# Parametric dependence of exponents and eigenvalues in focusing porous media flows

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We study the hole-filling problem for the porous medium equation $u_t = \frac{1}{m}\Delta u^m$ with $m > 1$ in two space dimensions. It is well-known that it admits a radially symmetric self-similar focusing solution $u = t^{2\beta-1}F(|x|t^{-\beta})$, and we establish that the self-similarity exponent $\beta$ is a monotone function of the parameter $m$. We subsequently use this information to examine in detail the stability of the radial self-similar solution. We show that it is unstable for any $m > 1$ against perturbations with 2-fold symmetry. Besides, we prove that as $m$ is varied there are bifurcations from the radial solution to self-similar solutions with $k$-fold symmetry for each $k = 3, 4, 5, \ldots$. These bifurcations are simple and occur at values $m_3 > m_4 > m_5 > \cdots \to 1$.

## 1 Introduction

In this paper we consider the hole-filling Aronson-Gravelau (AG) solutions of the porous medium equation

$$mu_t = \Delta u^m, \tag{1.1}$$

which we shall write in terms of the pressure variable

$$v = \frac{u^{m-1}}{m-1}$$

as

$$v_t = (m-1)v\Delta v + |\nabla v|^2. \tag{1.2}$$

Here $m$ is a fixed real and usually positive number. The space dimension is denoted by $N$.

Equation (1.1) arises in several applications and references go back as far as [22] in the context of gas flows in porous media ($m \geq 2$), from which the equation derives its name. It also arises in the context of high temperature hydrodynamics (with various values of $m$) [25], mathematical biology [15], superconductors (with sign changing solutions) [14], differential geometry ($m < 0$) [12] and in the study of flows of thin viscous films ($m = 4$) [11]. It is a prototypical nonlinear extension of the linear diffusion equation $u_t = \Delta u$.

We restrict our attention to $m > 1$ and nonnegative (weak) solutions. Our main interest is in the behaviour of the support of the solution. The Zel'dovich-Kompaneetz-Barenblatt (ZKB) point source solutions [24, 8], which, in terms of $v$, are given by

$$v(x,t)=t^{2\beta-1}F(\eta),\quad \eta=\frac{x}{t^{\beta}},\quad \beta=\frac{1}{(m-1)N+2},\quad F(\eta)=(C-\frac{\beta}{2}|\eta|^2)_+,$$

show that weak solutions have supports which propagate with a finite speed [5] given by the length of the gradient of $v$ at the boundary. Formally this behaviour is explained by dropping the first term on the right-hand side of (1.2) at the boundary of the support.

Using ZKB solutions as subsolutions and the comparison principle for weak solutions, one shows that the support of a solution eventually reaches every point in space. This means that any holes in the support which may exist initially, disappear in finite time. Assuming that such a hole vanishes at time $t=T$ in one point $x=x_0$ one can try to describe this process by zooming in at $(x_0,T)$ using self-similar variables

$$v(x,t)=(T-t)^{2\beta-1}w(\eta,\tau),\quad \eta=\frac{x-x_0}{(T-t)^{\beta}},\quad \tau=-\log(T-t). \tag{1.3}$$

In the new variables the pressure equation (1.2) reads

$$w_\tau=(m-1)w\Delta w+|\nabla w|^2-\beta\eta\cdot\nabla w+(2\beta-1)w. \tag{1.4}$$

The AG solutions are characterised by two properties: (i) they are radially symmetric equilibria $F(|\eta|)$ of (1.4) supported on the complement of a ball; (ii) they define self-similar solutions of (1.2) having a trace at $t=T$ (so that the solution may be continued for $t>T$).

The first property implies that $F(r)$ is a solution of the ordinary differential equation

$$(m-1)F(F''+\frac{N-1}{r}F')+F'^2-\beta rF'+(2\beta-1)F=0, \tag{1.5 a}$$

supported on an interval $[r_0,\infty)$ with $F(r)\to 0$ as $r\to r_0$. Formally, for a well-behaved solution we will have $F'(r_0^+)=\beta r_0$. In fact the weaker condition that $F(r)$ is positive and sub-linear as $r\to r_0$ already defines a unique local solution $F(r)$ which only depends on $r_0$ and this solution may be obtained by scaling the solution with $r_0=1$. Thus we need only consider the solution of (1.5 $a$) with

$$F(1)=0;\quad F'(1^+)=\beta. \tag{1.5 b}$$

With $m>1$ fixed this solution $F(r)$ depends only on the similarity exponent $\beta$.

The second property imposes an algebraic growth condition on $F(r)$ as $r=|\eta|\to\infty$, namely $F(r)\sim Cr^{\varepsilon}$ with $\varepsilon$ defined in (1.6) below. As we explain in Section 2, there is only one (positive) value of $\beta$, for which the solution of (1.5) has this property. The phase plane reduction we use to analyse (1.5 $a$) is different from the one used in [7], where the existence of a unique $\beta$ and corresponding self-similar solution was first proved.

Since the exponent $\beta$ is *not* explicitly determined from a conserved quantity of solutions (as in the case of the ZKB solutions), the AG solutions are self-similar solutions of the second kind. For $N>1$ we have the bounds $\frac{1}{2}<\beta<1$ (see [7]), and in the limit $t\uparrow T$

$$v(x,T)=C|x-x_0|^{\varepsilon},\quad \text{where}\quad 0<\varepsilon=\frac{2\beta-1}{\beta}<1\quad\text{and}\quad C>0. \tag{1.6}$$

Consequently solutions $v$ of (1.2) may not be Lipschitz continuous in space as long as they have holes in their supports. We note that $\beta = 1$ if $N = 1$. In [6] it is shown that $\varepsilon \to 0$ as $m \to \infty$ and $\varepsilon \to 1$ as $m \to 1$. Here we prove that $\varepsilon$, and thus $\beta$, is a strictly decreasing function of $m$. Moreover, we prove that the leading term asymptotics of $\varepsilon$ are given by

$$\frac{1}{m} \sim \frac{2}{\varepsilon} \exp\Big(-\gamma - \frac{2}{\varepsilon}\Big) \qquad \text{as} \quad m \to \infty,$$

where

$$\gamma = -\int_0^\infty \exp(-s) \log s \, ds$$

is Euler's constant, and

$$1 - \varepsilon \sim \frac{m-1}{4} \qquad \text{as} \quad m \to 1.$$

In [1] it has been shown that for radial solutions of (1.2) supported on a region between two concentric spheres, the inner sphere disappears as $t$ increases to some finite $T$ with a rate proportional to $(T-t)^\beta$. Moreover, in the variables (1.3), the solution converges to $F$ or one of its scalings. This phenomenon is called radial focusing.

It was conjectured that the $AG$-solutions describe the generic disappearance of holes in the supports of nonradial solutions and, in particular, that the $AG$-profiles are essentially stable. A first step in attempting to prove this conjecture is the linearised stability analysis of the $AG$-profiles $F$ as solutions of the partial differential equation (1.4). Roughly speaking, the self-similar variables (1.3) do not "see" what is happening away from the focusing point $(x_0, T)$ and so they do not see solutions which focus at other points in space-time. Any linearisation of (1.4) around $F$ will therefore have positive eigenvalues. Differentiating the $AG$-solutions with respect to $T$ and $x_0$ one may identify a priori the unstable eigenvalues $\omega = 1$ and $\omega = \beta$, even before finding the appropriate linearisation. Moreover, since the $AG$-solutions contain an additional free parameter $r_0$ we will also have an eigenvalue $\omega = 0$. The corresponding instabilities and neutral stability are unavoidable, and we say that the $AG$-profiles are essentially stable if there are no additional positive eigenvalues.

The two-dimensional hole-filling problem is studied numerically in [3] and [4]. The results give clear evidence of the instability of the Graveleau interface with respect to elongated perturbations (i.e., to perturbations with wave number $k = 2$). In addition, they clearly indicate the possibility of self-similar nonradial focusing with $k$-fold symmetry for $k = 3$ and $m$ fairly close to 1, with $k = 4, 5, 6, \ldots$ emerging as possible symmetries as $m \to 1$. The existence of these bifurcations is proved in [2] for sufficiently large wave numbers $k$. The analysis in [2] is independent of the number of space dimensions $N$. The net result of these investigations is the fact that the Graveleau profiles are not essentially stable. Here we investigate this instability in more detail for planar flows, and, in particular, answer several of the questions left open in [2].

In the appendix of [3] a linearisation around the radially symmetric AG solution is derived. Introducing polar coordinates $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$ the solution $w$ is represented as a graph

$$p = w(r, \theta, \tau)$$

in the $(r,p)$-plane, parametrised by $\theta$ and $\tau$. The relation between $p$ and $r$ is then inverted and written as

$$r = S(p,\theta,\tau),$$

and the equation for $S(p,\theta,\tau)$, which reads

$$\begin{aligned}(SS_p)^2S_\tau = \quad & (m-1)p\left(S^2S_{pp} - SS_p^2 + S_{pp}S_\theta^2 - 2S_pS_\theta S_{p\theta} + S_{\theta\theta}S_p^2\right) \\ & +\beta S^3S_p^2 - S^2S_p - S_pS_\theta^2 - (2\beta-1)pS^2S_p^3,\end{aligned} \tag{1.7}$$

is linearised around $\Psi = F^{-1}$. This produces a linear second order equation of the form

$$\xi_\tau = L\xi$$

with coefficients independent of $\theta$ and $\tau$ and singularities in $p=0$ and $p=\infty$ (see Section 2). Separation of variables yields solutions of the form

$$\xi_{ki}(p,\theta,\tau) = \exp(\omega_{ki}\tau)A_{ki}(p)\cos k\theta, \quad k,i = 0,1,2,\ldots,$$

where $k$ is the wave number and $i$ is the number of sign changes of the radial part $A_{ki}(p)$.

Our main interest is in the eigenvalues $\{\omega_{ki}(m)\}$. The a priori considerations explained above give

$$\omega_{00}(m) = 1,\ \omega_{01}(m) = 0, \quad \text{and} \quad \omega_{10}(m) = \beta(m) \qquad \text{for all } m \in (1,\infty).$$

As shown in [2], for each fixed $m$ the eigenvalues are monotone in both $k$ and $i$. Thus the only eigenvalues which can be positive (or nonnegative) are the $\omega_{k0}(m)$ for $k > 1$. The corresponding radial part of the eigenfunctions $A_{k0}(p)$ does not change sign. One of the main results of this investigation is that

$$\omega_{20}(m) > 0 \quad \text{for all } m \in (1,\infty).$$

Thus the $AG$-profiles are not essentially stable for any value of $m$.

Besides, for each $k > 2$ we may find the values $m = m_k$ for which the stability changes by solving

$$\omega_{k0}(m) = 0. \tag{1.8}$$

As explained and proved in [2], each such value of $m$ leads to the bifurcation from the radial branch of self-similar focusing solutions with $k$-fold symmetry. Although we are essentially only interested in integer values of the wave number $k$, it enters the equation for $A$ as a real parameter. We prove that equation (1.8) defines a smooth function $k = k(m)$ with

$$k(m) \to \infty \quad \text{as} \quad m \to 1, \qquad \text{and} \qquad k(m) \to 2 \quad \text{as} \quad m \to \infty.$$

Moreover, $k(m)$ is strictly monotone and hence invertible, so that $m_k$ is well-defined (single-valued). It also implies that $\frac{d\omega_{k0}}{dm}(m_k) > 0$, hence as $m$ is varied a *simple* bifurcation from the radial selfsimilar profile occurs at each of the values $m_3 > m_4 > m_5 > m_6 > \cdots$.

Let us briefly indicate the methods we use. We first reduce the ODE (1.5 $a$) for $F$ to a quadratic system of first order equations for the new unknown dependent variables

$$X = \frac{rF'}{F}, \quad Y = \frac{\beta r^2}{(m-1)F}$$

as functions of $\log r$. Starting from the resulting system for $X$ and $Y$ we obtain detailed information about the relation between the AG exponent $\beta$ and the exponent $m$ in (1.2). In terms of $\varepsilon$ as given in (1.6) and

$$\delta = \frac{1}{m-1}, \tag{1.9}$$

we obtain that $\delta$ is a strictly increasing function of $\varepsilon$ and show that

$$\delta \sim \frac{2}{\varepsilon}\exp\Big(-\gamma - \frac{2}{\varepsilon}\Big) \qquad \text{as} \quad \varepsilon \to 0, \tag{1.10}$$

and

$$4\delta \sim \frac{1}{1-\varepsilon} \quad \text{as} \qquad \varepsilon \to 1. \tag{1.11}$$

All this is done in Section 2 and relies on the analysis of one (non-autonomous) first order equation derived from the (autonomous) system for $X$ and $Y$.

Next, in Section 3 we reduce the equation for $A = A_{ki}(p)$ to a single first order equation which may be appended to the system for

$$X = \frac{rF'}{F} = \frac{\Psi}{p\Psi'} \quad \text{and} \quad Z = \frac{Y}{X} = \frac{\beta}{m-1}\Psi\Psi'$$

as functions of $\log r = \log \Psi$, where $\Psi$ is the inverse of $F$. The appropriate variable is

$$U = \frac{\Psi A'}{\Psi' A}.$$

Note that $A$ has no sign changes if $i = 0$.

From the equation for $U$ we can deduce the behaviour of well-behaved solutions at $p = 0$ and $p = \infty$, and give a direct ODE proof of the existence of a simple first eigenvalue $\omega_{k0}$ for every $k$. The positivity of $\omega_{20}$ is easily established. Moreover, setting $\omega_{k0}(m) = 0$ we prove monotonicity of $m_k$ for $k > 2$.

The first order equation for the "eigenfunction" $U$ naturally involves the profile of the self-similar solution about which one linearises. To analyse this equation we shall need qualitative information about the profile. Similarly, to study the dependence of the eigenvalues on $m$, we need detailed information about the solution of the $U$-equation. This means that in the next sections the theorems often include qualitative properties of solutions which may not seem relevant at first glance, but which are crucial to the subsequent analysis.

## 2 The AG solution in the phase plane

In this section we analyse the ODE

$$(m-1)F(F'' + \frac{1}{r}F') + {F'}^2 - \beta r F' + (2\beta - 1)F = 0, \tag{2.1 a}$$

for the AG profiles. Here we have already restricted attention to space dimension $N = 2$. The AG profiles are all scalings of a solution $F(r)$ with

$$F(1) = 0, \quad F'(1^+) = \beta, \quad F > 0 \text{ on } (1,\infty), \quad \lim_{r\to\infty} F(r) r^{\frac{1-2\beta}{\beta}} > 0. \tag{2.1 b}$$

In [7] equation (2.1 $a$) is transformed into a two-dimensional autonomous system and the AG profiles are identified as corresponding to a connection between a saddle and a saddle-node which is shown to exist for a unique $\beta$, which lies between $\frac{1}{2}$ and 1. The analysis here is based on a different transformation which also leads to a two-dimensional quadratic autonomous system. Setting

$$X = \frac{rF'}{F}, \quad Y = \frac{\beta r^2}{(m-1)F}, \quad t = \log r, \quad \delta = \frac{1}{m-1}, \quad \varepsilon = \frac{2\beta - 1}{\beta}, \tag{2.2}$$

we obtain

$$\dot{X} = -(\delta+1)X^2 + Y(X-\varepsilon); \qquad \dot{Y} = Y(2-X), \tag{2.3}$$

where we restrict the attention to orbits with $Y > 0$. The parameter $\delta > 0$ ranges from zero ($m \to \infty$) to infinity ($m \to 1$). The other parameter $\varepsilon$ must satisfy $0 < \varepsilon < 2$ in view of $\beta > \frac{1}{2}$. We note that for $N \neq 2$ the right-hand side of the $\dot{X}$-equation would also contain the term $(2-N)X$.

Transformations of this type have been used in [17], [21], [19], [20] and [18]. The resulting reduction requires a scaling invariance of the original second order equation (or system of equations). The transformations may be derived by introducing an $X$ as in (2.2) for each unknown $F$ in the system. For higher order equations one has to introduce also $\frac{r^2F''}{F}$ etc. (see e.g. [9]).

We briefly recall the phase-plane analysis of the system (2.3), see [17], [19] and [7]. There are two finite critical points, namely

$$(X,Y) = O = (0,0) \qquad \text{and} \qquad (X,Y) = P = (2, \tfrac{4(1+\delta)}{2-\varepsilon}).$$

The origin $O$ is a saddle-node. The only orbits coming out with $Y > 0$ are contained in the second quadrant and escape to infinity in finite time with $\phi \to \pi$ in terms of polar coordinates $X = R\cos\phi$, $Y = R\sin\phi$. The corresponding solutions $F(r)$ all hit zero with infinite slope and this disqualifies them as possible solutions of (2.1). This includes the solutions of (2.1 $a$) with finite $F(0) > 0$ which are contained in the orbit corresponding to the eigenvalue 2. The other orbits coming out of $O$ contain solutions of (2.1 $a$) which are singular in $r = 0$.

The point $P$ is a source for $1 < \varepsilon < 2$ but as $\varepsilon$ drops below 1 it undergoes a Hopf bifurcation with an unstable periodic orbit emerging, see [7]. Whether or not this periodic orbit is unique is not relevant here. Orbits going into $P$ or possibly a periodic orbit around $P$ contain solutions $F$ which grow too fast for (2.1 $b$) to hold and orbits coming out exist globally backwards in $t$ and can therefore not contain solutions $F$ with $F(1) = 0$.

There are four critical points at infinity, characterised by $R = \infty$ and, respectively,

$$\phi = 0, \quad \tan\phi = \delta, \quad \phi = \frac{\pi}{2}, \quad \phi = \pi. \tag{2.4}$$

All these points may be found and classified using either a Poincaré transformation (see [23]) or rewriting (2.3) as a system for $\rho$ and $\phi$ where $R = \frac{\rho}{1-\rho}$ (see [17]). The points with $\phi = 0$ and $\phi = \pi$ are respectively a source and a sink with orbits coming in/going out in finite time containing solutions $F$ hitting zero with infinite slope.

The two relevant points at infinity are the ones with $\tan\phi = \delta$ and $\phi = \frac{\pi}{2}$. The first one is a saddle with one orbit coming out of infinity in finite time. Normalising the $t$ at which

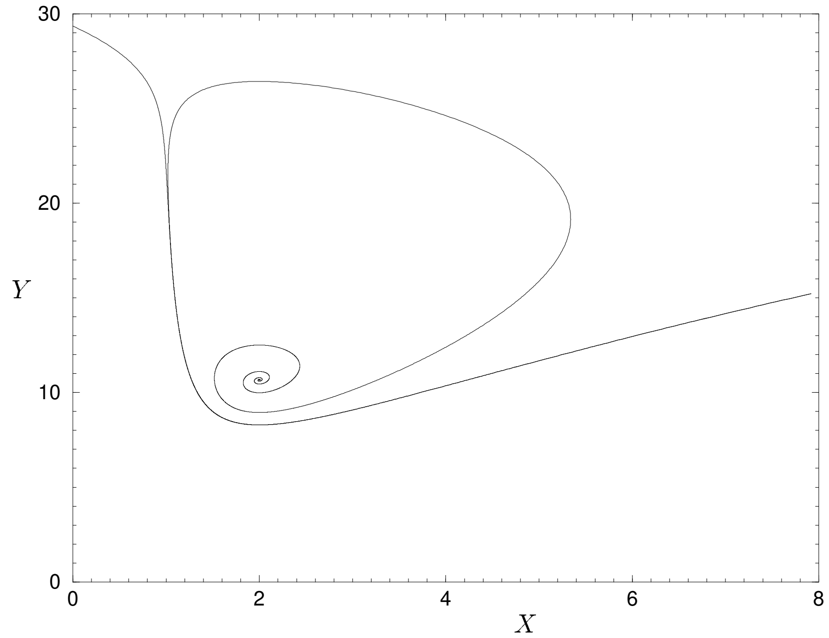


FIGURE 1. This figure shows two orbits close to the connection in the $(X, Y)$-phase plane for $m = \frac{3}{2}$ $(\delta = 2)$. We have taken $\epsilon$ related to $\delta$ by (1.11), i.e. $\epsilon = \frac{7}{8}$, and solved starting from $X = 8$ with $Y = 15.3165$ and $Y = 15.3169$.

it leaves infinity by $t = 0$ it follows from (2.2) that the corresponding $F$ has $F(1) = 0$ and $F'(1^+) = \beta$. The second one is a saddle-node with one unique orbit going in and this is the only orbit that escapes slowly to infinity, meaning as $t \to \infty$. The corresponding solutions $F$ have the appropriate algebraic growth required in (2.1 *b*). This follows from $X(t) \to \varepsilon$, $Y(t) \to \infty$ and some additional manipulations that we omit here.

In terms of (2.3) the result in [7] may now be reformulated as

**Theorem 2.1** *Let $\delta > 0$. For every $0 < \varepsilon < 2$ there is a unique orbit of (2.3) coming out of infinity in the first quadrant with $Y \sim \delta X$ and a unique orbit going into infinity in the first quadrant with $X \to \varepsilon$. There is a unique $\varepsilon$ for which these two orbits coincide. This $\varepsilon$ satisfies $0 < \varepsilon < 1$. The solutions contained in the connection exist on an interval $(t_0, \infty)$, where $t_0$ is a free parameter. Setting $t_0 = 0$ we obtain the solution of (2.1).*

The statement of this theorem is illustrated in Figure 1 which shows a picture of the phase plane for $m = \frac{3}{2}$ with $\delta$ and $\epsilon$ related by (1.11). The connection is situated in the vicinity of the two orbits drawn. In the subsections below we examine properties of the relation between $\varepsilon$ and $\delta$.

### 2.1 Monotonicity

The orbit of (2.3) representing the AG profiles has a monotone $X$-component so we may use $X$ as the independent variable in a first order equation for $Y$. Since $Y$ is unbounded it is convenient to introduce a new dependent variable which is bounded. In fact, setting

$$V = \frac{Y(X-\varepsilon)}{X^2} - \delta = 1 + \frac{\dot{X}}{X^2}, \quad \tau = \frac{2}{X}, \tag{2.5}$$

we arrive at

$$\frac{dV}{d\tau} = \frac{\delta + V}{1-V}\left(1 - \frac{V}{\tau} - \frac{1-V}{\lambda - \tau}\right), \quad \text{where} \quad \lambda = \frac{2}{\varepsilon} = \frac{2\beta}{2\beta - 1}, \tag{2.6 a}$$

with boundary conditions

$$V(0^+) = \lim_{\tau \downarrow 0} V(\tau) = 0, \quad V(\lambda^-) = \lim_{\tau \uparrow \lambda} V(\tau) = 1. \tag{2.6 b}$$

Note that $\frac{1}{2} < \beta < 1$ implies $\infty > \lambda > 2$.

The first condition in (2.6 $b$) is immediate from $X \to \infty$, $Y/X \to \delta$ and (2.5). As for the second condition, $V(\tau)$ is defined for all $0 < \tau < \lambda$ and the monotonicity of $X$ implies that $V(\tau) < 1$ in view of (2.5). In view of the right-hand side of of (2.6 $a$) such a solution either converges to 1 or to $-\infty$ as $\tau \uparrow \lambda$. The latter is excluded because combined with $X(t) \to \varepsilon$ it would follow from (2.5) that $\dot{X}(t) \to -\infty$ as $t \to \infty$, and this is impossible.

The equation for $V$ is singular in $\tau = 0$, $\tau = \lambda$ and $V = 1$. The orbits mentioned in the first statement of Theorem 2.1 correspond to locally defined solutions of (2.6 $a$) respectively starting from $\tau = 0$ and ending at $\tau = \lambda$. We reformulate Theorem 2.1 in terms of (2.6 $a$) and give a direct proof.

**Theorem 2.2** *Let $\delta > 0$ and $\lambda > 1$. There exists a unique solution $V = V_l(\tau)$ of (2.6 a) with $V_l(0) = 0$ defined in a right neighbourhood of $\tau = 0$. This solution has*

$$V_l'(0^+) = \frac{\delta}{1+\delta}(1 - \frac{1}{\lambda}). \tag{2.7}$$

*Also there exists a unique solution $V = V_r(\tau)$ with $V_r(\lambda) = 1$ defined in a left neighbourhood of $\tau = \lambda$. This solution has*

$$V_r'(\lambda^-) = 1 - \frac{1}{\lambda}.$$

*For each $\lambda > 2$ there exists a unique $\delta = \delta(\lambda) > 0$ such that (2.6) has a solution, i.e. $V_l \equiv V_r$. This solution has $V' > 0$ on $[0, \lambda]$. Finally, $\delta(\lambda)$ is a decreasing function of $\lambda$.*

**Proof** To prove local existence and uniqueness on the left we rewrite (2.6 $a$) as

$$\frac{dV}{d\tau} + \left(\frac{1}{\lambda - \tau} + \frac{\delta}{\tau} - 1 - \delta\right)V = \delta\left(1 - \frac{1}{\lambda - \tau}\right) + (1+\delta)\left(1 - \frac{1}{\tau}\right)\frac{V^2}{1-V},$$

which implies, for any solution with $V(\tau)$ bounded as $\tau \to 0$, that

$$V(\tau) = \int_0^\tau e^{(1+\delta)(\tau - s)}\left(\frac{s}{\tau}\right)^\delta \frac{\lambda - \tau}{\lambda - s}\left[\delta\left(1 - \frac{1}{\lambda - s}\right) + (1+\delta)\left(1 - \frac{1}{s}\right)\frac{V(s)^2}{1 - V(s)}\right] ds. \tag{2.8}$$

Local existence of a unique solution $V(\tau)$ of (2.8) follows from a standard contraction argument in a small ball centred around the origin in $C_b((0,T])$ (the space of bounded continuous functions on $(0,T]$) with $T$ sufficiently small. The contraction estimate uses the fact that

$$\int_0^\tau e^{(1+\delta)(\tau-s)}\left(\frac{s}{\tau}\right)^\delta \frac{\lambda-\tau}{\lambda-s}ds \sim \frac{\tau}{\delta+1}, \qquad \int_0^\tau e^{(1+\delta)(\tau-s)}\left(\frac{s}{\tau}\right)^\delta \frac{\lambda-\tau}{\lambda-s}\frac{ds}{s} \sim \frac{1}{\delta}, \tag{2.9}$$

as $\tau \to 0$.

Next we note that any solution $V(\tau)$ which remains bounded as $\tau \to 0$ has a well-defined limit $V(0)$. Taking $\tau \to 0$ in (2.8) this limit satisfies

$$V(0) = -\frac{1+\delta}{\delta}\frac{V(0)^2}{1-V(0)},$$

whence either $V(0) = 0$ or $V(0) = -\delta$. The smallness condition on the ball $C_b((0,T])$ needed to make the right-hand side of (2.8) a contraction excludes the possibility of $V(0) = -\delta$ for the fixed point solution. In fact, $V(0) = -\delta$ corresponds to $Y/X \to 0$, i.e. to the source $\phi = 0$ in (2.4).

Moreover, the part of the integral on the right-hand side of (2.8) containing $V$ is $o(V(\tau))$ as $\tau \to 0$ while the part not containing $V$ is asymptotic to

$$\frac{\delta}{\delta+1}\big(1-\frac{1}{\lambda}\big)\tau$$

as $\tau \to 0$. Thus $V$ satisfies (2.7). By the implicit function theorem, it depends smoothly on all parameters.

For the solution on the right we turn the picture around by setting

$$V = 1 - W, \quad \sigma = \lambda - \tau, \tag{2.10}$$

and obtain

$$\frac{dW}{d\sigma} = \frac{1+\delta-W}{W}\left(1 - \frac{1-W}{\lambda-\sigma} - \frac{W}{\sigma}\right), \tag{2.11}$$

with initial condition $W(0) = 0$. Clearly the right-hand side of (2.11) suggests that

$$W'(0) = 1 - \frac{1}{\lambda}, \tag{2.12}$$

so we set

$$W(\sigma) = \big(1-\frac{1}{\lambda}\big)\sigma(1-G(\sigma)),$$

whence

$$\begin{aligned}\frac{dG}{d\sigma} + \left(-\frac{1+\delta}{(\lambda-1)^2\sigma} + \frac{(1+\delta)\lambda}{(\lambda-1)\sigma^2} - \frac{2\lambda-\lambda^2+\delta}{(\lambda-1)^2(\lambda-\sigma)}\right)G \\ &= a(\sigma) + b(\sigma)\frac{G^2}{1-G} \\ &= \frac{1}{(\lambda-1)^2}\left(\frac{\lambda^2-3\lambda-\lambda\delta+3+2\delta}{\sigma} + \frac{(\lambda-2)(+\lambda-2-\delta)}{\lambda-\sigma}\right) \\ &+ \frac{1+\delta}{(\lambda-1)^2}\left(\frac{1}{\sigma} - \frac{\lambda(\lambda-1)}{\sigma^2} + \frac{1}{\lambda-\sigma}\right)\frac{G^2}{1-G}.\end{aligned}$$

The integrating factor for this equation is

$$\sigma^{-\frac{1+\delta}{(\lambda-1)^2}}(\lambda-\sigma)^{\frac{2\lambda-\lambda^2+\delta}{(\lambda-1)^2}}e^{-\frac{(1+\delta)\lambda}{(\lambda-1)\sigma}},$$

so, assuming $G(\sigma)$ bounded as $\sigma \to 0$, we have

$$G(\sigma) = \int_0^\sigma \left(\frac{s}{\sigma}\right)^{\frac{1+\delta}{(\lambda-1)^2}} \left(\frac{\lambda - s}{\lambda - \sigma}\right)^{\frac{2\lambda-\lambda^2+\delta}{(\lambda-1)^2}} e^{\frac{(1+\delta)\lambda}{(\lambda-1)}\left(\frac{1}{\sigma}-\frac{1}{s}\right)} \left(a(s) + \frac{b(s)G(s)^2}{1-G(s)}\right) ds. \tag{2.13}$$

Here we leave it to the reader to verify that solutions with $G(\sigma)$ unbounded as $\sigma \to 0$ cannot qualify to give solutions $W(\sigma)$ with $W(0) = 0$.

In (2.13) we have

$$a(s) = \frac{A(s)}{s}, \quad b(s) = \frac{B(s)}{s^2},$$

with $A(s)$ and $B(s)$ smooth near $s = 0$. We observe that, for any $\alpha > 0$ and $\beta > 0$,

$$\int_0^\sigma \left(\frac{s}{\sigma}\right)^\alpha e^{\beta\left(\frac{1}{\sigma}-\frac{1}{s}\right)} \frac{ds}{s} \sim \frac{\sigma}{\beta}, \qquad \int_0^\sigma \left(\frac{s}{\sigma}\right)^\alpha e^{\beta\left(\frac{1}{\sigma}-\frac{1}{s}\right)} \frac{ds}{s^2} \sim \frac{1}{\beta},$$

as $\sigma \to 0$ (cf. (2.9)). As a consequence, local existence of a unique solution $G(\sigma)$ of (2.13) follows again from a standard contraction argument in a small ball centred around the origin in $C_b((0,T])$ with $T$ sufficiently small. Taking $\sigma \to 0$ in (2.13) and reasoning as above this solution satisfies

$$\lim_{\sigma\to 0} G(\sigma) = G(0) = \frac{B(0)}{\beta}\frac{G(0)^2}{1-G(0)} = -\frac{G(0)^2}{1-G(0)},$$

whence $G(0) = 0$. Thus, in addition to $W(0) = 0$, $W$ satisfies (2.12).

To obtain the connection, we fix $\lambda > 2$ and shoot from the left with the solution $V(\tau) = V(\tau;\delta)$, $\delta > 0$ being the shooting parameter. Examining the $(\tau, V)$-plane, the isocline $V' = 0$ is given by

$$V = \frac{\tau(\lambda - 1 - \tau)}{\lambda - 2\tau}, \tag{2.14}$$

which consists of two branches, one to the left and one to the right of $2\tau = \lambda$. Note that a connection must have $V' > 0$ on $(0, \lambda)$. If $V(\tau)$ does not connect to $V(\lambda) = 1$ it has to follow one of two scenarios: either it hits $V = 1$ before $\tau = \lambda$, or it crosses the right branch of (2.14) (after which it must hit $V = 0$ before $\tau = \lambda$). By standard continuity arguments these two scenarios occur for, respectively, $\delta$ sufficiently large and $\delta$ sufficiently small, and the sets of $\delta$-values for which they occur are open. Thus there must exist at least one $\delta > 0$ for which neither of the two occurs. For such $\delta$ the solution must connect to $V(\lambda) = 1$.

Suppose that there exists another $\delta$-value for which the connection exists. It follows immediately from (2.6 $a$) that the connection with the larger $\delta$, which starts with a larger slope, cannot cross the other connection. Consider the flow of (2.6 $a$) with the larger $\delta$. All solutions between the two connections are trapped and converge to $V = 1$ as $\tau \to \lambda$, contradicting the uniqueness result for solutions of (2.6 $a$) with $V(\lambda) = 1$. Thus $\delta$ is unique and depends only on $\lambda$.

We finish by showing that $\delta = \delta(\lambda)$ is a decreasing function. Let $\lambda > \lambda_0 > 2$ and let $\delta \geq \delta(\lambda_0)$. In view of (2.7) and (2.6 $a$), the corresponding solution of (2.6 $a$) with $V(0) = 0$ starts above and cannot cross the connection corresponding to $\lambda_0$. In particular, it cannot connect to $V(\lambda) = 1$. Thus $\delta(\lambda) < \delta(\lambda_0)$. This completes the proof of Theorem 2.2. $\square$

In Figure 2 we show a plot of two solutions of (2.6 $a$) with $\varepsilon = \frac{1}{5}$ and two values of $\delta$

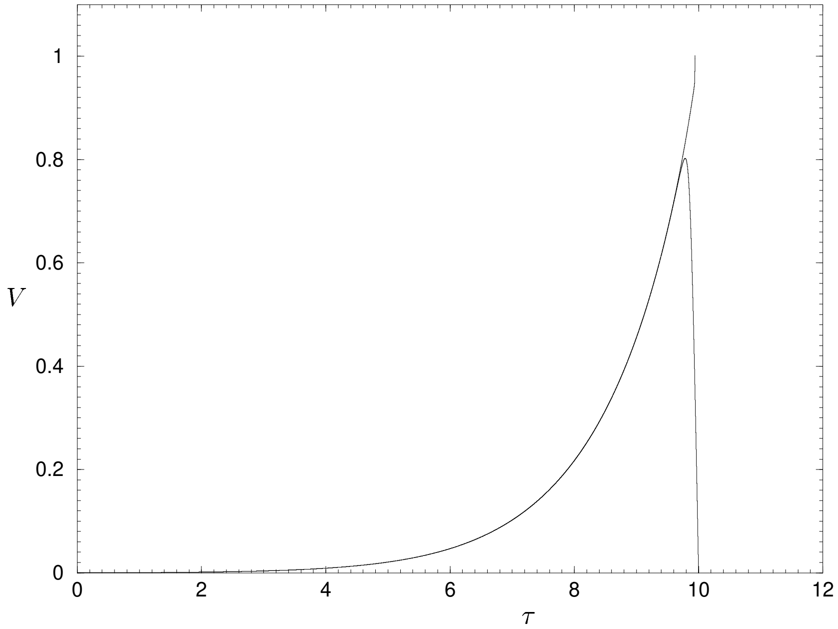


FIGURE 2. Two solutions close to the connection in the $(\tau, V)$-plane, $\varepsilon = \frac{1}{5}$. The values of $\delta$ used in the computation were $1.095 \cdot \delta_0$ and $1.096 \cdot \delta_0$ with $\delta_0$ given by the asymptotic formula (1.10).

close to the value given by (1.10). The actual connection lies close to the two graphs drawn and connects $(0,0)$ to $(\lambda, 1)$.

A direct consequence of Theorem 2.2 is

**Corollary 2.3** *The AG-exponent $\beta$ is an increasing function of $m$.*

There is also a monotonicity of the profiles but to see it we have to consider the problem on a fixed interval. Setting

$$\tau = \lambda t, \quad V(\tau) = v(t), \tag{2.15}$$

we consider the AG profiles as corresponding to the solution of

$$v' = \frac{dv}{dt} = \frac{\delta + v}{1 - v}\left(\lambda - \frac{v}{t} - \frac{1-v}{1-t}\right), \quad v(0) = 0, \quad v(1) = 1. \tag{2.16}$$

We first list some properties of the solution of (2.16) for $\lambda > 2$ fixed, which will be very useful later on.

**Proposition 2.4** *Let $\lambda > 2$. For $\delta = \delta(\lambda) > 0$ as in Theorem 2.2 the solution of (2.16) satisfies*

$$v'(0^+) = \frac{\delta}{\delta + 1}(\lambda - 1), \quad v'(1^-) = \lambda - 1. \tag{2.17}$$

$$v'(t) > 0, \quad 0 < v(t) < t \quad (0 < t < 1). \tag{2.18}$$

*Moreover, the function $A(t)$ defined by*

$$A(t) = \frac{v(t)}{t} + \frac{1-v(t)}{1-t}, \tag{2.19}$$

*satisfies*

$$A(0^+) = v'(0^+) + 1 = \frac{\delta}{1+\delta}(\lambda - 1) + 1, \quad A(1^-) = v'(1^-) + 1 = \lambda, \tag{2.20}$$

$$A'(t) > 0 \quad (0 < t < 1). \tag{2.21}$$

**Proof** From Theorem 2.2 we have (2.17), as well as $1 > v > 0$ and $v' > 0$ on $(0,1)$. To prove $v(t) < t$ we note that

$$v'|_{v=t} = \frac{\delta + t}{1-t}(\lambda - 2) > 1 \iff t > \frac{1 - \delta(\lambda - 2)}{\lambda - 1}. \tag{2.22}$$

We claim that

$$v'(0^+) = \frac{\delta}{1+\delta}(\lambda - 1) < 1. \tag{2.23}$$

Suppose not, then $\delta(\lambda - 2) \geq 1$ and, by (2.22), $v'|_{v=t} > 1$ for $0 < t < 1$. Thus the solution with $v(0) = 0$ can intersect $v = t$ at most once. If it does so then it has to be below $v = t$ for $t$ small, whence all solutions between it and $v = t$ are trapped as $t \to 0$, and therefore come out of $v(0) = 0$. This contradicts the uniqueness of the solution starting at $v(0) = 0$. Thus the solution with $v(0) = 0$ does not intersect $v = t$ and by the same argument it has to be above $v = t$. However, then the connection has $v'(1) \leq 1$ which is impossible in view of (2.17) and $\lambda > 2$. This proves (2.23) which implies that $v(t)$ starts below $v = t$. Now suppose it crosses $v = t$. Then by (2.22) it stays above $v = t$, which is again impossible. This proves (2.18).

As for $A(t)$, from its definition and (2.17) we have (2.20). To prove (2.21) we note that $A'(\frac{1}{2}) = 4 > 0$, that $v'(t) > 0$ implies that $A(t) < \lambda$, and that in any point where $A'(t) \leq 0$, we necessarily have $v''(t) > 0$. On the other hand, a direct computation shows that the level curves of

$$A = \frac{v}{t} + \frac{1-v}{1-t}, \tag{2.24}$$

see Figure 3, are strictly concave in $0 < v < t < 1$. For $t < \frac{1}{2}$ we have $A < 2$ and increasing $A$ increases $v$, while for $t > \frac{1}{2}$ we have $A > 2$ and increasing $A$ decreases $v$.

Suppose $A'(t_0) = 0$ for some $\frac{1}{2} < t_0 < 1$. Then the graph of $v(t)$ and the level curve $A = A(t_0)$ touch in $t = t_0$. The strict concavity of $A = A(t_0)$ and $v''(t_0) > 0$ imply that $A''(t_0) < 0$ whence $A'(t) < 0$ on some interval $(t_0, t_1)$. If $A'(t_1) = 0$ and $t_1 < 1$, we repeat this argument and conclude that $A''(t_1) < 0$ and $A'(t) < 0$ on some second interval $(t_1, t_2)$ as well as on $(t_0, t_1)$. This is impossible so it follows that $A'(t) < 0$ and $A(t) < A(t_0) < \lambda$ on $(t_0, 1)$, contradicting (2.20). Thus $t_0$ cannot exist and $A' > 0$ on $[\frac{1}{2}, 1)$.

Likewise suppose $A'(t_0) = 0$ for some $0 < t_0 < \frac{1}{2}$. By the same reasoning as above we now conclude that $A''(t_0) > 0$ and that $A' < 0$ on $(0, t_0)$. Hence the graph of $v$, which starts in the origin, where also the level curves of $A$ with $A < 2$ start, must intersect a level curve of $A$ with $A = A_1 < A(0)$ for some $t_1 \in (0, t_0)$. But then the graph of $v$

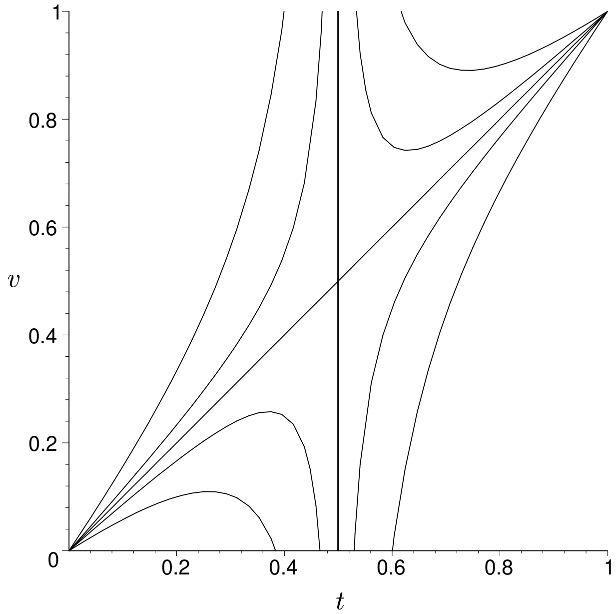


FIGURE 3. Level curves of $A$. The two straight lines correspond to $A = 2$. The curves without extrema have $A > 2$, the curves with extrema have $A < 2$.

lies above this level curve for $t \in (0, t_1)$, while the first is convex ($v'' > 0$ where $A' \le 0$) and the latter concave. This is impossible, so again we conclude that $t_0$ cannot exist and $A' > 0$ also on $(0, \frac{1}{2})$. This completes the proof of Proposition 2.4. $\square$

**Proposition 2.5** *Let $\delta = \delta(\lambda)$ and $v = v(t; \lambda)$ be the solution of (2.16). Then $\delta$ and $v$ are smooth functions of $\lambda$. Moreover, denoting derivatives with respect to $\lambda$ by subscripts, we have $\delta_\lambda < 0$ and $v_\lambda(t) < 0$.*

**Proof** We noted in the proof of Theorem 2.2 that the solutions starting on the left and on the right depend smoothly on $\delta$ and $\lambda$. Standard implicit function theorem arguments (see also below) applied to these two solutions at any $\lambda$ with corresponding $\delta$ and any $t \in (0, 1)$, imply that $\delta = \delta(\lambda)$ and $v = v(t; \lambda)$ are smooth in $\lambda$ and that

$$v'_\lambda = \left(\frac{1+\delta}{(1-v)^2}(\lambda - \frac{1}{t}) + \frac{1}{t} - \frac{1}{1-t}\right) v_\lambda + \frac{1}{1-v}\left(\lambda - \frac{v}{t} - \frac{1-v}{1-t}\right)\delta_\lambda + \frac{\delta + v}{1-v}, \quad (2.25)$$

with

$$v_\lambda(0) = v_\lambda(1) = 0. \quad (2.26)$$

Note that the coefficient of $\delta_\lambda$ is of fixed sign. This is in fact what guarantees that the invertibility condition holds in the implicit function theorem application mentioned above, whereby $\delta$ is smooth in $\lambda$. The sign of the coefficient of $\delta_\lambda$ coincides with that of $v'$ which is positive. The last term in (2.25) also being positive we conclude that $\delta_\lambda$ has to be negative because otherwise the whole inhomogeneous term in (2.25) is positive, impossible in view of (2.26).

Next we use the monotonicity of $A(t)$ in Proposition 2.4. Writing the inhomogeneous term in (2.25) as $F_I(t)/(1-v(t))$ with

$$F_I(t) = (\lambda - A(t))\delta_\lambda + \delta + v(t),$$

we see that $F_I'(t) > 0$ so that $F_I(t)$ has at most one sign change. In view of (2.26) again it cannot be of fixed sign so we conclude that there exists $t^* \in (0,1)$ such that $F_I < 0$ on $(0,t^*)$ and $F_I > 0$ on $(t^*,1)$. Solving (2.25) from the left and from the right we find $v_\lambda < 0$ on both $(0,t^*]$ and $[t^*,1)$. This completes the proof of Proposition 2.5. □

### 2.2 Behaviour as $m \to 1$

We continue the analysis in terms of the solutions of (2.16). Note that in view of (2.23)

$$\delta(\lambda - 2) < 1.$$

**Proposition 2.6** *Let $\delta = \delta(\lambda)$ and $v = v(t;\lambda)$ be the solutions of (2.16). Then $\delta(\lambda) \uparrow \infty$ as $\lambda \downarrow 2$.*

**Corollary 2.7** *In terms of $m$ and $\beta$ this means $\beta \uparrow 1$ as $m \downarrow 1$.*

**Proof** Suppose the assertion is false. Then $\delta \uparrow \delta^* < \infty$. Taking the limit in (2.16) we obtain a solution $v = v^*$ of

$$v' = \frac{dv}{dt} = \frac{\delta^* + v}{1-v}\left(2 - \frac{v}{t} - \frac{1-v}{1-t}\right), \quad v(0) = 0, \quad v(1) = 1,$$

satisfying also

$$v'(0^+) = \frac{\delta^*}{\delta^* + 1} < 1, \quad v'(1^-) = 1.$$

Starting from $t = 0$ we see that $v^*(t) < t$ for all $0 < t < 1$ while starting from $t = 1$ we find $v^*(t) > t$. This contradiction completes the proof of Proposition 2.6. □

**Proposition 2.8** *Let $\delta = \delta(\lambda)$ and $v = v(t;\lambda)$ be the solutions of (2.16). Then $v(t;\lambda) \uparrow t$ uniformly on $[0,1]$ as $\lambda \downarrow 2$.*

**Proof** By Proposition 2.5 the solutions $v(t;\lambda)$ increase to a limit as $\lambda \downarrow 2$. Let $A$ be defined by (2.24) (and $A(t)$ by (2.19)). The monotonicity, (2.17) and Proposition 2.6 imply that

$$v'(0^+) \uparrow 1, \quad A(0) \uparrow 2, \quad v'(1^-) \downarrow 1, \quad A(1) \downarrow 2,$$

as $\lambda \downarrow 2$. Since the graph of $v$ is above all the level curves of $A$ with respectively $A = A(0)$ for $0 < t < \frac{1}{2}$ and $A = A(1)$ for $\frac{1}{2} < t < 1$, it follows that $v(t;\lambda) \uparrow t$ for every $t \neq \frac{1}{2}$, and thus, in view of $v'(t;\lambda) > 0$, also for $t = \frac{1}{2}$. The convergence is uniform in view of Dini's theorem. This completes the proof of Theorem 2.8. □

**Proposition 2.9** *Let $\delta = \delta(\lambda)$ and $v = v(t;\lambda)$ be the solutions of (2.16). Then $2\delta(\lambda - 2) \to$ as $\lambda \to 2$.*

**Corollary 2.10** $1 - \beta \sim (m-1)/4$ *as $m \to 1$.*

**Proof** Recall the ODE for $v$ in (2.16),

$$v' = \frac{dv}{dt} = \Phi(t,v) \stackrel{\text{def}}{=} \frac{\delta+v}{1-v}\left(\lambda - \frac{v}{t} - \frac{1-v}{1-t}\right).$$

The assertion in the proposition is equivalent to $\Phi(\frac{1}{2},\frac{1}{2}) = (2\delta+1)(\lambda-2) \to 1$. Suppose this is false, then there exists a sequence $\lambda \to 2$ (dropping the index of the sequence from the notation) such that $\Phi(\frac{1}{2},\frac{1}{2})$ stays away from 1. Observing that

$$\Phi = \frac{\delta+v}{1-v}(\lambda - A), \quad \lambda \to 2, \quad A(t,t) = A(\tfrac{1}{2}, v) = 2,$$

we see that, by continuity, $\Phi(t,v)$ stays away from 1 in one of the two intersections $O \cap \{A < 2\}$ or $O \cap \{A > 2\}$ where $O$ is small neighbourhood of $(\frac{1}{2},\frac{1}{2})$. Clearly this makes it impossible to have $v(t;\lambda) \to t$ as $\lambda \to 2$. This contradiction completes the proof of Proposition 2.9. □

### 2.3 Behaviour as $m \to \infty$

In view of Corollary 2.3, (1.9) and (2.6 $a$), $m \to \infty$ corresponds to $\delta \to 0$ and $\lambda \to \infty$. We compute the asymptotic behaviour of $\delta$ and $V$ in Theorem 2.2 as $\lambda \to \infty$ in two steps. First we compute a nontrivial limit for the solution and then we use the limit solution to describe the asymptotics of $\delta$.

For the first step we turn the picture around as in the proof of Theorem 2.2 by (2.10), i.e. set $V = 1 - W$, $\sigma = \lambda - \tau$, and consider (2.11):

$$\frac{dW}{d\sigma} = \frac{1+\delta-W}{W}\left(1 - \frac{1-W}{\lambda-\sigma} - \frac{W}{\sigma}\right).$$

From Theorem 2.2 we know that

$$W(0) = 0, \quad W'(0) = 1 - \frac{1}{\lambda}, \tag{2.27}$$

and

$$W(\lambda) = 1, \quad W'(\lambda) = \frac{\delta}{1+\delta}\big(1 - \frac{1}{\lambda}\big). \tag{2.28}$$

We recall that in (2.27) and (2.28) the second condition for the derivative follows from the first condition.

**Proposition 2.11** *As $\delta \to 0$ and $\lambda \to \infty$ the solution $W = W(\sigma;\lambda,\delta)$ of (2.11) with (2.27) converges to the unique solution $\hat{W}(\sigma)$ of*

$$\frac{d\hat{W}}{d\sigma} = \frac{1-\hat{W}}{\hat{W}}\left(1 - \frac{\hat{W}}{\sigma}\right), \quad \hat{W}(0) = 0. \tag{2.29}$$

*The convergence is uniform on bounded intervals $[0,T]$.*

Rewriting (2.13) with a parameter dependence on $\frac{1}{\lambda}$ the proof of Proposition 2.11 is straightforward and left to the reader. The limit problem (2.29) is solvable by integration. Setting

$$1 - \hat{W} = \sigma y,$$

we write an equation for $\sigma$ as function of $y$ giving

$$\frac{d\sigma}{dy} = \sigma - \frac{1}{y}, \quad \sigma y \to 1 \quad \text{as} \quad y \to \infty. \tag{2.30}$$

The boundary condition at infinity follows from $\hat{W}(0) = 0$. Problem (2.30) has the unique explicit solution

$$\sigma = \exp(y) \int_y^\infty \exp(-s) \frac{ds}{s} = -\log y + \exp(y) \int_y^\infty \exp(-s) \log s ds,$$

so that

$$\sigma + \log y \to \int_0^\infty \exp(-s) \log s ds = -\gamma \quad \text{as} \quad y \to 0,$$

whence

$$1 - \hat{W}(\sigma) \sim \sigma \exp(-\gamma - \sigma) \quad \text{as} \quad \sigma \to \infty. \tag{2.31}$$

Now that we have a nontrivial limit for the profile, we use it to derive the asymptotics of $\delta$. In order to be able to use the limit solution for the asymptotics of $\delta$ we change the equations slightly before taking the limit. This is done by setting

$$W = (1+\delta)\tilde{W}, \quad \sigma = (1+\delta)\tilde{\sigma}, \quad \frac{1}{1+\delta} = 1 - \tilde{\delta}, \quad \lambda = (1+\delta)\tilde{\lambda}, \tag{2.32}$$

whence, omitting the tildes,

$$\frac{dW}{d\sigma} = \frac{1-W}{W}\left(1 - \frac{1-\delta-W}{\lambda - \sigma} - \frac{W}{\sigma}\right), \tag{2.33}$$

and the AG connection has

$$W(0) = 0 \quad \text{and} \quad W(\lambda) = 1 - \delta. \tag{2.34}$$

The limit equation is the same as before (and also Proposition 2.5 carries over with a slightly different proof). From (2.31) and (2.34) we guess that

$$\delta = 1 - W(\lambda) \approx 1 - \hat{W}(\lambda) \sim \lambda \exp(-\gamma - \lambda).$$

**Theorem 2.12**

$$\lim_{\lambda \to \infty} \frac{\delta \exp(\gamma + \lambda)}{\lambda} = 1.$$

*The limit is for $\tilde{\delta}$ and $\tilde{\lambda}$ but translating back to $\delta$ and $\lambda$ the formula remains the same.*

**Proof** We compare the solution $W(\sigma)$ with the solution $\hat{W}(\sigma)$ of (2.29) and the solution $W_1(\sigma; \lambda)$ of (2.33) with $\delta = 0$ and $W_1(0) = 0$. In view of the monotonicity of the right-hand side of (2.33) with respect to $\lambda$ and $\delta$ we have, by reasoning analogous to the proof of Theorem 2.2, that $W_1(\sigma) < W(\sigma) < \hat{W}(\sigma)$ for $0 < \sigma < \lambda$. The first inequality holds until $W_1(\sigma)$ hits zero. This happens after it has achieved a maximum in some $\sigma_\lambda$ between $\sigma = \lambda - 1$ and $\sigma = \lambda$ at the intersection with the isocline

$$\frac{1-W}{\lambda - \sigma} = 1 + \frac{W}{\sigma}. \tag{2.35}$$

Thus

$$1 - \hat{W}(\lambda) < \delta = 1 - W(\lambda) < 1 - W(\sigma_\lambda) < 1 - W_1(\sigma_\lambda; \lambda). \tag{2.36}$$

In (2.36) we have from (2.31) that $1 - \hat{W}(\lambda) \sim \lambda \exp(-\gamma - \lambda)$ as $\lambda \to \infty$. To finish the proof we have to show that also

$$\limsup_{\lambda \to \infty} \left(1 - W_1(\sigma_\lambda; \lambda)\right) \frac{\exp(\lambda)}{\lambda} \le \exp(-\gamma). \tag{2.37}$$

For (2.37) we need $\sigma_\lambda$ to get close to $\lambda$ which will follow from

$$W_1(\sigma_\lambda; \lambda) \to 1. \tag{2.38}$$

To prove (2.38), let $\epsilon > 0$. Then there exists $T > 0$ such that $0 < 1 - \hat{W}(\sigma) < \epsilon$ for all $\sigma \ge T$ because $\hat{W}(\sigma) \uparrow 1$ as $\sigma \to \infty$. But then, since Proposition 2.11 applies to $W_1(T; \lambda)$, which, being below $W = 1 - \delta$, increases with $\lambda$, there also exists $\Lambda > 0$ such that $0 < \hat{W}(T) - W_1(T; \lambda) < \epsilon$ for all $\lambda \ge \Lambda$. Thus, if also $\lambda - 1 \ge T$, we have $0 < 1 - W_1(\sigma_\lambda; \lambda) < 1 - W_1(T; \lambda) < 1 - \hat{W}(T) + \hat{W}(T) - W_1(T; \lambda) < 2\epsilon$. This proves (2.38). As a consequence we have in, view of (2.35), that

$$\lambda - \sigma_\lambda \to 0 \quad \text{as} \quad \lambda \to \infty. \tag{2.39}$$

Next we introduce

$$A_1(\sigma, \lambda) = \frac{\exp(\sigma)}{\sigma}(1 - W_1(\sigma, \lambda)),$$

which satisfies

$$\frac{dA_1}{d\sigma} = \frac{\sigma \exp(-\sigma) A_1^2}{1 - \sigma \exp(-\sigma) A_1} \left( \frac{1}{\lambda - \sigma} - 1 \right). \tag{2.40}$$

In view of (2.39) we have for every fixed $b > 0$ the existence of a $\Lambda$ such that $\sigma_\lambda > \lambda - b$ for all $\lambda \ge \Lambda$. In (2.37) we may then use

$$(1 - W_1(\sigma_\lambda; \lambda)) \frac{\exp(\lambda)}{\lambda} < A_1(\lambda - b; \lambda) \frac{\lambda - b}{\lambda} \exp(b).$$

Since $b > 0$ may be chosen arbitrarily small it remains to show that

$$A_1(\lambda - b; \lambda) \to \exp(-\gamma).$$

Note that for fixed $\sigma$

$$A_1(\sigma, \lambda) \to \hat{A}(\sigma) = \frac{\exp(\sigma)}{\sigma}(1 - \hat{W}(\sigma)).$$

and

$$\hat{A}(\sigma) \to \exp(-\gamma) \quad \text{as} \quad \sigma \to \infty.$$

Now fix $\epsilon > 0$. There exists $T > 0$ such that $|\hat{A}(\sigma) - \exp(-\gamma)| < \epsilon$ for all $\sigma \ge T$. Subsequently there also exists $\Lambda$ such that $|A_1(T; \lambda) - \hat{A}(T)| < \epsilon$ for all $\lambda \ge \Lambda$. Thus

$$|A_1(T; \lambda) - \exp(-\gamma)| < 2\epsilon. \tag{2.41}$$

For every $\lambda \ge \Lambda$ let $T_\lambda \le \lambda - b$ be the maximal value of $\sigma$ such that $|A_1(\sigma; \lambda) - \exp(-\gamma)| < 3\epsilon$ on the interval $[T, T_\lambda]$. Then on this interval

$$\frac{A_1(\sigma; \lambda)^2}{1 - \sigma \exp(-\sigma) A_1(\sigma; \lambda)} \le \frac{(\gamma + 3\epsilon)^2}{1 - T \exp(-T)(\gamma + 3\epsilon)} \le M, \tag{2.42}$$

for some $M > 0$. We may take $M$ to be fixed as we take smaller $\epsilon$- and larger $T$-values. It follows from (2.40) and (2.42) that

$$|A_1(\sigma;\lambda) - A_1(T,\lambda)| < M\big(\frac{1}{b}+1\big)(T+1)\exp(-T), \tag{2.43}$$

for every $T \leq \sigma \leq T_\lambda$. On the right-hand side of (2.43) $M$ and $b$ are independent of $T$ so we may a priori adjust the choice of $T$ above to ensure that also

$$M\big(\frac{1}{b}+1\big)(T+1)\exp(-T) < \epsilon. \tag{2.44}$$

From (2.41), (2.43) and (2.44) we then conclude for every $\lambda \geq \Lambda$ that

$$|A_1(\sigma;\lambda) - \exp(-\gamma)| < 3\epsilon,$$

for every $T \leq \sigma \leq T_\lambda = \lambda - b$, and in particular that

$$|A_1(\lambda - b;\lambda) - \exp(-\gamma)| < 3\epsilon.$$

This establishes the limit in Theorem 2.12. Inverting (2.32) this formula does not change and thus the proof is complete. □

## 3 The eigenvalue problem

In this section we consider the linearisation of (1.7) around the steady state $\Psi(p)$ and analyse solutions of the form

$$\xi(p,\theta,\tau) = \exp(\omega\tau)A(p)\cos(k\theta), \qquad k = 0,1,2,\ldots.$$

Here $\omega$ is the eigenvalue. Throughout this section $\beta$ (and hence $\varepsilon$) is the exponent corresponding to the AG solution and $\Psi(p)$ is the inverse function of the solution $F$ of (2.1).

The radial part $A(p)$ is a solution of the linear second order ODE

$$\begin{aligned}(m-1)p\Psi^2A'' + \big[2\beta\Psi^3\Psi' - \Psi^2 - 2(m-1)p\Psi\Psi' - 3(2\beta-1)p(\Psi\Psi')^2\big]A'\\ + \big[(m-1)(1-k^2)p + (\beta-\omega)\Psi^2\big](\Psi')^2A = 0.\end{aligned} \tag{3.1}$$

The coefficients contain the inverse AG solution $\Psi$ and its derivative $\Psi'$. As explained in [3] the eigenfunctions $A(p)$ should satisfy the conditions

$$A'(0^+) = \frac{\omega-\beta}{\beta^2}A(0), \qquad A(p) \sim Cp^{\frac{\beta-\omega}{2\beta-1}} \quad (p\to\infty). \tag{3.2}$$

For each $k = 0,1,2,\ldots$ this gives a decreasing sequence of real eigenvalues $\omega$, of which we only have to consider the first one and its corresponding *positive* eigenfunction $A(p)$.

We analyse the system consisting of the nonlinear second order ODE for $\Psi(p)$ and (3.1) using the independent variables

$$X = \frac{rF'}{F} = \frac{\Psi}{p\Psi'}, \quad Z = \frac{Y}{X} = \frac{\beta}{m-1}\Psi\Psi', \quad U = \frac{\Psi A'}{\Psi' A}$$

as functions of the new independent variable $\log r = \log\Psi$. With

$$\delta = \frac{1}{m-1}, \quad \varepsilon = \frac{2}{\lambda} = \frac{2\beta-1}{\beta}, \quad \mu = \frac{\omega}{\beta},$$

we obtain a system of three equations:

$$\dot{X} = -X\left((1+\delta)X - Z(X-\varepsilon)\right); \qquad \dot{Z} = Z\left(2 + \delta X - Z(X-\varepsilon)\right); \tag{3.3}$$

$$\dot{U} = k^2 - (U-1)^2 + \varepsilon Z(\mu - 1 + U) + Z(X-\varepsilon)(\mu - 1 - U). \tag{3.4}$$

In terms of $X$ and $Z$ the AG solution (see Theorem 2.1) corresponds to a solution of (3.3) with

$$X(0) = \infty, \quad Z(0) = \delta, \quad X(\infty) = \varepsilon, \quad Z(\infty) = \infty.$$

**Proposition 3.1** *Let $\omega$ be an eigenvalue with a positive eigenfunction $A(p)$. Then the corresponding solution $U(s)$ of (3.4) satisfies*

$$U(0) = \mu - 1, \quad U(\infty) = 1 - \mu. \tag{3.5}$$

**Proof** The boundary condition at $s = 0$ follows immediately from (3.2) and (2.1 $b$). Note that any solution $U(s)$ not satisfying $U(0) = \mu - 1$ converges to $\pm\infty$ as $s \to 0$ because $X(s) \sim 1/s$ and $Z(s) \to \delta$ for the AG connection. Likewise, if $U(s)$ does not satisfy $U(\infty) = 1 - \mu$, it must converge to $\pm\infty$ as $s \to \infty$ because $X(s) \to \varepsilon$ and $Z(s) \to \infty$. But then also $pA'/A \to \pm\infty$ which excludes algebraic behaviour. Thus we conclude that $U(\infty) = 1 - \mu$ and $pA'/A \to (1-\mu)/\varepsilon$, i.e. we have algebraic behaviour as $p \to \infty$. The additional argument giving the stronger statement that $A(p) \sim Cp^{(1-\mu)/\varepsilon}$ is omitted. □

### 3.1 The first eigenvalue

The first eigenvalue corresponds to the unique $\mu$-value for which (3.4) has a solution satisfying (3.5). As in Theorem 2.1 this $\mu$-value may be obtained by shooting from the left and from the right. We will use the variables $v$ and $t = \frac{\varepsilon}{X}$ introduced in (2.15), with the AG solution being a solution of (2.16), i.e.

$$\frac{dv}{dt} = \frac{\delta + v}{1 - v}\left(\lambda - \frac{v}{t} - \frac{1-v}{1-t}\right), \tag{3.6 a}$$

with boundary conditions

$$v(0) = 0, \quad v(1) = 1. \tag{3.6 b}$$

Equation (3.4) transforms into

$$\frac{dU}{dt} = \frac{1}{1-v}\left(\frac{k^2 - (1-U)^2}{2}\lambda - (v+\delta)\Big(\frac{U-\mu+1}{t} + \frac{1-\mu-U}{1-t}\Big)\right), \tag{3.7 a}$$

the boundary conditions being

$$U(0) = \mu - 1, \quad U(1) = 1 - \mu. \tag{3.7 b}$$

We now prove the existence of a unique first eigenvalue $\mu_k$ for all $k \geq 0$ and obtain the bounds $2 - k < \mu_k < 1$ for $k > 1$. Besides, we establish some properties of the "eigenfunction" $U$; in particular, we put some effort into proving that $\mu - 1 \leq U \leq 1 - \mu$, which will be very useful in what follows.

**Theorem 3.2** *Let $\delta = \delta(\lambda)$ as in Theorem 2.2, let $v(t)$ be the solution of (3.6) and let $\mu$ and $k \geq 0$ be real numbers. There exists a unique solution $U = U_l(t)$ of (3.7 a) with $U_l(0) = \mu - 1$ defined in a right neighbourhood of $t = 0$. This solution has*

$$U_l'(0^+) = \frac{(k^2 - (2-\mu)^2)\lambda}{2(\delta+1)} - \frac{2(1-\mu)\delta}{\delta+1}.$$

*Also there exists a unique solution $U = U_r(t)$ with $U_r(1) = 1 - \mu$ defined in a left neighbourhood of $t = 1$. This solution has*

$$U_r'(1^-) = \frac{(k^2 - \mu^2)\lambda}{2(\delta+1)} - 2(1-\mu).$$

*For each $k \geq 0$ there exists a unique $\mu = \mu_k$ such that (3.7) has a solution, i.e. $U_l \equiv U_r$. This $\mu_k$ is between $\mu = 1$ and $\mu = 2 - k$ if $k > 1$. This solution $U$ lies between its boundary values $\mu - 1$ and $1 - \mu$. Finally, $\mu_k$ is monotonically decreasing in $k$.*

**Corollary 3.3** *(instability of the AG solution) $\mu_2 > 0$.*

**Proof** The local existence and uniqueness from both sides is proved along the same lines as in the proof of Theorem 2.2. The value of the derivative in $t = 1$ follows because the second factor on the right-hand side of (3.7 $a$) must vanish as $t \to 1$. This occurs because the first factor is proportional to a multiple of $1/(1-t)$ as $t \to 1$. The value of the derivative at $t = 0$ is immediate from taking the limit $t \to 1$ in (3.7 $a$). By monotonicity arguments, $U_l$ is increasing in $\mu$ while $U_r$ is decreasing in $\mu$. This gives uniqueness of the $\mu$-value for which the two coincide.

To see that such a $\mu$-value must exist, we first assume that $k > 1$. If $\mu = 1$ the right-hand side of (3.7 $a$) is positive at $U = 1 - \mu = 0$, and also $U_l'(0^+) > 0$ and $U_r'(1^-) > 0$. This guarantees that, for as long as they exist, $U_l > 0 > U_r$ on $(0,1)$. In particular $U_l > U_r$ and clearly the set of $\mu$ for which this is the case is open. To find $\mu$ such that the other inequality holds we choose $\mu = 2 - k$, making the right-hand side of (3.7 $a$) negative at $U = \mu - 1 = 1 - k$ and $U_l'(0^+) < 0$, so $U_l < 1 - k < 0 < U_r$. Again the set of $\mu$-values for which $U_l < U_r$ holds is open. Thus there must exist a $\mu \in (2-k, 1)$ such that $U_l$ and $U_r$ are not ordered which is only possible if they coincide. This completes the proof for $k > 1$.

For $k = 1$ the solution is simply $\mu = 1$ and $U \equiv 0$. Concerning $0 \leq k < 1$ we note that all the signs in the argument above are reversed so now the existence of $\mu \in (1, 2-k)$ for which $U_l$ and $U_r$ coincide follows.

Following the similar arguments as for the monotonicity of $\delta(\lambda)$ in $\lambda$ (Proposition 2.5), it is not difficult to show that $\frac{d\mu}{dk} < 0$.

Finally, we prove the solution $U$ lies between its boundary values. We consider the case $k > 1$ ($k < 1$ is analogous). The function

$$\frac{k^2 - (2-\mu)^2}{4}\lambda - \big(v(t) + \delta\big)\frac{1-\mu}{1-t}$$

is increasing in $t$. Hence the right-hand side of (3.7 $a$) at $U = \mu - 1$ changes sign at most once on $[0,1)$ (from positive to negative). It follows that the solution $U \geq \mu - 1$, and in particular $U'(0+) \geq 0$, or equivalently

$$(k^2 - \mu^2)\lambda \geq 4(1-\mu)(\delta + \lambda). \tag{3.8}$$

The argument to show that $U \leq 1-\mu$ on $[0,1]$ is more involved. First, we note that it follows from (3.8) that

$$U'(1^-) = \frac{(k^2-\mu^2)\lambda}{2(\delta+1)} - 2(1-\mu) > 0.$$

Next, we assert that the right-hand side of (3.7 *a*) at $U = 1-\mu$ change sign exactly once for $t \in (0,1)$, from positive to negative. This claim implies that the solution $U \leq 1-\mu$ on $[0,1]$.

Consider

$$P(t) \stackrel{\text{def}}{=} \frac{k^2-\mu^2}{4(1-\mu)}\lambda - \frac{v(t)+\delta}{t}.$$

Clearly the right-hand side of (3.7 *a*) at $U = 1-\mu$ is zero if and only if $P = 0$. Notice that $P(0) = -\infty$ and $P(1) > 0$. Setting $P = 0$ one obtains $\frac{k^2-\mu^2}{4(1-\mu)}\lambda t = v + \delta < t + \delta$, which shows that $P(t)$ can only have zeros for

$$t \leq \frac{\delta}{\frac{k^2-\mu^2}{4(1-\mu)}\lambda - 1} \leq \frac{\delta}{\lambda+\delta-1}, \tag{3.9}$$

where we have used (3.8).

We now prove that if $P(t) = 0$ then $P'(t) > 0$, from which we infer that $P$ has at most and hence precisely one zero. In fact $P'(t) > 0$ for $t \leq \frac{\delta}{\lambda+\delta-1}$, the bound on zeros of $P$ derived above. Differentiation gives

$$P'(t) = -\frac{v'}{t} + \frac{v+\delta}{t^2} = \frac{v+\delta}{(1-v)t^2}\big(-t[\lambda - A(t)] + 1 - v\big),$$

where $A(t) \stackrel{\text{def}}{=} \frac{v}{t} + \frac{1-v}{1-t}$. Qualitative properties of $A$ were obtained in Proposition 2.4. Define

$$Q(t) \stackrel{\text{def}}{=} \frac{(1-v)t^2}{v+\delta}P'(t) = -t[\lambda - A(t)] + 1 - v.$$

Using the surprisingly useful property $A(t) > A(0) = \frac{\delta}{\delta+1}(\lambda-1)+1$ we obtain

$$Q(t) > -t[\lambda - A(t)] + 1 - t > 1 - t[1 + \lambda - A(0)] = 1 - t\Big[\lambda - \frac{\delta}{\delta+1}(\lambda-1)\Big]$$

and using (3.9) we continue the estimate

$$Q(t) > 1 - \frac{\delta}{\lambda+\delta-1}\Big[\lambda - \frac{\delta}{\delta+1}(\lambda-1)\Big] = \frac{\lambda-1}{(\lambda+\delta-1)(\delta+1)} > 0.$$

Hence $P'(t) > 0$ and the proof of the assertion is finished. This also completes the proof of Theorem 3.2. □

### 3.2 Dependence of the eigenvalues on $m$

We first examine the behaviour of the eigenvalues for large $m$.

**Proposition 3.4** *Let $\mu_k$ be as in Theorem 3.2. Then $\mu_k \to 2-k$ as $\lambda \to \infty$.*

**Proof** We turn to the time variable $\sigma = \lambda(1-t)$. We note that, as $\lambda \to \infty$ and $\mu \to \hat{\mu}$, the solution $\tilde{U}_r(\sigma) = U_r(1-\sigma/\lambda)$ converges to the unique solution of

$$\frac{d\hat{U}}{d\sigma} = \frac{1}{\hat{W}}\left(\frac{(1-\hat{U})^2-k^2}{2} + (1-\hat{W})\frac{1-\hat{\mu}-\hat{U}}{\sigma}\right), \quad \hat{U}(0) = 1-\hat{\mu}. \tag{3.10}$$

Here $\hat{W}(\sigma)$ is the function defined in Proposition 2.11. Suppose the theorem is false for $k > 1$. Then there must exists a sequence $\lambda \to \infty$ along which $\mu_k$ converges to a limit $\hat{\mu}$ contained in $(2-k, 1]$ and the solution $\tilde{U}(\sigma) = U(1-\sigma/\lambda)$ converges to the solution of (3.10), uniformly on compact intervals contained in the maximum interval on which $\hat{U}$ exists. This limit solution is easily seen to have $\hat{U}(\sigma) \to 1-k < \mu-1$ as $\sigma \to \infty$ and this forces $\tilde{U}(\sigma)$ to drop below $\mu - 1$ before $\sigma$ reaches $\lambda$, provided $\lambda$ is large, in contradiction with Theorem 3.2.

For $0 \le k < 1$ the argument is similar, the only difference being that the limit solution blows up in finite time if $\hat{\mu}$ is in $[1, 2-k)$. This completes the proof of Proposition 3.4. □

**Corollary 3.5**

$$\lim_{m\to\infty} \omega_{k0} = 1 - \frac{k}{2}.$$

Before we examine the other limit $m \to 1$ we note that changes in stability can only occur when $\mu_k = 0$ for some integer $k$ and $m > 1$. However, in (3.7 *a*), there is no reason to restrict $k$ to the integers. Thus we consider (3.7 *a*) with $\mu = 0$ for all real $k > 0$, and write $u = \frac{1}{2}(U+1)$ to simplify notation:

$$u' = \frac{1}{1-v}\left[\left(\frac{k^2}{4} - (1-u)^2\right)\lambda - (v+\delta)\left(\frac{u}{t} + \frac{1-u}{1-t}\right)\right]. \tag{3.11 a}$$

The boundary conditions are

$$u(0) = 0 \qquad \text{and} \qquad u(1) = 1. \tag{3.11 b}$$

**Theorem 3.6** *Let $\delta = \delta(\lambda)$ as in Theorem 2.2, let $v(t)$ be the solution of (3.6). Then there exists a unique $k = k(\lambda) > 2$ such that (3.11) has a solution. This solution $u$ lies between its boundary values 0 and 1. The function $k(\lambda)$ is smooth, $k(\lambda) \to 2$ as $\lambda \to \infty$ and $k(\lambda) \to \infty$ as $\lambda \to 2$.*

**Proof** Existence and uniqueness of $k$ are proved along the lines of the proof of Theorem 3.2, the right-hand side of (3.11 *a*) being monotone in $k$. With $\mu = 0$ we have that $u_l < 0$ if $k \le 2$, $u_l$ and $u_r$ are monotone in $k$, and choosing $k$ large $u_l$ has to grow above 1. That $u$ lies between its boundary values 0 and 1 follows from Theorem 3.2. The smoothness of $k(\lambda)$ follows again from the implicit function theorem (cf. Proposition 2.5). The limit $\lambda \to \infty$ is similar to Proposition 3.4. For the other limit $\lambda \to 2$ we argue by contradiction and assume $k$ remains bounded along a subsequence. Since $\delta \to \infty$ and, in view of Proposition 2.8, $v(t) \to t$, it follows that the right-hand side of (3.11 *a*) goes to $+\infty$, uniformly on sets where $u$ stays away from $u = 1$. This makes it impossible for $u$ to connect from $u = 0$ to $u = 1$ if $\lambda$ is close to 2, and this is a contradiction. $\square$

**Remark 3.7** *Since both $k$ and $\delta$ tend to infinity as $\lambda \to 2$, in this limit the terms in the right-hand side of (3.11 a) with $k$ and $\delta$ must balance. This implies that $k^2 \sim 2\delta$, i.e.*

$$k^2 \sim \frac{2}{m-1} \quad \textit{as} \quad m \to 1.$$

Finally, we prove that the function $k(\lambda)$ defined in Theorem 3.6 is monotone, i.e. $\frac{dk}{d\lambda} < 0$ for $\lambda \in (2, \infty)$. This implies that it is invertible and $\frac{d\lambda}{dk} < 0$. Besides, since $\mu(\lambda, k(\lambda)) = 0$, we have $\frac{\partial\mu}{\partial\lambda} = -\frac{\partial\mu}{\partial k}\frac{dk}{d\lambda} < 0$ at $\mu = 0$. A transformation to the original parameter $m$ (i.e. $\frac{dm}{d\lambda} = -\delta^{-2}\delta_\lambda > 0$) finishes the proof of the assertions at the end of Section 1.

**Theorem 3.8** *Let $k(\lambda)$ be as in Theorem 3.6. Then $\frac{dk}{d\lambda} < 0$ for $\lambda \in (2, \infty)$.*

**Proof** We denote differentiation with respect to $\lambda$ by subscripts. We recall that $v_\lambda < 0$ and $\delta_\lambda < 0$ by Proposition 2.5. Differentiating equation (3.11 $a$) with respect to $\lambda$ gives

$$\begin{aligned} u'_\lambda = \frac{1}{1-v}\Big\{ v_\lambda \Big[ u' - \Big(\frac{u}{t} + \frac{1-u}{1-t}\Big)\Big] - \delta_\lambda \Big(\frac{u}{t} + \frac{1-u}{1-t}\Big) \\ + \Big(\frac{k^2}{4} - (1-u)^2\Big) + \frac{k_\lambda k \lambda}{2} + u_\lambda(\ldots)\Big\}, \end{aligned} \tag{3.12}$$

where the dots represent the $u$-derivative of the second factor in the right-hand side of (3.11 $a$). The boundary conditions are $u_\lambda(0) = 0$ and $u_\lambda(1) = 0$, and

$$\begin{aligned} u'_\lambda(0) &= \Big(\frac{k^2}{4} - 1\Big)\Big(\frac{1}{\delta+1} - \frac{\delta_\lambda}{(\delta+1)^2}\Big) - \frac{\delta_\lambda}{(\delta+1)^2} + \frac{k k_\lambda \lambda}{2(\delta+1)}, \\ u'_\lambda(1) &= \frac{k^2}{4}\Big(\frac{1}{\delta+1} - \frac{\delta_\lambda}{(\delta+1)^2}\Big) + \frac{k k_\lambda \lambda}{2(\delta+1)}. \end{aligned}$$

Now suppose by contradiction that $k_\lambda \geq 0$ for some $\lambda$. Then $u'_\lambda(0) > 0$ and $u'_\lambda(1) > 0$, hence $u_\lambda(t)$ has a zero where it goes from positive to negative. At this point the fifth term in the right-hand side of (3.12) is zero, the second and third are positive and the fourth is nonnegative. If we show that

$$u' - \Big(\frac{u}{t} + \frac{1-u}{1-t}\Big) \leq 0 \tag{3.13}$$

then we conclude that $u'_\lambda > 0$, a contradiction.

We thus study

$$H(t) \stackrel{\text{def}}{=} u' - \Big(\frac{u}{t} + \frac{1-u}{1-t}\Big).$$

Clearly $H(0) = -1$ and $H(1) = -1$. We claim that $H(t) \leq 0$, i.e. (3.13) holds. Arguing by contradiction, we suppose $H(t) > 0$ for some $t$, and we define

$$t_0 \stackrel{\text{def}}{=} \inf\{t \,|\, H(t) > 0\}$$

and

$$t_1 \stackrel{\text{def}}{=} \sup\{\tilde{t} > t_0 \,|\, H(t) > 0 \text{ on } (t_0, \tilde{t})\}.$$

We have $0 < t_0 < t_1 < 1$ and $H$ has the following properties: $H(t_0) = H(t_1) = 0$, $H(t) > 0$ on $(t_0, t_1)$, $H'(t_0) \geq 0$ and $H'(t_1) \leq 0$.

Using the differential equation we obtain

$$H(t) = \frac{1+\delta}{1-v}\left\{\Big(\frac{k^2}{4} - (1-u)^2\Big)\frac{\lambda}{1+\delta} - \Big(\frac{u}{t} + \frac{1-u}{1-t}\Big)\right\}.$$

Define

$$G(t) \stackrel{\text{def}}{=} \frac{1-v}{1+\delta}H(t) = \Big(\frac{k^2}{4} - (1-u)^2\Big)\frac{\lambda}{1+\delta} - \Big(\frac{u}{t} + \frac{1-u}{1-t}\Big).$$

All the properties of $H$ listed above naturally carry over to $G$.

Differentiate:

$$G'(t) = 2(1-u)\tfrac{\lambda}{1+\delta}u' - \left[\Big(\frac{1}{t} - \frac{1}{1-t}\Big)u' - \frac{u}{t^2} + \frac{1-u}{(1-t)^2}\right],$$

and in the points $t_0$ and $t_1$ one has $u' = \frac{u}{t} + \frac{1-u}{1-t}$, hence

$$\begin{aligned}
G'(t_i) &= 2(1-u)\tfrac{\lambda}{1+\delta}\Big(\frac{u}{t_i} + \frac{1-u}{1-t_i}\Big) - \left[\Big(\frac{1}{t_i} - \frac{1}{1-t_i}\Big)\Big(\frac{u}{t_i} + \frac{1-u}{1-t_i}\Big) - \frac{u}{t_i^2} + \frac{1-u}{(1-t_i)^2}\right] \\
&= 2(1-u)\tfrac{\lambda}{1+\delta}\frac{u(1-t_i) + t_i(1-u)}{t_i(1-t_i)} - \frac{1-2u}{t_i(1-t_i)} \\
&= \frac{B(t_i)}{t_i(1-t_i)},
\end{aligned}$$

for $i = 0, 1$. Here

$$B(t) \stackrel{\text{def}}{=} 2(1-u)\tfrac{\lambda}{1+\delta}\big[u(1-t) + t(1-u)\big] + 2u - 1,$$

and the properties of $G$ imply that $B(t_0) \geq 0$ and $B(t_1) \leq 0$. However, we will show that this leads to a contradiction.

First we note that since $H(t) > 0$ on $(t_0, t_1)$, we have $u'(t) > 0$ on $[t_0, t_1]$. Also, notice that $B(t) > 0$ if $u(t) \geq \frac{1}{2}$, i.e. $B(t) \leq 0$ implies $u(t) < \frac{1}{2}$.

Differentiate again:

$$\begin{aligned}
B'(t) &= 2(1-u)\tfrac{\lambda}{1+\delta}(1-2u) + 2u'(t)\Big[1 + \tfrac{\lambda}{1+\delta}\big(-u(1-t) - t(1-u) + (1-u)(1-2t)\big)\Big] \\
&= 2(1-u)\tfrac{\lambda}{1+\delta}(1-2u) + 2u'(t)\Big[1 + \tfrac{\lambda}{1+\delta}\big(4tu - 2u - 3t + 1\big)\Big].
\end{aligned}$$

We use the fact that $\frac{B-u}{1-u} + 1 = \frac{\lambda}{1+\delta}(-4tu + 2u + 2t)$ to rewrite

$$B'(t) = 2(1-u)\tfrac{\lambda}{1+\delta}(1-2u) + 2u'(t)\Big[\tfrac{\lambda}{1+\delta}(1-t) - \frac{B-u}{1-u}\Big]. \tag{3.14}$$

Now either $B(t_0) > 0$, or $B(t_0) = 0$ hence $u(t_0) < \frac{1}{2}$ and $B'(t_0) > 0$ (just substitute $B = 0$ above and use that $u'(t_0) > 0$). In both cases $B$ is positive in a right neighbourhood of $t_0$ and

$$t_2 \stackrel{\text{def}}{=} \sup\{t > t_0 \mid B(t) > 0\}$$

is well-defined. Since $B(t_1) \leq 0$, one has $B(t_2) = 0$, hence $u(t_2) < \frac{1}{2}$, and $B'(t_2) \leq 0$. Moreover, $t_2 \leq t_1$ and thus $u'(t_2) > 0$. Evaluating (3.14) in $t_2$ and using this information, we obtain a contradiction.

We have thus proved (3.13) and conclude that $k_\lambda < 0$. $\square$

## 4 Conclusion

Our results on the parameter dependence of the AG exponent $\beta$ settles the long standing question of its monotonicity. The asymptotic behaviour of $\beta$ for $m \to 1$ and $m \to \infty$ was computed in [13] using formal PDE methods. Their results are confirmed by our ODE computation and proof.

Several questions concerning the parameter dependence of the critical eigenvalues of the linearisation of the porous medium pressure equation about the $AG$-profiles are raised in the concluding remarks to [2]. This paper answers these questions for two spatial dimensions. First, concerning the linearised stability with respect to perturbations with wave number $k = 2$, we have shown that the $AG$-solutions are unstable with respect to such perturbations for all $m \in (1, \infty)$. Thus there are no self-similar focusing solutions with 2-fold symmetry bifurcating from the circular $AG$-branch, and as shown in [2], there are also no further radially symmetric branches. Solutions whose support is the complement of an elongated hole are studied in [4].

As for the bifurcations with higher wave number, our analysis show that all of them indeed occur, that the bifurcations are simple and that they occur for a sequence

$$m_3 > m_4 > m_5 > m_6 > \cdots \to 1,$$

as was suggested by the numerical results in [3].

Although the bifurcation problem was studied in [2] for arbitrary dimension $N \geq 2$, most of the analysis of the properties of the AG solutions has been limited to $N = 2$, which, as far as the parameter dependence is concerned, is a little exceptional because the $X$-equation derived by setting (2.2) contains a term with coefficient $2 - N$. We conjecture though that the main results of this paper may be proved in exactly the same fashion for $N \geq 2$.

Finally we mention that other degenerate nonlinear diffusion equations give rise to similar questions, see e.g. [6] for the case of the $p$-Laplacian equation. Our ODE-reduction of the linear stability question may be expected to be applicable to any equation with a scaling invariance similar to that of (1.1).

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