A model problem for unsaturated groundwater flow with dynamic capillary pressure

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Abstract

Modelling vertical non-steady groundwater flow with dynamic capillary pressure leads to a nonlinear diffusion-convection equation extended with a nonlinear third order term with two space derivatives and one time derivative. We consider the special case in which the original equation is the Burgers' equation and the third order term is linear. We investigate stability of travelling wave solutions of the resulting pseudo-parabolic Burgers' equation, for which we first obtain well-posedness results.

1 Introduction

In this paper we consider the pseudo-parabolic Burgers' equation

$$u_t = u_{xx} + 2uu_x + \varepsilon^2 u_{xxt} \quad \text{on} \quad \mathbb{R} \times [0,T] \tag{1.1}$$

with initial data

$$u(0,x) = u_0(x) \quad \text{in} \quad \mathbb{R}. \tag{1.2}$$

Here ε is a positive parameter. Setting $\varepsilon = 0$ equation (1.1) reduces to the classical Burgers' equation, see e.g. [11], which, depending on u_0 has solutions exhibiting several possible large time behaviours: convergence to a selfsimilar source type solution, a rarefaction wave or a travelling wave, see [13].

After a change of notation, equation (1.1) is a special case of

$$S_t = \{S^\alpha + S^\beta S_x + \varepsilon^2 S^\alpha (S^\gamma S_t)_x\}_x, \tag{1.3}$$

in which α, β and γ are nonnegative constants. We studied (1.3) in [6] as a model equation for Darcy flow in porous media with a dynamic capillary pressure relation, the general form of the equation being

$$S_t = \{K(S) + K(S)\ (-p_c(S) + L(S)\, S_t)_x\}_x\,. \tag{1.4}$$

In (1.4) the unknown is the water saturation S in a vertically placed one-dimensional porous medium, $K(S)$ the conductivity, $p_c(S)$ the capillary pressure function and $L(S)$ a damping coefficient. Taking powerfunctions for these nonlinear functions of S leads to (1.3) under the assumption that S is small. We note that equation

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(1.4) combines conservation of mass, the law of gravity and Darcy's law with the pressure-saturation relation

$$p_c(S) = p_a - p_w + L(S)\,S_t, \tag{1.5}$$

where p_a and p_w denote the air and water pressures. The dynamic term with the time derivative accounts for the third order term in (1.3). This extension of the Darcy flow model is based on the approach introduced by Hassanizadeh & Gray [9] and previous experimental work done by Stauffer [16], among others, see [8] for an overview. Earlier modifications can be found in [1] and [2]. In the context of homogenization and hysteresis, see also [17], [4] and [10].

Returning to (1.3), we note that it is a nonlinear diffusion-convection equation with an additional nonlinear third order term involving two space derivatives and one time derivative. Equations with such a term are sometimes called pseudo-parabolic, see e.g. [7]. In principle this term allows to rewrite the equation as a nonlinear ordinary differential equation (ODE) in a suitable function space, which involves the solution operator of an elliptic equation in the space variable. However, in view of the degeneracies in (1.3) as $S \to 0$, it is far from clear how this has to be implemented if S is not bounded away from zero. Of course one may lift the initial data by a small parameter δ and then take the limit $\delta \to 0$, but in the absence of a comparison principle (except in very special cases) the limit solution has not been shown to exist.

This is more than just a technical complication in view of the results of a study of the possibility of having travelling wave solutions fitting in the familiar picture of propagating fronts separating dry and wet regions. The local analysis in [12] indicates that this depends heavily on the parameters in (1.3), as is confirmed by the formal arguments in the same paper. We note that the equation studied in [12] corresponds to a horizontal medium, i.e. without gravity. Including gravity (pointing in the negative x-direction) has no effect on the issue of dry-wet fronts but it does in general allow for global travelling wave solutions connecting two saturation levels, say $S(-\infty) = \delta > 0$ and $S(+\infty) > \delta$ at respectively $x = -\infty$ and $x = +\infty$. Indeed, in [6] we showed that equation (1.3) has such travelling waves moving with speed c given by the Rankine-Hugoniot condition

$$c = \frac{S(+\infty)^\alpha - S(-\infty)^\alpha}{S(+\infty) - S(-\infty)}. \tag{1.6}$$

These travelling waves converge to a limit as $\delta \to 0$ only if the equation (with or without convection) allows local travelling waves with dry-wet fronts. Of the two possible types of such fronts, see [12], the limit established in [6] then selects the flattest one, consistent with the formal asymptotics in [12].

In this paper we aim for a better understanding of the effect of the third order term on the dynamics of diffusion and convection. Intuitively one expects that this effect is more notable if the large time behaviour of solutions of the diffusion-convection equation is characterised by profiles which do not become flat (in terms of their dependence on x). This is why we restrict our study of the large time behaviour to the travelling wave case, meaning that $S(+\infty) > S(-\infty) \geq 0$. In fact we shall only consider the large time behaviour of solutions of (1.1), the simplest pseudo-parabolic equation allowing convection-driven travelling waves, with initial data satisfying

$$u_0(-\infty) = 0, \quad u_0(+\infty) = 1. \tag{1.7}$$

We will show that such solutions converge to a travelling wave solution

$$u(x,t) = \phi(x+t), \tag{1.8}$$

provided the travelling wave profile ϕ is monotone. This depends on $\varepsilon > 0$: travelling wave solutions connecting zero to one and travelling with speed one exist for all $\varepsilon > 0$, but only when $0 < \varepsilon < \frac{1}{2}$ the profiles are monotone, see Section 3.

Our stability result is of a global character and therefore we first require well-posedness results for the initial value problem. To this end we reformulate equation (1.1) in Section 2 as

$$u_t = F_\varepsilon(u) = A_\varepsilon u + B_\varepsilon u^2, \tag{1.9}$$

where A_ε, B_ε are linear operators defined by

$$A_\varepsilon u = (I - \varepsilon^2 \frac{d^2}{dx^2})^{-1} u_{xx}\,; \quad B_\varepsilon u = (I - \varepsilon^2 \frac{d^2}{dx^2})^{-1} u_x, \tag{1.10}$$

and study local well-posedness of the ODE (1.9) in several Banach spaces, namely in $L^1 \cap L^2$, $L^1 \cap H^1$, L^∞, L^2 and H^1. Here $L^p(\mathbb{R}) = L^p$ with norm $u \to |u|_p$ and $H^1 = H^1(\mathbb{R})$ is the Sobolev space with norm $u \to ||u|| = \sqrt{|u|_2^2 + |u'|_2^2}$. We note that although formally equation (1.1) preserves the integral (conservation of mass), the map $u \to B_\varepsilon u^2$ is not well defined on L^1, hence the choice of $L^1 \cap L^2$ with norm $u \to |u|_{1,2} = |u|_1 + |u|_2$, and $L^1 \cap H^1$ with norm $u \to ||u||_{1,2} = |u|_1 + ||u||$.

Since travelling wave solutions do not belong to L^p if $1 \le p < \infty$, we also consider (1.1) in affine spaces of the form $\mathcal{H} + X$, where $\mathcal{H}$ is any smooth function such that $\mathcal{H}(-\infty) = 0$, $\mathcal{H}(+\infty) = 1$. It is no restriction to assume that $\mathcal{H}'$ is nonnegative and compactly supported. In Section 4 we obtain local well-posedness in $\mathcal{H} + X$ for $X = L^2, L^1 \cap L^2, H^1, L^1 \cap H^1$.

In Section 5 we establish mass conservation: if u_1 and u_2 are solutions of (1.1) with $u_1 - u_2$ in $L^1 \cap L^2$ then

$$\frac{d}{dt} \int_{\mathbb{R}} (u_1(x,t) - u_2(x,t))dx = 0 \text{ for all } t.$$

This allows us to follow [13] by introducing

$$v(x,t) = \int_{-\infty}^{x} (u(s,t) - \phi(s+t))ds. \tag{1.11}$$

The function v is well defined if u is in $\mathcal{H} + L^1$. For solutions u with values in $\mathcal{H} + X$, $X = L^1 \cap L^2$, shifting either ϕ or u_0 we may restrict attention to solutions v of

$$v_t = v_{xx} + v_x^2 + 2v_x\phi + \varepsilon^2 v_{xxt}, \tag{1.12}$$

with $v(+\infty) = 0$.

Equation (1.12) is derived in Section 6 and analysed in Section 7. Again we establish local well-posedness in several natural spaces, in particular in H^1 and $H^2 = \{v \in L^2;\ v', v'' \in L^2\}$. In Section 8 we prove the identities

$$\frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}} (v^2 + \varepsilon^2 v_x^2)dx = -\int_{\mathbb{R}} (1-v)v_x^2 dx - \int_{\mathbb{R}} \phi' v^2 dx, \tag{1.13}$$

and

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(v_x^2+\varepsilon^2 v_{xx}^2)dx = -\int_R v_{xx}^2 dx + \int_{\mathbb{R}} \phi' v_x^2 dx, \tag{1.14}$$

and obtain a global well-posedness result for solutions of (1.12) in H^2. To formulate and prove stability results we need $\phi' \geq 0$ and $v(x,0)$ sufficiently small in H^1 guaranteeing $v < 1$. Using (1.13) this gives convergence of the integral $\int_0^\infty ||v(\cdot,t)||^2 dt$ and thereby of $\int_0^\infty \int_{-\infty}^\infty |u(x,t)-\phi(x+t)|^2 dxdt$. If in addition $v(x,0)$ is in H^2 we adapt methods from [14] and show that $v(\cdot,t)\to 0$ in H^2 whence $||u(\cdot,t)-\phi(\cdot+t)||\to 0$ as $t\to\infty$.

A natural question is of course whether the monotonicity of ϕ is essential. In the context of Korteweg-de Vries type equations there are examples where a switch from monotone to oscillatory behaviour of the travelling wave leads to instability, but this depends on the exponent in the nonlinearity, see [14]. Numerical experiments so far have been inconclusive.

So far we have formulated asymptotic stability in $\mathcal{H}+H^1$, in particular we assume solutions to be continuous in space. This can be relaxed a little bit. In Section 9 we examine solutions with jump discontinuities and show that jumps are fixed in space and decay in time. Moreover we extend our stability results to solutions having one single jump in $x=0$.

We conclude this introduction with the observation that we have avoided a transformation of the problem to travelling wave variables. Such a change is common in the study of stability properties of travelling wave solutions of Burgers' and other "normal" equations. Here it would lead to an equation with yet another third order term involving three space derivatives, which cannot be seen as an ODE in a function space.

2 Local well-posedness in Banach spaces

In this section we show that the initial value problem for the ODE (1.9), which as we recall reads

$$u_t = F_\varepsilon(u) = A_\varepsilon u + B_\varepsilon u^2,$$

is locally well-posed in the Banach spaces $L^1\cap L^2$, $L^1\cap H^1$, L^∞, L^2 and H^1. The operators A_ε and B_ε may be rewritten as

$$A_\varepsilon u = (I-\varepsilon^2\frac{d^2}{dx^2})^{-1}u_{xx} = \frac{1}{\varepsilon^2}\left((I-\varepsilon^2\frac{d^2}{dx^2})^{-1}-I\right)u = \frac{1}{\varepsilon^2}\left(g_\varepsilon * u - u\right), \tag{2.1}$$

and

$$B_\varepsilon u = (I-\varepsilon^2\frac{d^2}{dx^2})^{-1}u_x = g_\varepsilon * u_x, \tag{2.2}$$

where

$$g_\varepsilon(x) = \frac{1}{2\varepsilon}e^{-\frac{|x|}{\varepsilon}} \tag{2.3}$$

is the Green's function for the problem

$$w - \varepsilon^2 w'' = f. \tag{2.4}$$

That is to say,

$$w(x) = (G_\varepsilon f)(x) = (g_\varepsilon * f)(x) = \int_{\mathbb{R}} g_\varepsilon(x-y)f(y)dy \tag{2.5}$$

is the solution of (2.4). Since for any $u \in L^p$ with $1 \leq p \leq \infty$, the derivative $(g_\varepsilon * u)_x$ is well defined and $(g_\varepsilon * u)_x = g'_\varepsilon * u$ we have

$$B_\varepsilon u(x) = (g'_\varepsilon * u)(x) = \frac{1}{\varepsilon^2}\int_{\mathbb{R}} g_\varepsilon(x-y)\int_x^y u(s)ds dy \quad \text{for all} \quad x \in \mathbb{R}. \tag{2.6}$$

Theorem 2.1 *Let X denote any of the Banach spaces $L^1 \cap L^2$, $L^1 \cap H^1$, L^∞, L^2 and H^1. Then for all $u_0 \in X$ there exists $T > 0$ such that there exists a unique solution $u \in C^1([0,T]; X)$ of (1.9).*

Theorem 2.1 will follow from Picard's Theorem: if X is a Banach space and $F : X \to X$ is Lipschitz continuous in a neighbourhood of $u_0 \in X$, then there exists $T > 0$ and a unique solution $u \in C^1([0,T];\ X)$ of $u_t = F(u)$ with $u(0) = u_0$. We shall show that F_ε is locally Lipschitz continuous on each of the Banach spaces listed above. This is done within the next five lemmas and is based on the following properties of g_ε:

$$|g_\varepsilon|_1 = \int_{\mathbb{R}} g_\varepsilon(y)dy = 1, \quad |g_\varepsilon|_2^2 = \int_{\mathbb{R}} (g_\varepsilon(y))^2 dy = \frac{1}{4\varepsilon}, \tag{2.7}$$

$$|g'_\varepsilon|_1 = \int_{\mathbb{R}} |g'_\varepsilon(y)|dy = \frac{1}{\varepsilon}, \quad |g'_\varepsilon|_2^2 = \int_{\mathbb{R}} (g'_\varepsilon(y))^2 dy = \frac{1}{4\varepsilon^3}, \tag{2.8}$$

$$\int_{\mathbb{R}} g_\varepsilon(y)|y|dy = \varepsilon, \quad \int_{\mathbb{R}} g'_\varepsilon(y)dy = 0. \tag{2.9}$$

Throughout $B(X,Y)$ denotes the Banach space of bounded linear operators from a Banach space X to a Banach space Y, with the usual convention that $B(X) = B(X,X)$.

Lemma 2.2 *The map F_ε is locally Lipschitz continuous in L^∞.*

Proof. Since $A_\varepsilon = \frac{1}{\varepsilon^2}(G_\varepsilon - I)$ it follows from (2.7) that A_ε is a bounded linear operator in L^∞ with $\|A_\varepsilon\|_{B(L^\infty)} \leq \frac{2}{\varepsilon^2}$. Thus it is (uniformly) Lipschitz continuous on L^∞. By (2.6) and (2.9) we have

$$|(B_\varepsilon u)(x)| \leq \frac{1}{\varepsilon^2}\int_{-\infty}^{\infty} g_\varepsilon(x-y)\,|x-y|\,|u|_\infty\,dy = \frac{1}{\varepsilon}|u|_\infty,$$

which implies that the operator B_ε is a bounded linear operator in L^∞. The map $u \to u^2$ clearly maps L^∞ into L^∞, and is locally Lipschitz continuous: if $u_1, u_2 \in L^\infty$ are such that $\|u_i\|_\infty \leq R$, for some $R > 0$, then

$$|u_1^2 - u_2^2|_\infty = |(u_1+u_2)(u_1-u_2)|_\infty \leq 2R|u_1-u_2|_\infty.$$

□

Lemma 2.3 *The map F_ε is locally Lipschitz continuous in L^2.*

Proof. Since A_ε is the Hille-Yosida approximation of the operator $A = -\frac{d^2}{dx^2}$, which is maximal monotone in the Hilbert space L^2, A_ε is bounded linear operator in L^2 with $|A_\varepsilon|_{B(L^2)} \leq \frac{1}{\varepsilon^2}$, see [5]. In view of $B_\varepsilon u = g'_\varepsilon * u$, the inequality $|f * g|_p \leq |f|_p \, |g|_1$ and (2.8), the linear operator B_ε is bounded in L^1 and bounded as an operator from L^1 to L^2, with

$$\|B_\varepsilon\|_{B(L^1,L^2)} \leq \left(\int_{\mathbb{R}} |g'_\varepsilon(y)|^2 dy \right)^{\frac{1}{2}} = \frac{1}{2\varepsilon^{\frac{3}{2}}}. \tag{2.10}$$

For $u_1, \ u_2 \in L^2$ with $|u_i|_2 \leq R$ we have

$$|B_\varepsilon(u_1^2 - u_2^2)|_2 \ \leq \ \|B_\varepsilon\|_{B(L^1,L^2)} |(u_1 + u_2)(u_1 - u_2)|_1 \ \leq \ \frac{R}{\varepsilon^{\frac{3}{2}}} |u_1 - u_2|_2.$$

□

Lemma 2.4 *The map F_ε is locally Lipschitz continuous in H^1.*

Proof. Since $A = -\frac{d^2}{dx^2}$ is maximal monotone on the Hilbert space H^1, its Hille-Yosida approximation A_ε is a bounded linear operator in H^1 with $\|A_\varepsilon\|_{B(H^1)} \leq \frac{1}{\varepsilon^2}$. For u_1, u_2 in H^1 with $||u_i|| \leq R$ we now have

$$|B_\varepsilon(u_1^2 - u_2^2)|_2 \leq \frac{R}{2\varepsilon^{\frac{3}{2}}} ||u_1 - u_2||,$$

and

$$\begin{aligned} | \left(B_\varepsilon(u_1^2 - u_2^2) \right)_x |_2 \leq \|B_\varepsilon\|_{B(L^1,L^2)} | \left((u_1 + u_2)(u_1 - u_2) \right)_x |_1 \leq \\ \|B_\varepsilon\|_{B(L^1,L^2)} (|(u_1 + u_2)_x|_2 |u_1 - u_2|_2 + |u_1 + u_2|_2 |(u_1 - u_2)_x|_2) \leq \\ \frac{R}{\varepsilon^{\frac{3}{2}}} ||u_1 - u_2||. \end{aligned}$$

Thus $u \to B_\varepsilon u^2$ is locally Lipschitz continuous. □

Lemma 2.5 *The map F_ε is locally Lipschitz continuous in $L^1 \cap L^2$.*

Proof. The inequality $|g_\varepsilon * u|_p \leq |g_\varepsilon|_1 \, |u|_p$ for all $1 \leq p \leq \infty$ and (2.7) imply that A_ε is a bounded linear operator in L^1 with $\|A_\varepsilon\|_{B(L^1)} \leq \frac{2}{\varepsilon^2}$. Consequently A_ε is also bounded in $L^1 \cap L^2$ with $\|A_\varepsilon\|_{B(L^1 \cap L^2)} \leq \frac{3}{\varepsilon^2}$. Now B_ε is a bounded linear operator in $L^1(\mathbb{R})$ with

$$\|B_\varepsilon\|_{B(L^1)} \leq \int_{\mathbb{R}} |g'_\varepsilon(y)| dy = \frac{1}{\varepsilon}, \tag{2.11}$$

so by (2.11) and (2.10) $u \to B_\varepsilon u^2$ maps $L^1 \cap L^2$ to itself. Let $u_1, \ u_2 \in L^1 \cap L^2$ with $|u_i|_{1,2} \leq R$. Then

$$\begin{aligned} |B_\varepsilon(u_1^2 - u_2^2)|_{1,2} \ \leq \ \|B_\varepsilon\|_{B(L^1,L^2)} |(u_1 + u_2)(u_1 - u_2)|_1 + \\ \|B_\varepsilon\|_{B(L^1)} |(u_1 + u_2)(u_1 - u_2)|_1 \ \leq \\ 2R \left(\frac{1}{\varepsilon} + \frac{1}{2\varepsilon^{\frac{3}{2}}} \right) |u_1 - u_2|_2 \ \leq \ 2R \left(\frac{1}{\varepsilon} + \frac{1}{2\varepsilon^{\frac{3}{2}}} \right) |u_1 - u_2|_{1,2}. \end{aligned}$$

□

Lemma 2.6 *The map F_ε is locally Lipschitz continuous in $L^1 \cap H^1$.*

Proof. Again A_ε is a bounded linear operator on $L^1 \cap H^1$ with $\|B_\varepsilon\|_{B(L^1\cap H^1)} \leq \frac{3}{\varepsilon^2}$. As in Lemma 2.5 one has that B_ε is locally Lipschitz continuous in $L^1 \cap H^1$ with Lipschitz constant $\frac{R}{\varepsilon^{\frac{3}{2}}}$ on the ball with radius R in $L^1 \cap H^1$. $\square$

3 Travelling waves

The analysis of travelling wave solutions of (1.1) is similar to the analysis of travelling wave solutions of (1.3) in [6]. Substituting

$$u(x,t) = \phi(x+ct) \tag{3.1}$$

in (1.1), we have for $\phi(x)$, after an integration in x, that

$$c(\phi(x) - \phi(-\infty)) = \phi'(x) + c(\phi(x)^2 - \phi(-\infty)^2) + \varepsilon^2 c\phi''(x), \tag{3.2}$$

so that

$$c = \frac{\phi(\infty)^2 - \phi(-\infty)^2}{\phi(\infty) - \phi(-\infty)}. \tag{3.3}$$

Restricting attention to $\phi(-\infty) = 0$ and $\phi(\infty) = 1$ we have $c = 1$ and (3.2) can be written as a Lienard type system of two equations:

$$\begin{cases} \varepsilon^2\phi' = \psi - \phi \\ \psi' = \phi(1-\phi) \end{cases}$$

The travelling wave solutions connecting $\phi(-\infty) = 0$ to $\phi(\infty) = 1$ are unique up to translation and correspond to a unique orbit connecting the saddle $(0,0)$ to the sink $(1,1)$. Note that $(0,0)$ has eigenvalues

$$\lambda_{1,2} = -\frac{1}{2\varepsilon^2}\left(1 \pm \sqrt{1+4\varepsilon^2}\right), \quad \lambda_1 < 0 < \lambda_2, \tag{3.4}$$

and $(1,1)$ has eigenvalues

$$\mu_{1,2} = -\frac{1}{2\varepsilon^2}(1 \pm \sqrt{1-4\varepsilon^2}), \quad \mu_2 < \mu_1 < 0 \text{ if } \varepsilon^2 < \frac{1}{4}, \quad \mathrm{Re}\mu_{1,2} < 0. \tag{3.5}$$

The unique orbit coming out of $(0,0)$ into the first quadrant connects to $(1,1)$. This follows from arguments very similar to the arguments in [6] and relies in particular on the negativity of the divergence of the Lienard vector field.

If $\varepsilon^2 < \frac{1}{4}$ the eigenvalues at $(1,1)$ are negative: $(1, -\varepsilon^2\mu_2)$ is an eigenvector of the slow eigenvalue μ_1 and $(1, -\varepsilon^2\mu_1)$ is an eigenvector of the fast eigenvalue μ_2. The set $\{\phi > 0,\ 0 < \psi < -\varepsilon^2\mu_2\phi + (1+\varepsilon^3\mu_2)\}$, contained in the region where $\phi' > 0$, is then invariant and contains the connecting orbit. Therefore ϕ is monotone if $\varepsilon^2 \leq \frac{1}{4}$. In this case the invariant region gives an explicit upper bound for ϕ', namely

$$\phi' = \frac{\psi - \phi}{\varepsilon^2} \leq \frac{1+\varepsilon^2\mu_2}{\varepsilon^2} = \frac{1-\sqrt{1-4\varepsilon^2}}{2\varepsilon^2}. \tag{3.6}$$

Theorem 3.1 *Equation (1.1) has a travelling wave solution connecting $u = 0$ in $x = -\infty$ to $u = 1$ in $x = \infty$. This solution is unique up to translation and of the form $u(x,t) = \phi(x+t)$. If $\varepsilon^2 < \frac{1}{4}$ the profile ϕ is monotone increasing and its derivative is bounded by (3.6).*

4 Local well-posedness in affine Banach spaces

In this section we show that the initial value problem for the ODE (1.9) is locally well-posed in the affine Banach spaces $Y = \mathcal{H}+X$, $X = L^1\cap L^2, L^1\cap H^1, L^2, H^1$. We recall that $\mathcal{H}$ is a smooth function with $\mathcal{H}(-\infty) = 0$, $\mathcal{H}(\infty) = 1$ and $\mathcal{H}'$ nonnegative and compactly supported. We say that u is a solution of (1.9) in $C^1([0,T];\ Y)$ if $\bar{u} = u - \mathcal{H}$ is a solution in $C^1([0,T];\ X)$ of the equation

$$\bar{u}_t = F_\varepsilon(\bar{u}) + 2B_\varepsilon(\mathcal{H}\bar{u}) + F_\varepsilon(\mathcal{H}). \tag{4.1}$$

Theorem 4.1 *Let $Y = \mathcal{H}+X$, where X is any of the spaces $L^1\cap L^2, L^1\cap H^1, L^2, H^1$. Then for all $u_0 \in Y$ there exists $T > 0$ and a unique solution of problem (1.1) $u \in C^1([0,T];\ Y)$.*

Proof. If we show that the operator

$$\bar{u} \to F_\varepsilon(\bar{u}) + 2B_\varepsilon(\mathcal{H}\bar{u}) + F_\varepsilon(\mathcal{H})$$

is locally Lipschitz from X to X, the theorem follows again from Picard's theorem. From Section 2 we know that $\bar{u} \to F_\varepsilon(\bar{u})$ is locally Lipschitz in X for each of the choices of X. We only need to prove that $F_\varepsilon(\mathcal{H}) \in X$ and that the linear map $\bar{u} \to 2B_\varepsilon(\mathcal{H}\bar{u})$ is a bounded operator in X.

Clearly $F_\varepsilon(\mathcal{H}) = A_\varepsilon\mathcal{H} + B_\varepsilon\mathcal{H}^2 = G_\varepsilon\mathcal{H}'' + G_\varepsilon(\mathcal{H}^2)'$, $F_\varepsilon(\mathcal{H})$ is in X for any of the choices of X, because $\mathcal{H}''$ and $(\mathcal{H}^2)'$ are compactly supported (smooth) functions. As for $\bar{u} \to 2B_\varepsilon(\mathcal{H}\bar{u})$, we saw in (2.11) that B_ε is a bounded linear operator in L^1. It is also bounded in L^2 and H^1 with

$$\|B_\varepsilon\|_{B(L^2)} \le \int_{\mathbb{R}} |g_\varepsilon'(y)|dy = \frac{1}{\varepsilon}, \tag{4.2}$$

$$\|B_\varepsilon\|_{B(H^1)} = \int_{\mathbb{R}} |g_\varepsilon'(y)|dy = \frac{1}{\varepsilon}, \tag{4.3}$$

respectively. Thus B_ε is bounded in $L^1 \cap L^2$ and in $L^1 \cap H^1$. Finally $|\mathcal{H}|_\infty = 1$, $|\mathcal{H}'|_\infty < \infty$,

$$|\mathcal{H}\bar{u}|_2 \le |\bar{u}|_2,$$

$$\begin{aligned} \|\mathcal{H}\bar{u}\| \ &\le\ |\mathcal{H}\bar{u}|_2 + |\mathcal{H}\bar{u}_x|_2 + |\mathcal{H}'\bar{u}|_2 \ \le \\ |\bar{u}|_2 + |\bar{u}_x|_2 + |\mathcal{H}'|_\infty\, |\bar{u}|_2 \ &\le\ \|\bar{u}\|_{H^1}\,(1 + |\mathcal{H}'|_\infty) \end{aligned}$$

Combining (4.2) and (4.3) with the above estimates we get that $\bar{u} \to 2B_\varepsilon(\mathcal{H}\bar{u})$ is a bounded linear operator on each X. This completes the proof. $\square$

5 Conservation of mass

In this section we prove that equation (1.1) preserves the integral if we consider solutions in $\mathcal{H} + L^1 \cap L^2$. Note that unlike in the case of the Burgers' equation, this is not the same as being contracting in L^1. Again this is due to the absence of a comparison principle.

Proposition 5.1 *Let u_1, u_2 be two solutions of equation (1.9) in $C^1([0,T];\ \mathcal{H}+L^1\cap L^2)$. Then*

$$\int_{\mathbb{R}}(u_1(x,t)-u_2(x,t))dx=\int_{\mathbb{R}}(u_1(x,0)-u_2(x,0))dx \quad \textit{for all} \quad t\in[0,T]. \tag{5.1}$$

Proof. Consider the composite map

$$\mathcal{F}:t\in[0,T]\to u(\cdot,t)\in L^1\cap L^2\to\int_{\mathbb{R}}u(x,t)dx\in\mathbb{R}$$

where $u=u_1-u_2$. Since by definition $u\in C^1([0,T];L^1\cap L^2)$, we have $\mathcal{F}\in C^1([0,T],\mathbb{R})$. The chain rule implies that

$$\mathcal{F}'(t)=\int_{\mathbb{R}}A_\varepsilon(u_1(x,t)-u_2(x,t))dx+\int_{\mathbb{R}}B_\varepsilon(u_1(x,t)^2-u_2(x,t)^2)dx. \tag{5.2}$$

We claim that both right hand side terms in (5.2) are zero for all $t\in[0,T]$.

To see that the first term is zero, we recall that $A_\varepsilon=\frac{1}{\varepsilon^2}(G_\varepsilon-I)$ and note that $\int_{\mathbb{R}}G_\varepsilon u=\int_{\mathbb{R}}u$ for all $u\in L^1$. This is immediate from the definition of G_ε as convolution with the Green's function for (2.4): if $f\in L^1$ then both w and w' are in L^1 and $\int_{\mathbb{R}}w=\int_{\mathbb{R}}f$.

Before proving

$$\int_{\mathbb{R}}B_\varepsilon(u_1(x,t)^2-u_2(x,t)^2)dx=0,$$

we observe that it is well-defined. Indeed, if $u_i=\bar{u}_i+\mathcal{H}$, $i=1,2$ then $u_1^2-u_2^2=\bar{u}_1^2-\bar{u}_2^2+2\mathcal{H}(\bar{u}_1-\bar{u}_2)$. Since $\bar{u}_1^2-\bar{u}_2^2\in L^1$ and $2\mathcal{H}(\bar{u}_1-\bar{u}_2)\in L^1\cap L^2$ we have $u_1^2-u_2^2\in L^1$ and from Lemma 2.5 (2.11) we have $B_\varepsilon\in B(L^1)$. Thus the second right hand side term in (5.2) is well defined.

Now let $w\in L^1$ and consider the integral

$$\int_{\mathbb{R}}B_\varepsilon w(x)dx=\int_{\mathbb{R}}\left(\int_{\mathbb{R}}g_\varepsilon'(x-y)w(y)dy\right)dx.$$

Applying Fubini's theorem to $g_\varepsilon'(x-y)w(y)\in L^1(\mathbb{R}\times\mathbb{R})$ we obtain, in view of (2.9),

$$\int_{\mathbb{R}}B_\varepsilon w(x)dx=\int_{\mathbb{R}}w(y)\left(\int_{\mathbb{R}}g_\varepsilon'(x-y)dx\right)dy=0,$$

which in particular holds for $w=u_1(\cdot,t)^2-u_2(\cdot,t)^2\in L^1$. Thus also the second term in (5.2) is zero. □

6 Integrated equation

Now that we have conservation of the integral we may adapt ideas from [13]. Rather than solving (1.1) for the unknown $u(x,t)$ we consider an equation for the unknown $v(x,t)$, which as we recall is formally defined as

$$v(x,t)=\int_{-\infty}^{x}(u(s,t)-\phi(s+t))ds. \tag{6.1}$$

Thus if u is a solution of (1.1) in $C^1([0,T];\ \mathcal{H}+L^1\cap L^2)$, then by Proposition 5.1

$$\int_{\mathbb{R}}(u(s,t)-\phi(s+t))ds=\int_{\mathbb{R}}(u(s,0)-\phi(s))ds \quad \text{for all} \quad t\in[0,T], \tag{6.2}$$

which, without loss of generality, we take equal to zero, just by shifting ϕ. This allows us to work with function spaces for v having $v(\pm\infty,t)=0$ in some weak or strong sense. The equation for $v(x,t)$ is obtained by formally integrating (1.1). This yields (1.12), which, rewritten as an ODE in similar fashion as (1.9), reads

$$v_t+\frac{1}{\varepsilon^2}v=G_\varepsilon\left(\frac{1}{\varepsilon^2}v+v_x^2+2v_x\phi\right). \tag{6.3}$$

Proposition 6.1 *Let u be a solution in $C^1([0,T];\ \mathcal{H}+L^1\cap L^2)$. Then v defined by (6.1) is a solution of (6.3) defined on $[0,T]$.*

Proof. We rewrite equation (1.9) as

$$u_t+\frac{1}{\varepsilon^2}u=G_\varepsilon\left(\frac{1}{\varepsilon^2}u+(u^2)_x\right). \tag{6.4}$$

Subtracting from (6.4) the same equation for $\phi(x+t)$ we arrive at

$$z_t+\frac{1}{\varepsilon^2}z=\frac{1}{\varepsilon^2}G_\varepsilon z+G_\varepsilon(z^2+2z\phi)_x. \tag{6.5}$$

for $z(x,t)=u(x,t)-\phi(x+t)$. We define the operator $J:L^1\to L^\infty$ by

$$(Jf)(x)=\int_{-\infty}^x f, \tag{6.6}$$

and apply J to (6.5). Then

$$(Jz)_t+\frac{1}{\varepsilon^2}Jz=\frac{1}{\varepsilon^2}JG_\varepsilon z+JG_\varepsilon(z^2+2z\phi)_x,$$

and $v=Jz$ satisfies (6.3), provided $JG_\varepsilon f=G_\varepsilon Jf$ and $JG_\varepsilon f_x=G_\varepsilon f$ for all $f\in L^1$.

We note that J commutes with G_ε. Indeed, if $f\in L^1$, then $w=G_\varepsilon f$ has w, w' and w'' in L^1 and satisfies the equation $-\varepsilon^2w''+w=f$. Thus $Jw=JG_\varepsilon f$ satisfies

$$-\varepsilon^2(Jw)''+Jw=Jf,$$

whence $G_\varepsilon Jf=JG_\varepsilon f$. Finally $JG_\varepsilon f_x=G_\varepsilon f$ for all $f\in L^1(\mathbb{R})$ because

$$JG_\varepsilon f_x=J(g_\varepsilon'*f)=(Jg_\varepsilon')*f=G_\varepsilon f,$$

where, if we write the integrals explicitly, we have used Fubini's theorem applied to $(s,y)\to g_\varepsilon'(s-y)f(y)$ on $(-\infty,x)\times\mathbb{R}$. □

Remark 6.2 *Note that if $u\in L^1\cap H^1$ we only need G_ε commuting with J in the argument above. Applying J to (6.5) gives*

$$(J(u-\phi))_t+\frac{1}{\varepsilon^2}J(u-\phi)=JG_\varepsilon(u-\phi)+JG_\varepsilon(u^2-\phi^2)_x.$$

Since $(u^2-\phi^2)_x\in L^1$ this implies

$$v_t+\frac{1}{\varepsilon^2}v=G_\varepsilon v+G_\varepsilon(u^2-\phi^2),$$

and the result follows because $u^2-\phi^2=(u-\phi)(u+\phi)=v_x^2+2\phi v_x$.

7 Local well-posedness of the integrated equation

By Proposition 6.1 a solution u of (1.9) in $C^1([0,T];\ \mathcal{H}+L^1\cap L^2)$ defines a solution v of (6.3) in the Banach space $X=\{v\in L^\infty:\ v_x\in L^1\cap L^2\}$ with norm $||v||_X=|v|_\infty+|v_x|_{1,2}$. In this section we give a direct proof of local well-posedness of (6.3) in a number of Banach spaces.

Proposition 7.1 *The initial value problem for equation (6.3) is well-posed in each of the following Banach spaces.*

(i) $X=\{v\in L^\infty, v_x\in L^1\cap L^2\}$ *with norm* $|v|_\infty+|v_x|_{1,2}$.

(ii) $X=\{v\in L^\infty, v_x\in L^2\}$ *with norm* $|v|_\infty+|v_x|_2$.

(iii) $X=H^1$ *with norm* $|v|_2+|v_x|_2$.

(iv) $X=\{v\in L^2,\ v_x\in L^1\cap L^2\}$ *with norm* $|v|_2+|v_x|_{1,2}$.

For each of these spaces it is also well-posed in $X_1=\{v\in X,\ v_{xx}\in L^2\}$ *with norm* $\|v\|_{X_1}=\|v\|_X+|v_{xx}|_2$.

Proof. We rewrite the equation (6.3) as

$$v_t-A_\varepsilon v=G_\varepsilon(v_x^2+2\phi v_x). \tag{7.1}$$

We first observe that the linear operator A_ε is bounded in X and in X_1. This follows from $(A_\varepsilon v)_x=A_\varepsilon v_x$ and $(A_\varepsilon v)_{xx}=A_\varepsilon v_{xx}$ and the boundedness of A_ε on L^∞, L^1 and L^2, see Section 2.

Next we prove that the operator $v\to G_\varepsilon(v_x\phi)$ is bounded in X and in X_1. It is bounded on L^2 and L^∞ because for all $v\in L^p$ $(1<p\le\infty)$ we have, writing $\phi v_x=(\phi v)_x-\phi_x v$,

$$|G_\varepsilon(\phi v_x)|_p\le|g_\varepsilon'*(\phi v)|_p+|g_\varepsilon*(\phi_x v)|_p\le\frac{1}{\varepsilon}|\phi v|_p+|\phi_x v|_p\le(\frac{1}{\varepsilon}|\phi|_\infty+|\phi'|_\infty)|v|_p.$$

Moreover,

$$|(G_\varepsilon(\phi v_x))_x|_p=|g_\varepsilon'*(\phi v_x)|_p\le|g_\varepsilon'|_1|\phi v_x|_p\le\frac{1}{\varepsilon}|\phi|_\infty|v_x|_p.$$

Thus the operator $v\to G_\varepsilon(v_x\phi)$ is bounded in X for each of the choices of X. Finally if $v_x\in L^2$ and $v_{xx}\in L^2$, then

$$|(G_\varepsilon(\phi v_x))_{xx}|_2\le|g_\varepsilon'*(\phi v_{xx})|_2+|g_\varepsilon'*(\phi_x v_x)|_2\le\frac{1}{\varepsilon}(|\phi|_\infty|v_{xx}|_2+|\phi'|_\infty|v_x|_2),$$

so $v\to G_\varepsilon(v_x\phi)$ is also bounded in each X_1.

It remains to show that the map $v\to G_\varepsilon v_x^2$ is locally Lipschitz continuous in X and in X_1. It is well defined on X because with $f=v_x^2\in L^1$ the solution $w=G_\varepsilon v_x^2$ of (2.4) has $w,w_x\in L^2$ and $w,w_{xx}\in L^1$. If in addition $v_{xx}\in L^2$, then $f_x=(v_x^2)_x=2v_vv_{xx}\in L^1$, so that w_x has the same properties as just formulated for w and in particular $w_{xx}\in L^2$. The local Lipschitz continuity in each X follows from the estimates

$$|G_\varepsilon((v_1)_x^2-(v_2)_x^2)|_p\le|g_\varepsilon|_p|(v_1+v_2)_x|_2|(v_1-v_2)_x|_2,$$

which we use for $p = 2$ and $p = \infty$,

$$|(G_\varepsilon((v_1)_x^2 - (v_2)_x^2))_x|_1 = |g'_\varepsilon * ((v_1)_x^2 - (v_2)_x^2)|_1 = |B_\varepsilon((v_1)_x^2 - (v_2)_x^2)|_1 \leq \\ ||B_\varepsilon||_{B(L^1)}|(v_1 + v_2)_x|_2|(v_1 - v_2)_x|_2.$$

and

$$|(G_\varepsilon((v_1)_x^2 - (v_2)_x^2))_x|_2 = |g'_\varepsilon * ((v_1)_x^2 - (v_2)_x^2)|_2 = |B_\varepsilon((v_1)_x^2 - (v_2)_x^2)|_2 \leq \\ ||B_\varepsilon||_{B(L^1,L^2)}|(v_1 + v_2)_x|_2|(v_1 - v_2)_x|_2.$$

The local Lipschitz continuity in each X_1 , i.e. the estimate for the L^2-norm of the difference of the second order derivatives, is left to the reader. □

Remark 7.2 *As long as $v_x \in L^2$ the operator $v \to G_\varepsilon v_x^2$ is Lipschitz continuous in X. Thus if $v_x \in L^2$ for all $t > 0$ then the solution of (6.3) for $v_0 \in X$ exists globally. As for $v_0 \in X_1$, v exists globally if $v_x \in L^2$ and $v_{xx} \in L^2$ for all $t > 0$.*

8 Global existence and stability

In this section we establish two equalities for solutions of the integrated equation (6.3) and deduce from them global existence for small initial data and stability properties of the zero solution of equation (6.3). The first comes from testing the equation with v.

Lemma 8.1 *Any solution v of equation (6.3) in $C^1([0,T]; H^1)$ satisfies*

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(v^2 + \varepsilon^2 v_x^2) = -\int_{\mathbb{R}}\left((1-v)v_x^2 + \phi_x v^2\right). \tag{8.1}$$

Proof. We use again (7.1). Let $w = v_t - A_\varepsilon v$ and $f = v_x^2 + 2v_x\phi$. Then by assumption f is in L^1+L^2 and thus $w \in H^1$ is the (weak) solution of $-\varepsilon^2 w'' + w = f$, i.e.

$$\varepsilon^2 \int_{\mathbb{R}} w_x\varphi_x + \int_{\mathbb{R}} w\varphi = \int_{\mathbb{R}} f\varphi \quad \text{for all} \quad \varphi \in H^1.$$

Taking $\varphi = v$ and observing that

$$\int_{\mathbb{R}} fv = \int_{\mathbb{R}} vv_x^2 - \int_{\mathbb{R}} \phi_x v^2, \quad \varepsilon^2\int_{\mathbb{R}}(A_\varepsilon v)_x v_x + \int_{\mathbb{R}}(A_\varepsilon v)v = -\int_{\mathbb{R}} v_x^2,$$

we arrive at

$$\int_{\mathbb{R}}(vv_t + \varepsilon^2 v_x v_{xt}) = -\int_{\mathbb{R}}\left((1-v)v_x^2 + \phi_x v^2\right).$$

This equality is valid for each $t \in [0,T]$. □

The second equality is derived testing with v_x.

Lemma 8.2 *For any choice of X_1 in Proposition 7.1 any solution v of equation (6.3) in $C^1([0,T]; X_1)$ satisfies*

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\left(v_x^2 + \varepsilon^2 v_{xx}^2\right) = -\int_{\mathbb{R}} v_{xx}^2 + \int_{\mathbb{R}} \phi_x v_x^2 \tag{8.2}$$

Proof. We follow the proof of Lemma 8.1 above. Differentiating with respect to x we have with the same notation that $f_x = v_{xx}v_x + 2\phi v_{xx} + 2\phi_x v$ is in $L^1 + L^2$, $w_x = G_\varepsilon f_x$ is in H^1, so that

$$\varepsilon^2 \int_{\mathbb{R}} w_{xx}\varphi_x + \int_{\mathbb{R}} w_x \varphi = \int_{\mathbb{R}} f_x \varphi \quad \text{for all} \quad \varphi \in H^1.$$

Taking $\varphi = v_x$ and observing that

$$\int_{\mathbb{R}} f_x v_x = \int_{\mathbb{R}} \phi_x v_x^2, \quad \varepsilon^2 \int_{\mathbb{R}} (A_\varepsilon v)_{xx} v_{xx} + \int_{\mathbb{R}} (A_\varepsilon v)_x v_x = -\int_{\mathbb{R}} v_{xx}^2,$$

we arrive at

$$\int_{\mathbb{R}} (v_x v_{xt} + \varepsilon^2 v_{xx} v_{xxt}) = -\int_{\mathbb{R}} v_{xx}^2 + \int_{\mathbb{R}} \phi_x v_x^2.$$

This equality is again valid for each $t \in [0, T]$. □

Since the L^∞-norm is controlled by the H^1-norm, see (8.4) below, the first equality tells us that a solution in H^1 can be continued as long as $|v_x(\cdot)|_2$ remains bounded. The second equality shows that this is also the criterion for solutions in X_1 to be continued. In terms of u the condition for global existence is therefore that $|u(\cdot, t) - \phi(\cdot + t)|_2$ does not blow up in finite time. Next we show that this can be assured by a smallness condition on the initial data. It will be convenient to use the norm

$$||v||_\varepsilon = \left(\int_{\mathbb{R}} (v^2 + \varepsilon^2 v_x^2) \right)^{\frac{1}{2}},$$

which is equivalent to the standard H^1-norm:

$$||v||_\varepsilon \leq ||v|| \leq \frac{1}{\varepsilon} ||v||_\varepsilon. \tag{8.3}$$

Estimate (8.1) implies stability of the null solution in H^1, provided the travelling wave profile ϕ is monotone.

Proposition 8.3 *Let $\varepsilon^2 \leq \frac{1}{4}$. For every initial value $v_0 \in H^1$ with $||v_0||_\varepsilon < \sqrt{\varepsilon}$ there exists a unique solution $v : [0, \infty) \to H^1$ with $||v(\cdot, t)||_\varepsilon$ decreasing for all $t \geq 0$. Moreover*

$$\int_0^\infty |v_x(\cdot, t)|_2^2 dt < \infty,$$

whence also

$$\int_0^\infty |v(\cdot, t)|_\infty^4 dt < \infty,$$

This result is better than stability but slightly weaker than asymptotic stability: we do not get that $v(\cdot, t) \to 0$ in H^1.

Proof. In view of $\phi_x \geq 0$ and the estimate

$$|v^2|_\infty \leq \int_{\mathbb{R}} |2vv_x| \leq 2|v|_2 |v_x|_2 \leq \frac{1}{\varepsilon}|v|_2^2 + \varepsilon |v_x|_2^2 = \frac{1}{\varepsilon} ||v||_\varepsilon^2, \tag{8.4}$$

the assertion follows immediately from (8.1). In particular the solution has $1 - v$ bounded away from zero by a positive constant, C, so that upon integrating (8.1) we find the first estimate:

$$C \int_0^\infty \int_{\mathbb{R}} v_x^2 < \frac{1}{2} ||v_0||_\varepsilon^2.$$

Combining both estimates with the boundedness of $|v(\cdot, t)|_2$ gives the second estimate in the theorem. □

Next we obtain an asymptotic stability result using the stronger norms with $v_{xx} \in L^2$ and a combination of (8.1) and (8.2).

Proposition 8.4 *Let $\varepsilon^2 \leq \frac{1}{4}$ and $0 < \alpha < \frac{1}{\max \phi'}$. For every initial value $v_0 \in H^2$ with $||v_0||_\varepsilon < \sqrt{(1 - \alpha \max \phi')\varepsilon}$ (no assumption on the size of $|v_0''|_2$) there exists a unique solution $v : [0, \infty) \to H^2$ with not only*

$$\int_{\mathbb{R}} (v^2 + \varepsilon^2 v_x^2),$$

but also

$$\int_{\mathbb{R}} (v^2 + (\varepsilon^2 + \alpha) v_x^2 + \varepsilon^2 \alpha v_{xx}^2)$$

decreasing for all $t \geq 0$. Moreover, $t \to ||v_x(\cdot, t)||_\varepsilon$ is square integrable and converges to zero as $t \to \infty$. Finally, $|v(\cdot, t)|_\infty \to 0$ and $|v_x(\cdot, t)|_\infty \to 0$.

Proof. Combining (8.1) and (8.2) we have

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (v^2 + (\varepsilon^2 + \alpha) v_x^2 + \varepsilon^2 \alpha v_{xx}^2) = - \int_{\mathbb{R}} \phi_x v^2 - \int_{\mathbb{R}} (1 - v - \alpha \phi_x) v_x^2 - \alpha \int_{\mathbb{R}} v_{xx}^2. \quad (8.5)$$

As in the proof of Proposition 8.3, the first assertion follows immediately from (8.5). In particular the solution has $1 - v - \alpha \phi_x$ bounded away from zero by a positive constant. Note that Proposition 8.3 applies here as well.

To establish the asymptotic behaviour we note that we now have two decreasing functions. Taking the difference it follows that the function

$$t \to \int_{\mathbb{R}} (v_x(x, t)^2 + \varepsilon^2 v_{xx}(x, t)^2) dx$$

has a finite limit as $t \to \infty$. Integrating (8.5) over $(0, \infty)$ on t, it is also integrable over $(0, \infty)$ and thus the limit is zero. This proves the statement about v_x. The remaining assertion follows again using (8.4). □

We list the consequences that Proposition 8.3 and Proposition 8.4 have for solutions of (1.1).

Theorem 8.5 *Let $\varepsilon^2 \leq \frac{1}{4}$ and let $u_0 \in \mathcal{H} + L^1 \cap L^2$ be such that $v_0 \in L^2$, where*

$$v_0(x) = \int_{-\infty}^{x} (u(s, t) - \phi(s + t)) ds. \quad (8.6)$$

If v_0 is sufficiently small the solution exists globally and

$$t \to \int_{-\infty}^{\infty} |u(x,t) - \phi(x+t)|^2 dx$$

is both integrable and bounded on $[0,\infty)$. If in addition $v_0 \in H^2$ is sufficiently small, the solution has $(u(\cdot,t) - \phi(\cdot+t)) \to 0$ as $t \to \infty$ in H^1 and therefore also in L^∞ as $t \to \infty$. Without any restriction on ε and the norm of v_0 the solution is global if $v_0 \in H^2$.

9 Discontinuities

In this section we show that jump discontinuities persist as t evolves. The jumps do not move and decrease exponentially as t increases. We then focus on the case of initial data with a single jump at $x = 0$ and establish convergence of the solution to a travelling wave.

Theorem 9.1 *If $u_0 \in L^1 \cap L^\infty$ or $u_0 \in \mathcal{H} + L^1 \cap L^\infty$ has a jump discontinuity in $x = 0$, then so does the solution $u(\cdot,t)$ as long as it exists. Moreover*

$$u(0^+,t) \, - \, u(0^-,t) = \exp\left(\frac{-t}{\varepsilon^2}\right) \left(u_0(0^+) \, - \, u_0(0^-)\right),$$

in which the superscripts $-$ and $+$ indicate left and right limits.

Proof. We only discuss the case $u_0 \in \mathcal{H} + L^1 \cap L^\infty$ and write $u = \mathcal{H} + \bar{u}$, thus $\bar{u}$ satisfies the equation

$$\bar{u}_t + \frac{1}{\varepsilon^2}\bar{u} = \Phi_\varepsilon(\bar{u})$$

where $\Phi_\varepsilon(\bar{u}) = \frac{1}{\varepsilon^2} G_\varepsilon \bar{u} + B_\varepsilon \bar{u}^2 + 2B_\varepsilon(\mathcal{H}\bar{u}) + F_\varepsilon(\mathcal{H})$. Clearly

$$\Phi_\varepsilon : L^1 \cap L^\infty \to H^1 \text{ is continuous.} \tag{9.1}$$

By using the variation of constants formula we get

$$\bar{u}(\cdot,t) = \bar{u}(0,t) \exp\left(\frac{-t}{\varepsilon^2}\right) + \int_0^t \Phi_\varepsilon(\bar{u}(\cdot,t)) \, \exp\left(\frac{-(t-s)}{\varepsilon^2}\right) ds,$$

so $t \to \left(\bar{u}(\cdot,t) - \bar{u}_0 \exp\left(\frac{-t}{\varepsilon^2}\right)\right)$ is a continuously differentiable H^1-valued function. Since $u \in H^1 \mapsto u(0^+)$ and $u \in H^1 \mapsto u(0^-)$ are bounded linear functionals, the theorem follows. $\square$

In what follows we assume that $u_0 \in \mathcal{H} + L^1 \cap L^\infty$ and that u_0' is the sum of an L^2-function and a multiple of the δ-distribution with a point mass at the origin. Following the steps of the proof of Proposition 8.4 for the integrated equation (6.3) for (6.1) we shall prove that

$$|u - \phi|_\infty \to 0 \text{ as } t \to \infty.$$

Reasoning as in the proof of Theorem 9.1 we have $v_{xx} \in L^2(-\infty,0)$ and $v_{xx} \in L^2(0,\infty)$, as long as the solution exists, and that $v_t + \frac{1}{\varepsilon^2}v$ is continuous. However Lemma 8.2 does not apply because $v \notin H^2$. Instead we have the next lemma.

Lemma 9.2 *As long as the solution exists it satisfies*

$$\frac{1}{2}\frac{d}{dt}\left(\int_{-\infty}^{0}(v_x^2+\varepsilon^2 v_{xx}^2)+\int_{0}^{\infty}(v_x^2+\varepsilon^2 v_{xx}^2)\right)= \\ -\int_{-\infty}^{0} v_{xx}^2-\int_{0}^{\infty} v_{xx}^2+\int_{\mathbb{R}}\phi_x v_x^2+R(t), \tag{9.2}$$

where

$$R(t)=\exp\left(\frac{-t}{\varepsilon^2}\right)(u_0^+-u_0^-)\left(\frac{1}{3}((v_x^+)^2+v_x^+v_x^-+(v_x^-)^2)+\phi(t)(v_x^++v_x^-)-v_t(0,t)\right),$$

the superscripts $-$ and $+$ denoting the left and right limits at $x=0$.

Proof. The function $z=u-\phi$ satisfies equation (6.5) and thus $w=z_t+\frac{1}{\varepsilon^2}z$ is a solution of (2.4) with $f=\frac{1}{\varepsilon^2}z+(z^2)_x+2(\phi z)_x$. Using the formula

$$\varepsilon^2\int_0^\infty w'\varphi'+\varepsilon^2 w'(0^+)\varphi(0)+\int_0^\infty w\varphi=\int_0^\infty f\varphi$$

with $\varphi=z$ leads, after integrating by parts, to

$$\frac{1}{2}\frac{d}{dt}\int_0^\infty(z^2+\varepsilon^2 z_x^2)=-\int_0^\infty z_x^2+\int_0^\infty\phi_x z^2+R^+(t), \tag{9.3}$$

where

$$R^+(t)=-\frac{2}{3}(z^+)^3-\phi(t)(z^+)^2-\varepsilon^2 z^+(z_{xt}^++\frac{1}{\varepsilon^2}z_x^+).$$

Along the same lines we have

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^0(z^2+\varepsilon^2 z_x^2)=-\int_{-\infty}^0 z_x^2+\int_{-\infty}^0\phi_x z^2+R^-(t), \tag{9.4}$$

where

$$R^-(t)=\frac{2}{3}(z^-)^3+\phi(t)(z^-)^2+\varepsilon^2 z^-(z_{xt}^-+\frac{1}{\varepsilon^2}z_x^-).$$

Using $v_x=z$ we express $R^+(t)$ and $R^-(t)$ in terms of v. Since

$$\varepsilon^2 v_{xxt}^\pm+v_{xx}^\pm+\frac{2}{3}(v^\pm)_x^2+\phi v_x^\pm=v_t^\pm-\frac{1}{3}(v^\pm)_x^2-\phi v_x^\pm$$

it follows

$$R^+(t)=-(v_t^+v_x^+-\frac{1}{3}(v_x^+)^3-\phi(v_x^+)^2),$$
$$R^-(t)=(v_t^-v_x^--\frac{1}{3}(v_x^-)^3-\phi(v_x^-)^2).$$

Setting $R(t)=R^+(t)+R^-(t)$ completes the proof. □

With Lemma 8.1 and Lemma 9.2 we can now adapt Proposition 8.4.

Proposition 9.3 *Let $\varepsilon^2 < \frac{1}{4}$, $0 < \alpha < \frac{1}{\max \phi'}$. Let $u_0 \in \mathcal{H} + L^1 \cap L^\infty$ be such that u_0' is the sum of an L^2 function and a multiple of the δ-distribution with a point mass at $x = 0$. If v_0 is given by (8.6) and $\|v_0\|_\varepsilon < \sqrt{\varepsilon}$, there exists a unique global solution v of (6.3) with*

$$\|v\|_\varepsilon^2 = \int_{\mathbb{R}} (v^2 + \varepsilon^2 v_x^2)$$

decreasing for all $t \geq 0$. Moreover

$$\int_{\mathbb{R}} \left(v^2 + (\varepsilon^2 + \alpha) v_x^2\right) + \varepsilon^2 \alpha \left(\int_0^\infty v_{xx}^2 + \int_{-\infty}^0 v_{xx}^2 \right)$$

is eventually decreasing and

$$t \to \int_{\mathbb{R}} v_x^2 + \varepsilon^2 \int_0^\infty v_{xx}^2 + \varepsilon^2 \int_{-\infty}^0 v_{xx}^2$$

is integrable and converges to zero as $t \to \infty$. Finally $|v(\cdot, t)|_\infty \to 0$ and $|v_x(\cdot, t)|_\infty = |u(\cdot, t) - \phi(\cdot + t)|_\infty \to 0$ as $t \to \infty$.

Proof. Proposition 8.3 implies global existence and the monotonicity of $\|v\|_\varepsilon$. Combining (8.1) and (9.2) we get

$$\frac{1}{2} \frac{d}{dt} \left(\int_{\mathbb{R}} \left(v^2 + (\alpha + \varepsilon^2) v_x^2\right) + \varepsilon^2 \alpha \int_{-\infty}^0 v_{xx}^2 + \varepsilon^2 \alpha \int_0^\infty v_{xx}^2 \right) \leq$$
$$- \int_{\mathbb{R}} (1 - v - \alpha \max \phi') v_x^2 - \int_{\mathbb{R}} \phi_x v^2 - \alpha \int_{-\infty}^0 v_{xx}^2 - \alpha \int_0^\infty v_{xx}^2 + \alpha\, R(t). \tag{9.5}$$

If $R(t) \to 0$ as $t \to \infty$ we have

$$\lim_{t \to \infty} \left(\int_{\mathbb{R}} \left(v^2 + (\alpha + \varepsilon^2) v_x^2\right) + \varepsilon^2 \alpha \int_{-\infty}^0 v_{xx}^2 + \varepsilon^2 \alpha \int_0^\infty v_{xx}^2 \right) < \infty,$$

which implies the rest of the statements by the same arguments as in Proposition 8.4.

Thus it remains to show that $R(t) \to 0$ as $t \to \infty$. We observe that $|R(t)|$ is bounded for all $t \geq 0$,

$$|R(t)| \leq \exp\left(\frac{-t}{\varepsilon^2}\right) |u_0^+ - u_0^-| \left(|v_x^2|_\infty + 2|\phi|_\infty |v_x|_\infty + |v_t(0, t)| \right).$$

We shall prove that the last factor cannot increase as fast as $\exp\left(\frac{t}{\varepsilon^2}\right)$ as $t \to \infty$.

The last term inside brackets, $|v_t(0, t)|$, is bounded. Indeed, v satisfies equation (6.3), so, by the maximum principle we get

$$v_t + \frac{1}{\varepsilon^2} v \leq |v_x^2|_\infty + 2|\phi|_\infty |v_x|_\infty + \frac{1}{\varepsilon^2} |v|_\infty$$

for all $(x, t) \in \mathbb{R} \times [0, T]$. In particular

$$|v_t(0, t)| \leq |v_x^2|_\infty + 2|\phi|_\infty |v_x|_\infty + \frac{2}{\varepsilon^2},$$

where we have used that $\|v\|_\varepsilon$ decreases with t and $\|v_0\|_\varepsilon < \sqrt{\varepsilon}$. Thus

$$|R(t)| \leq \exp\left(\frac{-t}{\varepsilon^2}\right)|u_0^+ - u_0^-|\left(2|v_x|_\infty^2 + 4|\phi|_\infty|v_x|_\infty + \frac{2}{\varepsilon^2}\right). \tag{9.6}$$

Note that

$$\varepsilon|v_x|_\infty^2 \leq y(t) \overset{\text{def}}{=} \left(\int_{-\infty}^0 (v_x^2 + \varepsilon^2 v_{xx}^2) + \int_0^\infty (v_x^2 + \varepsilon^2 v_{xx}^2)\right),$$

so, in view of (9.6)

$$|R(t)| \leq \exp\left(\frac{-t}{\varepsilon^2}\right)|u_0^+ - u_0^-|\left(\frac{2}{\varepsilon}y(t) + \frac{4}{\sqrt{\varepsilon}}|\phi|_\infty\sqrt{y(t)} + \frac{2}{\varepsilon^2}\right). \tag{9.7}$$

From (9.2) we also get

$$\frac{1}{2}y'(t) \leq |\phi'|_\infty y(t) + |R(t)|y(t). \tag{9.8}$$

Next we obtain an estimate for $y(t)$ as $t \to \infty$ which, combined with (9.7) gives the behaviour of $|R(t)|$ as $t \to \infty$. By (9.8) and (9.7),

$$\frac{1}{2}y'(t) \leq |\phi'|_\infty y(t) + \exp\left(\frac{-t}{\varepsilon^2}\right)|u_0^+ - u_0^-|\left(\frac{2}{\varepsilon}y(t) + \frac{4}{\sqrt{\varepsilon}}|\phi'|_\infty\sqrt{y(t)} + \frac{2}{\varepsilon^2}\right). \tag{9.9}$$

With $\sigma(t) = \frac{4}{\varepsilon}\exp\left(\frac{-t}{\varepsilon^2}\right)|u_0^+ - u_0^-|$ as integrating factor,. $w(t) = e^{\sigma(t)}y(t)$ satisfies

$$w'(t) \leq 2|\phi'|_\infty w(t) + 2\exp\left(\frac{-t}{\varepsilon^2} + \frac{\sigma(t)}{2}\right)|u_0^+ - u_0^-|\left(\sqrt{w(t)} + \frac{2}{\varepsilon^2}\right).$$

Since $\exp\left(\frac{-t}{\varepsilon^2} + \frac{\sigma(t)}{2}\right) \to 0$ as $t \to \infty$, there exist $\tilde{t} > 0$ and a positive constant C such that

$$w(t) \leq \exp\Big(2t|\phi'|_\infty\Big)\, C \quad \text{for all } t \geq \tilde{t},$$

and, possibly with a larger constant,

$$y(t) \leq \exp\Big(2t|\phi'|_\infty\Big)\, C \quad \text{for all } t \geq \tilde{t}. \tag{9.10}$$

By substituting (9.10) into (9.7), we get

$$|R(t)| \leq \exp\left(\frac{-t}{\varepsilon^2} + 2t|\phi'|_\infty\right)|u_0^+ - u_0^-|\, r(t) \text{ for all } t \geq \tilde{t},$$

where

$$r(t) = \left(\frac{2}{\varepsilon}C + |\phi|_\infty \exp\Big(-t|\phi'|_\infty\Big)\sqrt{C} + \frac{2}{\varepsilon^2}\exp\Big(-2t|\phi'|_\infty\Big)\right)$$

is uniformly bounded in t. The upper bound (3.6) for ϕ' implies $2|\phi'|_\infty < \frac{1}{\varepsilon^2}$, thus $R(t) \to 0$ as $t \to \infty$. $\square$

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