

Singularities at $t = \infty$ in Equivariant Harmonic Map Flow

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1. Introduction

Many nonlinear parabolic equations in Geometry and Applied Mathematics (mean curvature flow, Ricci flow, harmonic map flow, the Yang-Mills flow, reaction diffusion equations such as $u_t = \Delta u + u^p$) have solutions which become singular either in finite or infinite time, meaning either that the evolving object (map, metric, surface, or function) becomes unbounded, or that one of its derivatives becomes unbounded. The analysis of the asymptotic behaviour of a solution of a nonlinear parabolic equation just before it becomes singular is known to be a difficult problem. The main general point of this note is that this analysis is considerably easier in the case where the singularity occurs in infinite time. The reason for this is that infinite time singularities are a “stable phenomenon” in the following sense. Given an initial data whose solution becomes singular at $t = \infty$, a slight modification of this initial data will generally still produce a solution which becomes singular *at the same time* (namely, $t = \infty$). In contrast, if a solution becomes singular at time $t = T < \infty$, then a small perturbation of the initial data will generally still produce a solution which becomes singular in finite time, but, usually, at a different time (e.g. simply replace the solution $u(t)$ by the solution $u(t + \epsilon)$.) This instability makes that standard tools for constructing solutions to PDEs (such as the contraction mapping principle or the method of sub and supersolutions) cannot directly provide precise information about solutions near their singularities. To make all this more specific we now consider the example of harmonic map flow from the disc $D^2 \subset \mathbb{R}^2$ to the sphere $S^2 \subset \mathbb{R}^3$.

A family of maps $F_t : D^2 \rightarrow S^2$ evolves according to the *harmonic map flow* if

$$(1.1) \quad \frac{\partial F_t}{\partial t} = \Delta F_t + |\nabla F_t|^2 F_t.$$

Here the right hand side is the component of $\Delta F_t \in \mathbb{R}^3$ which is tangential to S^2 . We will always assume “Dirichlet-type” boundary conditions, i.e. $F_t|_{\partial D^2}$ is fixed.

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As is well known (see [S85],[S90]) a classical solution of (1.1) exists for each C^1 initial data $F_0 : D^2 \rightarrow S^2$ that satisfies the boundary condition. If $\{F_t \mid 0 \leq t < T\}$ is a maximal classical solution then each F_t with $t > 0$ is C^∞ . If this solution only exists for finite time, i.e. if $T < \infty$ then

$$\limsup_{t \nearrow T} \sup_{D^2} |\nabla F_t| = \infty.$$

There exist examples of solutions which become singular in finite time [CDY],[CG].

If a solution exists for all time $t \in [0, \infty)$, and if its gradient does not blow up, i.e. if

$$\sup_{D^2 \times [0, \infty)} |\nabla F_t| < \infty,$$

then the maps F_t must converge to harmonic maps (in the sense of ω -limit sets: any sequence $t_j \nearrow \infty$ has a subsequence $\{t_{j_k}\}$ for which the maps $F_{t_{j_k}}$ converge in C^∞ to a harmonic map.) However, even if a solution does not become singular in finite time its gradient can still become unbounded as $t \nearrow \infty$. This will certainly happen if there is no limiting harmonic map $F_\infty : D^2 \rightarrow S^2$ for the maps F_t to converge to. In this note we consider an example of such a solution and determine the precise rate at which the gradient grows as $t \rightarrow \infty$.

1.1. A problem with long time blow-up. If one makes the ansatz

$$F_t(r, \theta) = \begin{pmatrix} \cos \theta \sin \varphi(r, t) \\ \sin \theta \sin \varphi(r, t) \\ \cos \varphi(r, t) \end{pmatrix}$$

one finds that harmonic map flow is equivalent to the following PDE for φ :

$$(1.2) \quad \frac{\partial \varphi}{\partial t} = \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r} - \frac{f(\varphi)}{r^2},$$

where $0 \leq r \leq 1$, and where $f(\varphi) = \frac{1}{2} \sin 2\varphi$.

We consider the equivariant harmonic map flow equation (1.2) with boundary condition

$$(1.3) \quad \varphi(0, t) \equiv 0 \quad \varphi(1, t) \equiv \pi.$$

Let $\varphi(r, t)$ be a solution of (1.2) whose initial data satisfy

$$(1.4) \quad \begin{cases} \varphi(\cdot, 0) \in C^1([0, 1]), \\ 0 < \varphi(r, 0) < \pi \text{ for } 0 \leq r \leq 1, \\ \varphi(0, 0) = 0, \quad \varphi(1, 0) = \pi, \end{cases}$$

Moreover, we will assume that

$$(1.5) \quad \exists! R_0 \in (0, 1) : \varphi(R_0, 0) = \frac{\pi}{2}.$$

LEMMA 1.1. *For each $t > 0$ there is a unique $R_\varphi(t) \in (0, 1)$ such that $\varphi(R_\varphi(t), t) = \pi/2$.*

(We postpone the short proof until the end of this section.) Under these conditions the initial map $F_0 : D^2 \rightarrow S^2$ maps the unit disc onto the unit sphere, collapsing the boundary ∂D^2 to one point (the south pole). We define

$$(1.6) \quad \varphi(r, t) = u\left(\frac{r}{R_\varphi(t)}, t\right).$$

THEOREM 1.2. *Assume (1.4) and (1.5). Then the solution φ exists for all $t > 0$. The radius $R_\varphi(t)$ converges to zero as $t \rightarrow \infty$, with*

$$R_\varphi(t) = e^{-(2+o(1))\sqrt{t}} \quad (t \rightarrow \infty).$$

Furthermore, one has

$$\lim_{t \rightarrow \infty} u(y, t) = U(y) = 2 \arctan y$$

uniformly on arbitrary but bounded intervals $0 \leq y \leq Y$.

The limiting map $U(y) = 2 \arctan y$ corresponds to stereographic projection from the plane to the sphere. It is to be expected from much more general results on harmonic map flow (see [S85]) that formation of a singularity should proceed by the “bubbling off” of a sphere in the way described here.

Indeed, if T were finite, then for some sequence $t_k \nearrow T$ and some sequence of points $P_k \in D^2$ the maps

$$(1.7) \quad \bar{F}_k(x) \stackrel{\text{def}}{=} F_{t_k}(P_k + \lambda_k x), \text{ with } \lambda_k = \left(\sup_{D^2} |\nabla F_{t_k}| \right)^{-1}$$

would converge to a harmonic map $\bar{F}_\infty : \mathbb{R}^2 \rightarrow S^2$. The only way in which this can happen is for the P_k to converge to the origin, and for the limiting map $\bar{F}_\infty$ to be stereographic projection onto the sphere. In corollary 6.4 we show that this is impossible.

Thus the general theory in [S85] implies that the solution exists for all $t < \infty$, and that its gradient remains uniformly bounded on any finite time interval $0 \leq t \leq t_0$.

The general theory does however not predict at what rate the gradient should blow up. In [BHK] van den Berg, Hulshof and King gave a formal derivation of what the blow-up rate for the gradient should be in all imaginable variations of boundary and initial conditions (for the rotationally symmetric case at least). Our main observation here is that one can rigorously prove the blow-up in the current setting by modifying the formal solutions in [BHK] until they become sub- and supersolutions for (1.2).

Since the convergence of $u(y, t)$ to $U(y)$ follows from more general theory, we will concentrate here on proving the asymptotic formula for $R_\varphi(t)$.

1.2. Proof of lemma 1.1. Both φ and $\hat{\varphi}(r, t) \equiv \pi/2$ are solutions of (1.2), so that their difference satisfies a linear parabolic equation to which we can apply the Sturmian theorem: the number of zeroes of $r \mapsto \varphi(r, t) - \pi/2$ does not increase with time. Since it starts out being 1, and since the boundary conditions in φ force $\varphi(r, t) - \pi/2$ to change sign at least once between $r = 0$ and $r = 1$, we conclude that for each $t > 0$ the function $r \mapsto \varphi(r, t) - \pi/2$ must vanish exactly once.

2. Constructing Formal Solutions

We consider the function

$$u(y, t) = \varphi(yR(t), t),$$

where $R(t)$ will be an approximation of $R_\varphi(t)$.

This function satisfies

$$(2.1) \quad R(t)^2 u_t = u_{yy} + \frac{u_y}{y} - \frac{f(u)}{y^2} + R R_t y u_y,$$

Since we expect $u(y, t) \rightarrow U(y)$ as $t \nearrow \infty$, we write

$$u = U(y) + v(y, t).$$

Assuming v is small compared with U , at least for $y \ll R(t)^{-1}$, one obtains the following equation for v

$$(2.2) \quad R^2 \frac{\partial v}{\partial t} = \mathcal{M}[v] - \frac{f''(U; v)}{y^2} v^2 + RR_t y U_y + RR_t y v_y$$

Here $\mathcal{M}$ is the differential operator

$$(2.3) \quad \mathcal{M} = \left(\frac{\partial}{\partial y}\right)^2 + \frac{1}{y} \frac{\partial}{\partial y} - \frac{f'(U(y))}{y^2} = \left(\frac{\partial}{\partial y}\right)^2 + \frac{1}{y} \frac{\partial}{\partial y} - \frac{1}{y^2} + \frac{8}{(1+y^2)^2}.$$

Also, we use the following notation:

$$f^{(n)}(u; v) = \int_0^1 n(1-\tau)^{n-1} f^{(n)}(u + \tau v) d\tau,$$

so that the Taylor-Maclaurin formula with remainder can be written as

$$f(u+v) = f(u) + f'(u)v + \frac{f''(u)}{2!}v^2 + \dots + \frac{f^{(n-1)}(u)}{(n-1)!}v^{n-1} + \frac{f^{(n)}(u; v)}{n!}v^n.$$

We set $v(y, t) = \alpha(t)\psi_1(y)$, where $\psi_1(y)$ is a solution of

$$(2.4) \quad \mathcal{M}\psi_1 = \psi_0(y) \stackrel{\text{def}}{=} yU'(y) = \frac{2y}{1+y^2}$$

which satisfies $\psi_1(0) = 0$, and $\alpha(t)$ is determined by the boundary condition at $r = 1$, i.e. at $y = R^{-1}$. Namely, we require $U(R^{-1}) + \alpha(t)\psi_1(R^{-1}) = \pi$, i.e.

$$(2.5) \quad \alpha(t) = \frac{2 \arctan R(t)}{\psi_1(1/R(t))}.$$

We will show in section 3 that ψ_1 is uniquely determined upto a multiple of $\psi_0(y)$, and has the following asymptotic behaviour

$$(2.6) \quad \psi_1(y) = y \log y + o(y \log y), \quad (y \nearrow \infty).$$

Equation (2.5) implies a relation between α and R , which in view of the asymptotic behaviour of ψ_1 implies

$$(2.7) \quad \alpha = (2 + o(1)) \frac{R^2}{\log \frac{1}{R}}.$$

We will choose

$$(2.8) \quad \alpha(t) = -R(t)R'(t).$$

The two relations (2.5) and (2.8) imply a differential equation for R , namely

$$(2.9) \quad \frac{dR}{dt} = -2 \frac{\arctan R}{R\psi_1(1/R)} = -(2 + o(1)) \frac{R}{\log 1/R}$$

where we have used $\arctan x = x + o(x)$, and (2.6). Integrating this differential equation we find

$$R(t) = \exp \left(-(2 + o(1)) \sqrt{t} \right).$$

(The integration constant can be absorbed in the $o(1)$.) The function $u(y, t) = U(y) + \alpha(t)\psi_1(y)$ is the formal solution found in [BHK].

3. Inverting $\mathcal{M}$

3.1. A formula for $\mathcal{M}^{-1}$. We observe that for each $\lambda > 0$ the function $U^\lambda(y) = U(\lambda y)$ satisfies

$$\frac{d^2 U^\lambda}{dy^2} + \frac{1}{y} \frac{dU^\lambda}{dy} - \frac{f(U^\lambda)}{y^2} = 0,$$

so that we can differentiate this equation and set $\lambda = 1$. One finds that

$$\psi_0(y) = yU'(y) = \frac{2y}{1+y^2}$$

satisfies

$$\mathcal{M}[\psi_0] = 0.$$

To solve $\mathcal{M}u = v$ we apply the standard method of “reduction of order.” One puts $u = w\psi_0$ and computes

$$\begin{aligned} \frac{1}{\psi_0} \mathcal{M}(\psi_0 w) &= w'' + \frac{1}{y} w' - q(y)w + 2 \frac{\psi_0'}{\psi_0} w' + \frac{1}{y} \frac{\psi_0'}{\psi_0} w + \frac{\psi_0''}{\psi_0} w \\ &= w'' + \left\{ \frac{1}{y} + \frac{2\psi_0'}{\psi_0} \right\} w' + \frac{\mathcal{M}\psi_0}{\psi_0} w = w'' + \left\{ \frac{1}{y} + \frac{2\psi_0'}{\psi_0} \right\} w' \end{aligned}$$

so that $\mathcal{M}u = v$ is equivalent to

$$w'' + \left\{ \frac{1}{y} + \frac{2\psi_0'}{\psi_0} \right\} w' = \frac{v}{\psi_0}.$$

Multiply with $y\psi_0(y)^2$ and integrate, to get

$$y\psi_0(y)^2 w'(y) = A + \int_0^y \eta \psi_0(\eta) v(\eta) d\eta,$$

and, integrating again,

$$u(y) = A\tilde{\psi}_0(y) + \left\{ B + \int_0^y \frac{1}{\eta \psi_0(\eta)^2} \int_0^\eta \eta' \psi_0(\eta') v(\eta') d\eta' d\eta \right\} \psi_0(y).$$

Here

$$\tilde{\psi}_0(y) = \int \frac{dy}{y\psi_0(y)^2}$$

is a solution of the homogeneous equation $\mathcal{M}\tilde{\psi}_0 = 0$ which is singular at $y = 0$. Since we shall always require solutions of $\mathcal{M}u = v$ to be regular at $y = 0$, we set the coefficient $A = 0$, and choose B so that the solution we find vanishes at $y = 1$. This leads to

$$(3.1) \quad u(y) = \mathcal{K}v(y) \stackrel{\text{def}}{=} \psi_0(y) \int_1^y \frac{1}{\eta \psi_0(\eta)^2} \int_0^\eta \eta' \psi_0(\eta') v(\eta') d\eta' d\eta.$$

LEMMA 3.1. *If $v(y) = (C + o(1))y^\alpha$ for $y \nearrow \infty$, then, assuming $\alpha \neq -1, -3$,*

$$\mathcal{K}v(y) = \frac{1 + o(1)}{(\alpha + 1)(\alpha + 3)} y^{\alpha+2}$$

as $y \nearrow \infty$. If $\alpha = -1$, then

$$\mathcal{K}v(y) = \left(\frac{1}{2}C + o(1) \right) y \log y$$

More generally, if $v(y) = (1 + o(1))y^\alpha (\log y)^\beta$, with $\beta > -1$, then for $\alpha \neq -1, -3$,

$$\mathcal{K}v(y) = \frac{1 + o(1)}{(\alpha + 1)(\alpha + 3)} y^{\alpha+2} (\log y)^\beta,$$

while for $\alpha = -1$, $\beta > -1$, one has

$$\mathcal{K}v(y) = \frac{1 + o(1)}{(\alpha + 3)(\beta + 1)} y (\log y)^{\beta+1}.$$

PROOF. For large y one has $\psi_0(y) = (2 + o(1))y^{-1}$, so, assuming $\alpha \neq -1, -3$,

$$\begin{aligned} \int_0^y \eta \psi_0(\eta) v(\eta) d\eta &= \int_0^y (2 + o(1)) \eta^\alpha (\log \eta)^\beta d\eta \\ &= \frac{2 + o(1)}{\alpha + 1} y^{\alpha+1} (\log y)^\beta, \end{aligned}$$

and thus

$$\begin{aligned} \int_1^y \frac{1}{\eta \psi_0(\eta)^2} \int_0^\eta \eta' \psi_0(\eta') v(\eta') d\eta' d\eta &= \int_1^y \frac{\eta}{4} \frac{2 + o(1)}{\alpha + 1} \eta^{\alpha+1} (\log \eta)^\beta d\eta \\ &= \frac{1 + o(1)}{2(\alpha + 1)(\alpha + 3)} y^{\alpha+3} (\log y)^\beta. \end{aligned}$$

Multiply with $\psi_0(y) \sim 2/y$, and the proposition follows. The case $\alpha = -1$ follows by a similar computation. $\square$

3.2. Expansions for derivatives. In general asymptotic expansions $f(y) = o(g(y))$ may not always be differentiated, however, the expansions for $\mathcal{K}$ do withstand differentiation.

LEMMA 3.2. *If $v(y) = (C + o(1))y^\alpha$ for $y \nearrow \infty$, then, assuming $\alpha \neq -1, -3$,*

$$\frac{d}{dy} \mathcal{K}v(y) = \frac{\alpha + 2 + o(1)}{(\alpha + 1)(\alpha + 3)} y^{\alpha+1}, \text{ and } \frac{d^2}{dy^2} \mathcal{K}v(y) = \frac{\alpha + 2 + o(1)}{\alpha + 3} y^\alpha$$

as $y \nearrow \infty$. If $\alpha = -1$, then

$$\frac{d}{dy} \mathcal{K}v(y) = \left(\frac{1}{2}C + o(1) \right) \log y, \text{ and } \frac{d^2}{dy^2} \mathcal{K}v(y) = \frac{\frac{1}{2}C + o(1)}{y}$$

for $y \nearrow \infty$.

PROOF. The expansions for first derivatives follow directly by differentiating the integrals which represent $\mathcal{K}v(y)$. The expansion for the second order derivatives are then obtained by using the differential equation $\mathcal{M}u = v$ which $u = \mathcal{K}v$ satisfies. $\square$

4. Construction of a sub and super solution.

4.1. Specification of ψ_1 . In (2.4) we defined ψ_1 as a solution to $\mathcal{M}[\psi_1] = \psi_0$, where $\psi_0(y) = yU'(y)$. We imposed one boundary condition, $\psi_1(0) = 0$, but otherwise left ψ_1 unspecified. Thus ψ_1 is determined upto a multiple of ψ_0 (which satisfies $\mathcal{M}[\psi_0] = 0$). Since ψ_0 is bounded (in fact, $\psi_0(y) \sim 2/y$ for $y \nearrow \infty$) any choice of ψ_1 will satisfy the same asymptotic condition (2.6) at ∞ .

We now make a specific choice of ψ_1 . First, let $\bar{\psi}_1 = \mathcal{K}[\psi_0]$. Then, in view of the asymptotic behaviour of $\bar{\psi}_1$ as $y \rightarrow \infty$, as well as the fact that $\bar{\psi}_1$ is C^1 at

$y = 0$, there will be a $K > 0$ such that $\bar{\psi}_1(y) \geq -K\psi_0(y)$ for all $y \geq 0$. We choose such a K and henceforth define

$$\psi_1(y) = \bar{\psi}_1(y) + K\psi_0(y).$$

It follows that there is a constant $c > 0$ such that $\psi_1(y) \geq cy$ for all $y \geq 0$.

4.2. The Ansatz. Let

$$\mathfrak{F}[v] = R^2 \frac{\partial v}{\partial t} - \mathcal{M}[v] - RR_t y U_y + \frac{f''(U; v)}{2y^2} v^2 - RR_t y v_y,$$

so that (2.2) can be written as $\mathfrak{F}[v] = 0$. We now let

$$v = v_1 + v_2 = -RR_t \psi_1(y) + v_2(t, y)$$

with v_2 undetermined for the moment, and compute

$$\begin{aligned} \mathfrak{F}[v_1 + v_2] &= -R^2 (RR_t)_t \psi_1 + R^2 \frac{\partial v_2}{\partial t} - \mathcal{M}[v_2] + \frac{f''(U; v)}{2y^2} v^2 \\ &\quad + (RR_t)^2 y \psi_1'(y) - RR_t y v_{2,y} \\ &= -R^3 R_{tt} \psi_1 + R^2 \frac{\partial v_2}{\partial t} - \mathcal{M}[v_2] + \frac{f''(U; v)}{2y^2} v^2 \\ &\quad + (RR_t)^2 (y \psi_1'(y) - \psi_1(y)) - RR_t y v_{2,y} \end{aligned}$$

i.e.

$$(4.1) \quad \mathfrak{F}[v_1 + v_2] = -\mathcal{M}[v_2] + R^2 \frac{\partial v_2}{\partial t} - RR_t y v_{2,y} + T_1 + T_2 + T_3$$

where

$$\begin{aligned} T_1 &= -R^3 R_{tt} \psi_1 \\ T_2 &= (RR_t)^2 (y \psi_1'(y) - \psi_1(y)) \\ T_3 &= \frac{1}{2} \frac{f''(U; v)}{y^2} v^2 \end{aligned}$$

4.3. Estimation of the remainder terms. Assuming that R satisfies (2.9), i.e.

$$(4.2) \quad \frac{dR}{dt} = -a(R), \text{ where } a(R) \stackrel{\text{def}}{=} \frac{2 \arctan R}{R \psi_1(1/R)} = (2 + o(1)) \frac{R}{\log 1/R},$$

we now estimate the terms T_j , beginning with the time derivatives of R .

LEMMA 4.1. *If R satisfies (4.2) then for large t one has*

$$\begin{aligned} -R_t &= (2 + o(1)) \frac{R}{\log 1/R} \\ R_{tt} &= -a'(R) R_t = (4 + o(1)) \frac{R}{(\log R)^2} \\ (RR_t)_t &= (8 + o(1)) \frac{R^2}{(\log R)^2} \end{aligned}$$

PROOF. This follows immediately from (4.2) and the fact that $\frac{da(R)}{dR} = (2 + o(1))(-\log R)^{-1}$ as $R \searrow 0$. $\square$

It will be convenient to use

$$L(y) \stackrel{\text{def}}{=} 1 + \log_+ y = \begin{cases} 1 & y \leq 1 \\ 1 + \log y & y \geq 1 \end{cases}$$

Then for all $y \geq 0$ we have

$$cyL(y) \leq \psi_1(y) \leq CyL(y) \text{ and } |\psi'_1(y)| \leq CL(y)$$

for certain constants¹ $0 < c < C < \infty$.

PROPOSITION 4.2. *For $t \rightarrow \infty$ one has*

$$|T_1| + |T_2| \leq C \frac{R^4}{(\log R)^2} yL(y)$$

PROOF. From $T_1 = -R^3 R_{tt} \psi_1$ we have

$$|T_1| \leq CR^4 (\log R)^{-2} \psi_1(y) \leq CR^4 (\log R)^{-2} yL(y).$$

For T_2 we have

$$|T_2| = (RR_t)^2 |y\psi'_1(y) - \psi_1(y)| \leq \frac{CR^4}{(\log R)^2} yL(y).$$

□

PROPOSITION 4.3. *If $|v_2| \leq v_1$ for all sufficiently large t , then one has*

$$|T_3| \leq \frac{CR^4}{(\log R)^2} yL(y).$$

PROOF. The hypothesis $|v_2| < v_1$ implies that $0 \leq v \leq 2v_1$. Since $v_1 = -RR_t \psi_1(y)$ we find that $v(y, t)$ is bounded by

$$|v(y, t)| \leq C |RR_t| yL(y) \leq \frac{CR^2}{|\log R|} yL(y).$$

Using $f''(u; v) = f''(u) + \frac{1}{3}v f^{(3)}(u; v)$ we split T_3 into two terms,

$$T_3 = T_4 + T_5 = -\frac{1}{2} \frac{f''(U(y))}{y^2} v^2 - \frac{1}{6} \frac{f^{(3)}(U; v)}{y^2} v^3.$$

Since $f''(U(y)) = 8y(1 - y^2)/(1 + y^2)^2$, we have

$$\begin{aligned} |T_4| &\leq \left| \frac{f''(U)}{2y^2} v^2 \right| \leq C (RR_t)^2 \frac{y|y^2 - 1|}{(1 + y^2)^2} \frac{(yL(y))^2}{y^2} \\ &\leq C \frac{R^4}{(\log R)^2} \frac{y}{1 + y^2} L(y)^2 \\ &\leq C \frac{R^4}{(\log R)^2} yL(y). \end{aligned}$$

To estimate T_5 we note that $f^{(3)}(U; v) = f^{(3)}(U + \theta v) = -4 \cos 2(U + \theta v)$ by the Mean Value Theorem, and hence

$$|T_5| = \left| \frac{1}{6} \frac{f^{(3)}(U; v)}{y^2} v^3 \right| \leq \frac{2}{3} \left| \frac{v^3}{y^2} \right| \leq C (RR_t)^3 yL(y)^3.$$

¹Here and elsewhere c and C stand for generic constants, whose value may change from line to line.

Using Lemma 4.1 we then get

$$|T_5| \leq \frac{CR^6}{(\log R)^3} yL(y)^3 \leq \frac{CR^6}{|\log R|} yL(y) \leq C \frac{R^4}{(\log R)^2} yL(y).$$

□

4.4. Choice of v_2 . We let ψ_2 be a solution of $\mathcal{M}[\psi_2] = yL(y)$, e.g. we could choose $\psi_2 = \mathcal{K}[yL(y)]$. According to Proposition 3.1 we have

$$(4.3) \quad \psi_2(y) = \left(\frac{1}{8} + o(1)\right) y^3 \log y \text{ as } y \nearrow \infty.$$

We set

$$v_2 = k \frac{R^4}{(\log R)^2} \psi_2(y),$$

where $k \in \mathbb{R}$ is a constant to be specified below. The function $u = U(y) + v_1(y, t) + v_2(y, t)$ will be a subsolution if

$$\mathfrak{F}[v_1 + v_2] = -\mathcal{M}[v_2] + R^2 \frac{\partial v_2}{\partial t} - RR_t y v_{2,y} + T_1 + T_2 + T_3 \leq 0.$$

The opposite inequality will generate a supersolution.

LEMMA 4.4. *For any $k \in \mathbb{R}$ there is a t_k such that $|v_2| \leq \frac{1}{2} v_1$ for $t \geq t_k$.*

PROOF. We have

$$|v_2(y, t)| \leq Ck \frac{R^4}{(\log R)^2} y^3 L(y).$$

Since $-RR_t \sim R^2/|\log R|$ and $yL(y) \leq C\psi_1(y)$, we get

$$|v_2(y, t)| \leq Ck \frac{R^2}{(\log R)^2} yL(y) \leq \frac{Ck}{|\log R|} (-RR_t) \psi_1(y) \leq \frac{Ck}{|\log R|} v_1(y, t).$$

Since $R(t) \rightarrow 0$ as $t \nearrow \infty$, we get $|v_2| \leq \frac{1}{2} v_1$ for large enough t . □

Proposition 4.3 therefore applies. Together with proposition 4.2 we find that

$$(4.4) \quad \begin{aligned} \mathfrak{F}[v_1 + v_2] &= -\mathcal{M}[v_2] + R^2 \frac{\partial v_2}{\partial t} - RR_t y v_{2,y} + T_1 + T_2 + T_3 \\ &\leq -k \frac{R^4}{(\log R)^2} yL(y) + R^2 \frac{\partial v_2}{\partial t} - RR_t y v_{2,y} + \frac{CR^4}{(\log R)^2} yL(y) \\ &\leq (C - k) \frac{R^4}{(\log R)^2} yL(y) + T_6 + T_7 \end{aligned}$$

where

$$T_6 = R^2 \frac{\partial v_2}{\partial t} \text{ and } T_7 = -RR_t y v_{2,y}.$$

4.5. Estimation of T_6 and T_7 . For T_6 we compute

$$\begin{aligned} \left| R^2 \frac{\partial v_2}{\partial t} \right| &= \left| R^2 \frac{\partial}{\partial t} \left(\frac{R^4}{(\log R)^2} \right) \psi_2(y) \right| \\ &= (4 + O((\log R)^{-1})) \frac{R^5 |R_t|}{(\log R)^2} |\psi_2(y)| \\ &\leq C \frac{R^6}{(\log R)^3} y^3 L(y) \\ &\leq C \frac{R^4}{(\log R)^3} y L(y) \end{aligned}$$

Next, we deal with T_7 . We have $|y\psi_2'(y)| \leq Cy^3L(y)$, which implies

$$\begin{aligned} |T_7| &\leq |-RR_t y v_{2,y}| \\ &\leq \frac{CR^2}{|\log R|} \frac{R^4}{(\log R)^2} y^3 L(y) \\ &\leq \frac{CR^4}{|\log R|^3} y L(y). \end{aligned}$$

In combination with (4.4) we therefore find that

$$\mathfrak{F}[v_1 + v_2] \leq (C + \frac{C}{|\log R|} - k) \frac{R^4}{(\log R)^2} y L(y).$$

This finally leads to the following result.

THEOREM 4.5. *If $k > 0$ is large enough, and if $R(t)$ satisfies $-R_t = -2 \frac{\arctan R}{R\psi_1(1/R)}$, then a t_0 exists such that*

$$u_+(y, t) = U(y) - RR_t \psi_1(y) + k \frac{R^4}{(\log R)^2} \psi_2(y)$$

is a subsolution for (2.1) for $t \geq t_0$, while

$$u_-(y, t) = U(y) - RR_t \psi_1(y) - k \frac{R^4}{(\log R)^2} \psi_2(y)$$

will be a supersolution for (2.1) for $t \geq t_0$.

Unfortunately the sub and super solution provided by this theorem are ordered in the wrong way: the subsolution lies *above* the supersolution and it is impossible to conclude that there is a solution between them.

5. The sub and supersolutions in the r variable

5.1. The functions $\varphi_{\pm}$. We choose sufficiently large k , and define $u_{\pm}(y, t)$ as above in Theorem 4.5. As always, $R(t)$ will be a solution of (2.9), or, equivalently, (4.2). To fix our choice of R we prescribe the initial condition

$$(5.1) \quad R(0) = \rho,$$

for some fixed $\rho \in (0, 1)$. We define

$$\varphi_{\pm}(r, t) = u_{\pm}\left(\frac{r}{R(t)}, t\right).$$

While these functions are sub and supersolutions for $t \geq t_0$, for some $t_0 < \infty$, they do not satisfy the boundary condition $\varphi = \pi$ at $r = 1$. Indeed we have obtained the differential equation (2.9) by imposing this boundary condition on the first two terms $U(y) + v_1(y, t)$ which make up $u_{\pm}$. We will now use the invariance of the Harmonic Map Flow equation under the parabolic similarity transformation $\varphi(r, t) \mapsto \varphi(\theta r, \theta^2 t)$ to turn $\varphi_{\pm}$ into sub and super solutions which satisfy the boundary conditions.

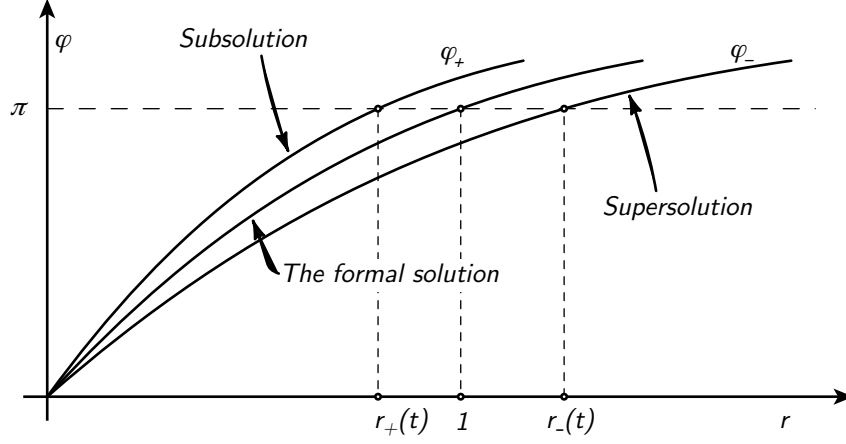


FIGURE 5.1. An unfortunate ordering of a sub and supersolution

5.2. $r_+(t)$ and $r_-(t)$. The functions $u_{\pm}(y, t)$ are defined for all $y > 0$, and we know they provide sub or super solutions when $0 < y \leq 2R(t)^{-1}$. Thus the $\varphi_{\pm}(r, t)$ are sub and super solutions for $0 < r \leq 2$. We now consider $\varphi_{\pm}(1, t)$ for large t . From

$$(5.2) \quad u_{\pm}(y, t) = U(y) - RR_t \psi_1(y) \pm k \frac{R^4}{(\log R)^2} \psi_2(y)$$

and $\psi_2(y) = (1 + o(1))y^3 \log y$ for $y \nearrow \infty$, we conclude

$$\begin{aligned} \varphi_{\pm}(1, t) &= u_{\pm}(R(t)^{-1}, t) = 2 \arctan R(t)^{-1} - RR_t \psi_1(R^{-1}) \pm k \frac{R^4}{(\log R)^2} \psi_2(R^{-1}) \\ &= \pi \pm k \frac{R^4}{(\log R)^2} \psi_2(R^{-1}) \\ &= \pi \pm (k/8 + o(1)) \frac{R^4}{(\log R)^2} R^{-3} \log R^{-1} \end{aligned}$$

so that

$$(5.3) \quad \varphi_{\pm}(1, t) = \pi \pm (k/8 + o(1)) \frac{R}{\log 1/R} \text{ for } t \nearrow \infty.$$

Next, we consider $\partial_r \varphi_\pm$ for $r < 2$ and for large t . Again using (5.2) we find that

$$\begin{aligned}
 \partial_r \varphi_\pm(r, t) &= R(t)^{-1} \partial_y u_\pm(y, t) \\
 (5.4) \quad &= R^{-1} \left\{ U'(y) - R R_t \psi'_1(y) \pm k \frac{R^4}{(\log R)^2} \psi'_2(y) \right\} \\
 &= \frac{2R}{R^2 + r^2} - R_t(1 + o(1)) \log \frac{r}{R} \pm \left(\frac{3k}{8} + o(1) \right) \frac{R}{(\log R)^2} r^2 \log \frac{r}{R}.
 \end{aligned}$$

If $\frac{1}{2} < r < 2$ then $\log r = o(\log R)$ so that $\log \frac{r}{R} = (1 + o(1)) \log 1/R$, and hence, using (4.2) once again,

$$\begin{aligned}
 \partial_r \varphi_\pm(r, t) &= (2 + o(1)) \frac{R}{r^2} + (2 + o(1)) \frac{R}{\log 1/R} \log 1/R \pm \left(\frac{3k}{8} + o(1) \right) \frac{R}{\log 1/R} r^2 \\
 &= (2 + o(1)) R \left(\frac{1}{r^2} + 1 \pm \frac{(3k/8 + o(1))r}{\log 1/R} \right)
 \end{aligned}$$

whence

$$(5.5) \quad \partial_r \varphi_\pm(r, t) = (2 + o(1)) (1 + r^{-2}) R$$

for $\frac{1}{2} < r < 2$, and for $t \nearrow \infty$.

LEMMA 5.1. *For large enough t there exist unique $r_\pm(t) \in (\frac{1}{2}, 2)$ such that $\varphi_\pm(r_\pm(t), t) = \pi$. One has*

$$r_\pm(t) = 1 \mp \frac{\frac{k}{16} + o(1)}{\log 1/R(t)} = 1 + o(1).$$

PROOF. This follows immediately from (5.3) and (5.5). Indeed, these equations imply that $\partial_r \varphi_\pm \geq (2 + o(1))R$ for $\frac{1}{2} < r < 2$, while $\varphi_\pm - \pi = \pm(k + o(1))R/(\log 1/R)$, so that $\varphi_\pm - \pi$ must vanish at some $r_\pm = 1 + o(1)$. But (5.5) implies $\partial_r \varphi_\pm = (2 + o(1))R$ for $r = 1 + o(1)$, which then leads to the stated asymptotic expression for $r_\pm(t)$. $\square$

6. Proof of Theorem 1.2

6.1. Every solution becomes singular. We consider a solution $\varphi(r, t)$ of (1.2) whose initial data satisfy (1.4) and (1.5). We will assume in addition that

$$(6.1) \quad \partial_r \varphi(0, 0) > 0, \text{ and hence } \varphi(r, 0) \geq \delta r$$

for all $r \in [0, 1]$ and some small enough $\delta > 0$. We may do this without loss of generality, since the strong maximum principle will force any solution $\varphi(r, t)$ of (1.2) which satisfies (1.4) to satisfy (6.1) immediately for $t > 0$. So if our chosen initial function $\varphi(r, 0)$ does not satisfy (6.1), then we replace it with $\varphi(r, t)$ for any small $t > 0$.

Choose $t_1 > 0$ so large that $r_+(t) \geq 1/2$ for $t \geq t_1$.

LEMMA 6.1. *There is an $\varepsilon \in (0, \frac{1}{2})$ such that $\varphi(r, 0) \geq \varphi_+(\varepsilon r, t_1)$ for all $r \in [0, 1]$. Moreover, for all $t \geq 0$ one has*

$$(6.2) \quad \varphi(r, t) \geq \varphi_+(\varepsilon r, t_1 + \varepsilon^2 t).$$

PROOF. This follows immediately from (6.1) and the fact that for all $t > 0$ one can find a constant $C(t) < \infty$ such that $\varphi_+(r, t) \leq C(t)r$ holds for all $r \in [0, 1]$.

We observe that $\hat{\varphi}(r, t) = \varphi_+(\varepsilon r, t_1 + \varepsilon^2 t)$ is a subsolution of (1.2). Also, it follows from $r_+(t) > \frac{1}{2} > \varepsilon$ for all $t \geq t_1$ that $\hat{\varphi}(1, t) = \varphi_+(\varepsilon, t_1 + \varepsilon^2 t) < 1$. Hence the Maximum Principle implies $\varphi \geq \hat{\varphi}$ for all $r \in [0, 1]$ and $t \geq 0$, as claimed. $\square$

We improve the previous lemma by showing that one can take ε arbitrarily close to $\varepsilon = 1$, possibly at the expense of increasing t_1 . Let $\theta \in (0, 1)$ be given, and choose $t_2 > 0$ such that $r_+(t) > \theta$ for all $t \geq t_2$.

LEMMA 6.2. *For large enough $t_3 \geq t_1$ one has $\varphi_+(\varepsilon r, t_3) \geq \varphi_+(\theta r, t_2)$ for all $r \in [0, 1]$. Furthermore*

$$\varphi(r, t_4 + t) \geq \varphi_+(\theta r, t_2 + \theta^2 t)$$

for all $t > 0$, where $t_4 = (t_3 - t_1)/\varepsilon^2$.

PROOF. Since $r_+(t_2) > \theta$ we have $\varphi_+(\theta r, t_2) < \varphi_+(\theta, t_2) < \pi$. On the other hand, for any $r > 0$ one has $\lim_{t \nearrow \infty} \varphi_+(\varepsilon r, t) = \pi$ with uniform convergence on any interval $\delta \leq r \leq 1$. Thus for any $\delta > 0$ there will be a $t = t(\delta) > 0$ such that $\varphi_+(\varepsilon r, t) \geq \varphi_+(\theta r, t_2)$ holds for $r \in [\delta, 1]$. On the short interval $[0, \delta]$ one has $\varphi_+(\theta r, t_2) \leq Cr$ for some fixed $C < \infty$. Hence, if t is chosen large enough one will also have $\varphi_+(\varepsilon r, t) \geq \varphi_+(\theta r, t_2)$ for $r \in [0, \delta]$.

It follows from (6.2), i.e. $\varphi(r, t) \geq \varphi_+(\varepsilon r, t_1 + \varepsilon^2 t)$ and $\varphi_+(\varepsilon r, t_3) \geq \varphi_+(\theta r, t_2)$ that, with $t_4 = (t_3 - t_1)/\varepsilon^2$, one has

$$\varphi(r, t_4) \geq \varphi_+(\theta r, t_2) \text{ for } r \in [0, 1].$$

Since $\tilde{\varphi}(r, t) = \varphi_+(\theta r, t_2 + \theta^2 t)$ is a subsolution for (1.2) which satisfies $\tilde{\varphi}(1, t) = \varphi_+(\theta, t_2 + \theta^2 t) < \pi$ (because $\theta < r_+(t_2 + \theta^2 t)$) for all $t > 0$, the Maximum Principle implies $\tilde{\varphi}(r, t) \leq \varphi(r, t_4 + t)$ for all $r \in [0, 1]$, $t \geq 0$. $\square$

Inspection of the subsolution φ_+ reveals that for large enough t there is a unique $\tilde{R}(t) = (1 + o(1))R(t)$ such that $\varphi_+(\tilde{R}(t), t) = \pi/2$. Since $\tilde{R}(t)$ and $R(t)$ differ by $o(R(t))$, the asymptotic expansion for $R(t)$ also applies to $\tilde{R}(t)$, i.e. $\tilde{R}(t) = \exp(-2\sqrt{t} + o(\sqrt{t}))$.

Lemma 6.2 with $r = R_\varphi(t_4 + t)$ asserts that

$$\frac{\pi}{2} = \varphi(R_\varphi(t_4 + t), t_4 + t) \geq \varphi_+(\theta R_\varphi(t_4 + t), t_2 + \theta^2 t),$$

whence

$$\theta R_\varphi(t_4 + t) \leq \tilde{R}(t_2 + \theta^2 t) = e^{-(2+o(1))\sqrt{t_2 + \theta^2 t}} = e^{-(2\theta+o(1))\sqrt{t}} \quad (t \nearrow \infty).$$

Division by θ and replacement of $t_4 + t$ by t leads to additional terms all of which can be absorbed in the $o(1)$ in the exponential. We therefore find that $R_\varphi(t) \leq e^{-(2\theta+o(1))\sqrt{t}}$ for all $\theta < 1$, and thus

$$(6.3) \quad R_\varphi(t) \leq e^{-(2+o(1))\sqrt{t}}.$$

It remains to prove the opposite asymptotic inequality.

Let $\theta > 1$ be given, and choose a $t_5 > 0$ so that $r_-(t) < \theta$ for all $t \geq t_5$.

LEMMA 6.3. *For sufficiently large $t_6 > t_5$ one has*

$$\varphi(r, 0) \leq \varphi_-(\theta r, t_6) \quad (0 \leq r \leq 1),$$

and hence

$$\varphi(r, t) \leq \varphi_-(\theta r, t_6 + \theta^2 t)$$

for all $t \geq 0$, $0 \leq r \leq 1$.

As we pointed out in the introduction, this implies global existence of our solution.

COROLLARY 6.4. *Any solution $\varphi(r, t)$ of (1.2) whose initial data satisfies (1.4), (1.5) exists for all $t > 0$.*

Indeed, on any finite time interval $[0, T]$ we have $\varphi(r, t) \leq \varphi_-(\theta r, t_6 + \theta^2 t)$, so that on some small interval $0 \leq r \leq \delta$ one has $\varphi(r, t) \leq Cr$ for some $C < \infty$. Thus the maps $\tilde{F}_{t_k}$ defined in (1.7) cannot converge to the stereographic projection.

PROOF OF LEMMA 6.3. Again, $\varphi_-(\theta r, t)$ converges uniformly to π on any interval $[\delta, 1]$ with $\delta > 0$, in fact, the convergence is in $C^1([\delta, 1])$. It follows from $r_-(t) < \theta$ that $\varphi_-(\theta, t) > \pi$. On the other hand we may assume w.l.o.g. that $\varphi(1, 0) = \pi$, and that $\varphi(r, 0) \leq \pi - \delta(1 - r)$ for some small $\delta > 0$. (As before, even if the initial φ fails to satisfy this condition, $\varphi(\cdot, t)$ will do so for any small $t > 0$.) So, for any given $\delta > 0$ we can find a $t > 0$ such that $\varphi(r, 0) \leq \varphi_-(\theta r, t)$ holds for $\delta \leq r \leq 1$. On the short interval $[0, \delta]$ the initial data are bounded by $\varphi(r, 0) \leq Cr$ for some $C < \infty$. Clearly, for sufficiently large $t > 0$ one will have $\varphi(r, 0) \leq \varphi_-(\theta r, t)$ for $r \in [0, \delta]$.

Let t_6 be such a large t . Then, since $\hat{\varphi}(r, t) = \varphi_-(\theta r, t_6 + \theta^2 t)$ is a subsolution, and since $\hat{\varphi}(1, t) = \varphi_-(\theta, t_6 + \theta^2 t) \geq \varphi_-(r_-(t_6 + \theta^2 t), t_6 + \theta^2 t) = \pi$, it follows from the Maximum Principle that $\hat{\varphi}(r, t) \geq \varphi(r, t)$ for all $r \in [0, 1]$, $t \geq 0$, as claimed. $\square$

This implies that

$$R_\varphi(t) \geq R(t_6 + \theta^2 t) = e^{-(2\theta + o(1))\sqrt{t}} \quad (t \nearrow \infty).$$

Once again, this holds for all $\theta > 1$, so that we have $R_\varphi(t) \geq \exp(-(2 + o(1))\sqrt{t})$ for $t \nearrow \infty$. combined with (6.3) we get $R_\varphi(t) = \exp(-(2 + o(1))\sqrt{t})$.

References

- [A] S.B.Angenent, *The zero set of a solution of a parabolic equation*, Journal für die reine and angewandte Mathematik **390**, 79-96, 1988.
- [BHK] Jan Bouwe van den Berg, Joost Hulshof and John R. King, *Formal asymptotics of bubbling in the harmonic map heat flow*, to appear SIAM Journal of Applied Mathematics.
- [CDY] K.-C. Chang, W.-Y. Ding and R. Ye, *Finite-time blow-up of the heat flow from surfaces*, J. Diff. Geom. **36** (1992), 507-515.
- [CG] J.M.Coron and J.M.Ghidaglia, *Explosion en temps fini pour le flot des applications harmoniques*, C.R.Acad. Sci.Paris, t.**308** Série I (1989) 339-344.
- [S85] M.Struwe, *On the evolution of harmonic mappings of Riemannian surfaces*, Comm. Math. Helv. **60** (1985) 558-581.
- [S90] M.Struwe, *VARIATIONAL METHODS*, Springer Verlag, Berlin, etc. (1990).

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