theta_\pm) = \theta' + \mu v,$$

and we consider solutions $(\mu, \theta, v) = (\mu_\alpha, \theta_\alpha, v_\alpha)$ along a sequence $\alpha \to 0$. We control uniformly θ and θ', as well as v, v', v'', thus we can pass to the limit along a subsequence. Both θ and v are convex for $x > 0$ and therefore also decreasing. This gives uniform estimates on θ and v in $W^{1,1}(0, \infty)$ and makes the convergence uniform on $[0, \infty)$. We conclude that the limit solution (μ_0, θ_0, v_0) satisfies

$$-v_0'' = \frac{\chi}{\mu_0}(\theta_0^4)'; \quad \mu_0(\theta_0 - \theta_\pm) = \theta_0' + \mu_0 v_0; \quad \mu_0 = \frac{F(\theta_0(0))}{\theta_+ - \theta_-}, \tag{3.11}$$

with

$$\theta_0(+\infty) = \theta_+, \quad v_0(+\infty) = 0. \tag{3.12}$$

Because $\chi \neq 0$ we then must have $\theta_0(-\infty) = \theta_0(+\infty)$ whence

$$\theta_0(-\infty) = \theta_+, \quad v_0(-\infty) = \theta_+ - \theta_-,$$

and we conclude that the limit solution satisfies

$$\theta'' = \mu\theta' + \chi(\theta^4 - \theta_+^4), \quad [\theta] = 0, \quad [\theta'] = -\mu(\theta_+ - \theta_-) = F(\theta), \tag{3.13}$$

where we have omitted the subscripts.

Proposition 3.5 *There exists a solution $(\mu, \theta) = (\mu_0, \theta_0)$ of (3.13). This limit solution has its maximum in $x = 0$. As $\chi \to 0$,*

$$\theta_0(0) \to 2\theta_+ - \theta_-, \tag{3.14}$$

achieving again the upperbound in (2.23).

Proof Existence follows from the above convergence. However we present a direct alternative demonstration. For $x < 0$ the solution must correspond to the orbit coming out of the saddle $(\theta_+, 0)$ into the first quadrant of the (θ, θ') phase plane for $\theta'' = \mu\theta' + \chi(\theta^4 - \theta_+^4)$. For $x > 0$ the solution must correspond to the orbit going into the same saddle from the fourth quadrant. To find the solution of (3.13) we have to find $\theta_* > \theta_+$ and $\mu > 0$ such that the length $H(\theta_*; \mu, \chi)$ of the corresponding vertical line segment between these two orbits satisfies

$$H(\theta_*; \mu, \chi) = \mu(\theta_+ - \theta_-) = F(\theta_*) = \gamma \exp(-\frac{\mathrm{Ze}}{\theta_*}).$$

Clearly H increases from zero to infinity as θ_* increases from zero to some finite value where one of the two orbits above has a vertical asymptote. Thus we have for each $\mu > 0$ (with $\chi > 0$ considered as fixed) a unique solution $\theta_* = \theta_*(\mu)$ of $H(\theta_*; \mu, \chi) = \mu(\theta_+ - \theta_-)$. We have to determine μ such that also

$$\theta_*(\mu) = F^{-1}(\mu(\theta_+ - \theta_-)) = \frac{\mathrm{Ze}}{\log\gamma - \log(\theta_+ - \theta_-) - \log\mu}. \tag{3.15}$$

We claim that

$$\lim_{\mu\to 0} \theta_*(\mu) = \theta_+ \text{ and } \lim_{\mu\to\infty} \theta_*(\mu) = 2\theta_+ - \theta_-, \tag{3.16}$$

which immediately implies the existence of at least one solution μ of (3.15).

It remains to establish our claim. The limit $\mu \to 0$ is straightforward and left to the reader. For the limit $\mu \to \infty$ we introduce a new variable $\hat{x} = \mu x$ so that the system becomes (primes now denoting derivatives with respect to $\hat{x}$)

$$\theta'' = \theta' + \frac{\chi}{\mu^2}(\theta^4 - \theta_+^4), \quad [\theta] = 0, \quad -[\theta'] = \theta_+ - \theta_-. \tag{3.17}$$

As $\mu \to \infty$ the two orbits considered above converge to the lines $\theta' = \theta - \theta_+$ and $\theta' = 0$ in the (θ, θ') phase plane corresponding to (3.17). Since $-[\theta'] = \theta_+ - \theta_-$ as we jump from one orbit to the other at $\theta = \theta_*$, if follows that in the limit $\theta_* - \theta_+ = \theta_+ - \theta_-$.

Thus the claim (3.16) and the existence of a solution to (3.15) follow. Finally as $\chi \to 0$, by the same argument as for $\mu \to \infty$ just above, the two orbits converge, in terms of (3.17), to $\theta' = \theta - \theta_+$ and $\theta' = 0$, so that $\theta_* \to 2\theta_+ - \theta_-$, whence (3.14).
■

We conclude this subsection by a physical discussion of the left boundary condition $\theta(-\infty) = \theta_-$ being lost at the limit: for α fixed, θ_α varies from θ_- up to θ_+, the preheating temperature, at some point $x_1 < 0$. The interval $(-\infty, x_1]$ is called the *radiative precursor zone*. The latter vanishes at the limit $\alpha \to 0$. We refer to [11, Fig. 2], [3, Fig. 1(c)], [6, Fig. 3(b)]. A typical profile is sketched in Figure 3.

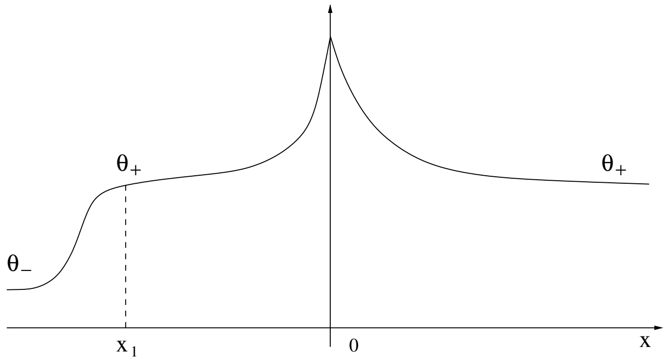

Figure 3: $\alpha \to 0$, $\alpha\beta$ fixed. The picture of θ is for α small. The preheating temperature θ_+ is achieved at $x_1 \to -\infty$.

3.4 The effect of an ignition temperature

In [11] the combustion-radiation model is studied with a reaction rate which vanishes if the temperature is below a certain treshold, usually refered to as the ignition temperature θ_{ig} which we assume to satisfy $\theta_{\mathrm{ig}} > \theta_-$. In terms of the travelling wave formulation in Section 2 this means replacing the reaction rate

$$F(\theta) = \gamma \exp(-\frac{\mathrm{Ze}}{\theta}) \tag{3.18}$$

in (2.2) by a smooth monotone function $F_{\mathrm{ig}}(\theta)$ such that

$$F_{\mathrm{ig}}(\theta) = 0 \text{ for } \theta \leq \theta_{\mathrm{ig}}, \quad 0 < F(\theta) < \gamma \text{ for } \theta > \theta_{\mathrm{ig}}, \quad F(\infty) = \gamma > 0. \tag{3.19}$$

A first observation is then that it is possible to have travelling waves with the temperature at the flame front being larger than the ignition temperature, while the ignition temperature itself is larger than the (dimensionless) adiabatic temperature, i.e.

$$\theta_+ < \theta(0) < \theta_{\mathrm{ig}}.$$

In physical terms, there is no adiabatic flame but the presence of (even a little bit of) dust may make the medium flammable.

It is not our intention here to give a complete analysis of the correspondingly modifed travelling wave problem. We restrict ourselves to the limit problem (3.13) in Section 3.3 together with $\chi = \alpha\beta$ small, which, replacing F by F_{ig}, reads

$$\theta'' = \mu\theta' + \chi(\theta^4 - \theta_+^4), \quad [\theta] = 0, \quad [\theta'] = -\mu(\theta_+ - \theta_-) = F_{\mathrm{ig}}(\theta). \tag{3.20}$$

We consider this problem with θ_+ and χ as parameters (note that varying θ_+ corresponds to varying Y_f) in the following three cases:

$$(i)\ \theta_- \leq \theta_+ \leq \frac{\theta_{\mathrm{ig}} + \theta_-}{2}; \quad (ii)\ \frac{\theta_{\mathrm{ig}} + \theta_-}{2} < \theta_+ < \theta_{\mathrm{ig}}; \quad (iii)\ \theta_+ > \theta_{\mathrm{ig}}. \tag{3.21}$$

Theorem 3.6 *For θ_+ in the range (3.21-i) there are no solutions of (3.20) with $\mu > 0$. For θ_+ in the range (3.21-ii) and $\chi > 0$ small there exist exactly two solutions with $\mu > 0$. For θ_+ in the range (3.21-iii) and $\chi > 0$ small there exists exactly one solution with $\mu > 0$. As a consequence we also have that for $\chi > 0$ small there exists a critical value $\theta_+^{\mathrm{c}}(\chi)$ in the range (3.21-ii) such that for $\theta_+ < \theta_+^{\mathrm{c}}$ there are no solutions of (3.20) with $\mu > 0$, while there is at least one such solution if $\theta_+ \geq \theta_+^{\mathrm{c}}$.*

Proof The solutions correspond to positive solutions μ of

$$F_{\text{ig}}^{-1}(\mu(\theta_+ - \theta_-)) = \theta_*(\mu), \tag{3.22}$$

where $\theta_*(\mu)$ is defined in the proof of Proposition 3.5 in terms of orbits of (3.17). Clearly there are no positive solutions in the range (3.21-i) because then $\theta_{\text{ig}} \geq 2\theta_+ - \theta_-$, the upper bound for $\theta(0)$. For the other two ranges we consider χ as a parameter, so, because of the combined appearance of χ and μ in (3.17),

$$\theta_* = \theta_*(\frac{\mu}{\sqrt{\chi}}).$$

It follows that for χ small θ_* increases very rapidly to its limit $2\theta_+ - \theta_-$ as μ is increasing from zero. Thus if θ_{ig} is between θ_+ and $2\theta_+ - \theta_-$ (range (3.21-ii)), we see that there are exactly two positive solutions of (3.22). On the other hand, in the same limit $\chi \to 0$, the positive solution of (3.22) is unique if $\theta_{\text{ig}} < \theta_+$ (range (3.21-ii)). ∎

The above analysis concerns a limit case close to the situation considered and depicted in [11, Fig. 3]. It confirms that the presence of an ignition temperature may lead to the appearance of a critical value θ_+^c for θ_+ with no solutions (extinction) for θ_+ below this value and multiple solutions for θ_+ above this value. In general this depends also on the parameters α and β and we note that in the transparent limit (Section 3.1) with β fixed this phenomenon does not occur.

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