Stability and Attractor for Quasi-Steady Equation of Cellular Flames

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Abstract

We continue to study a simple integro-differential equation: the Quasi-Steady equation (QS) of flame front dynamics. This second order quasi-linear parabolic equation with a non-local term is dynamically similar to the Kuramoto-Sivashinsky (KS) equation. In [FGS03], where it was introduced, its well-posedness and proximity for finite time intervals to the KS equation in Sobolev spaces of periodic functions were established. Here we demonstrate that QS possesses a universal absorbing set, and a compact attractor. Furthermore we show that the attractor is of a finite Hausdorff dimension.We discuss relationship with the Kuramoto-Sivashinsky and Burgers-Sivashinsky equations.

1 Introduction

In a recent paper [BFHS05] we studied the Quasi-Steady (QS) Model of cellular flames:

$$\Phi_t + \frac{1}{2}\Phi_x^2 = (I - \alpha A)\Phi_{xx}, \tag{1.1}$$

where

$$A := (I - \partial_x^2)^{-1}.$$

The equation in (1.1) arises as a certain truncation of a weakly-nonlinear version of a more detailed model of cellular flames (the $\kappa - \theta$ model) introduced in [FGS03].

Equation (1.1) was shown to be well posed in Sobolev spaces of periodic functions. Moreover, for a fixed time interval in a certain sense solutions of (1.1) become uniformly close to those of the Kuramoto-Sivashinsky equation (KS)

$$\Phi_t + \frac{1}{2}\Phi_x^2 = (1 - \alpha)\Phi_{xx} - \Phi_{xxxx}, \tag{1.2}$$

as the instability parameter $\varepsilon = 1 - \alpha$ approaches zero.

The latter result seems to imply a strong resemblance between the dynamics generated by the two equations, in particular it therefore suggests that QS is capable of generating a cellular structure and chaotic behavior for an appropriate range of parameters. Indeed, some preliminary results of a direct numerical simulation [BFHS05] confirm this hypothesis. Thus, "quasi-steady" in the name of (1.1) relates rather to the way it was originally introduced, and is in no way indicative of its rich dynamics.

However, one wonders whether the dynamical features such as stability/dissipativity and compact attractor of finite Hausdorff dimension that were established rigorously for KS, can be demonstrated for the QS model. The positive answer to this question is the main result of the current paper.

Since the time of writing the paper [BFHS05] our view of the nature of (1.1) has somewhat evolved, therefore we present a slightly different discussion of its basic properties that are expressed through its dynamics. First, it is more convenient to represent (1.1) in a slightly different form:

$$\Phi_t + \frac{1}{2}\Phi_x^2 = \Phi_{xx} + \alpha(I - A)\Phi. \tag{1.3}$$

This form is rather reminiscent of yet another dissipative system that has been a subject of discussion in recent years (see [RS87, G94, BKS01]), the so called Burgers-Sivashinsky (BS) equation:

$$\Phi_t + \frac{1}{2}\Phi_x^2 = \Phi_{xx} + (\alpha - 1)\Phi. \tag{1.4}$$

The coefficient $\alpha - 1$ (we assume $\alpha > 1$) is chosen in such a form here simply to adjust the neutral stability wave number (at $\omega = 0$) to the other two equations for easier comparison.

All three of these equations (1.2), (1.3), and (1.4) share the same basic quality revealed by linear stability analyses, namely long-wave destabilization which is suppressed by the dominant dissipative principal term for small wave lengths. The respective dispersion relations read

$$\omega(k) = (\alpha - 1)k^2 - \alpha k^4, \tag{KS}$$

$$\omega(k) = -k^2 + \frac{\alpha k^2}{1 + k^2}, \tag{QS}$$

$$\omega(k) = -k^2 + \alpha - 1. \tag{BS}$$

The neutral stability curve is the same for all three equations, it is the parabola

$$k(\alpha) = \sqrt{\alpha - 1}. \tag{1.5}$$

As one can easily see, in the long-wave part of the spectrum $k \ll 1$, the dispersion relation for QS is similar to that of KS

$$\omega(k) = -k^2 + \frac{\alpha k^2}{1 + k^2} = (\alpha - 1)k^2 - \alpha k^4 + \mathcal{O}(k^6)..., \tag{1.6}$$

while for the short waves $k \gg 1$ its decay (in contrast to KS) $\omega \sim -k^2$ is identical to that of BS.

In order to understand, which dispersion relation better reflects the physical reality we should turn to the original combustion problem. The real-life flames constitute a rather intricate physical system involving the fluid dynamics of the multicomponent gaseous mixture, the multistep chemical kinetics, as well as the molecular and radiative transfer. It transpires, however, that the cellular instability may be successfully captured by a simple reaction-diffusion system. In an appropriate range of physico-chemical parameters a subsequent reduction to a free interface problem can be performed [MS79]. The dispersion relation for this free-interface problem reads (see e.g. [BL00]):

$$\left(1 + 4\omega + 4k^2 - \alpha\right)\left(1 - \sqrt{(1 + 4\omega + 4k^2)}\right) - 2\alpha\omega = 0 \qquad (1.7)$$

After some elementary calculations one finds then that, while in the long-wave approximation the behavior of the exact dispersion relation the free-interface problem is mimicked by KS and QS, its decay at infinity is identical to BS and once again to QS. The three equation QS, KS and BS share the same nonlinearity and the destabilization-dissipation mechanism is identical for all three (as we show below). The nonlinearity disperses the energy from the unstable modes to the rest of the spectrum sufficiently for the principal term to dissipate its excess.

Therefore, the (linear part of) QS equation presents *a uniform approximation of the spectrum* of the original combustion problem. Fig. 1 illustrates our remarks concerning the comparative linear stability analyses of the three equations on the one hand and the free-interface problem on the other hand. Fig. 1-left represents the dispersion relations while Fig. 1-right is a magnification of its long wave region. It is necessary to remark that while KS is the model equation for the onset of cellular-chaotic dynamics, one may expect that some important features of global dynamics of the original physical system should be better described by the QS equation, as they may be influenced by the manner in which the energy created by the long-wave instability is channeled to the rest of the spectrum.

Summarizing our observations we remark that the QS equation represents as it were a middle ground between the KS and BS. Indeed, while the dynamics of the QS is qualitatively identical to KS regarding the basic ability to generate chaotic cellular solutions, QS is similar to BS in such features

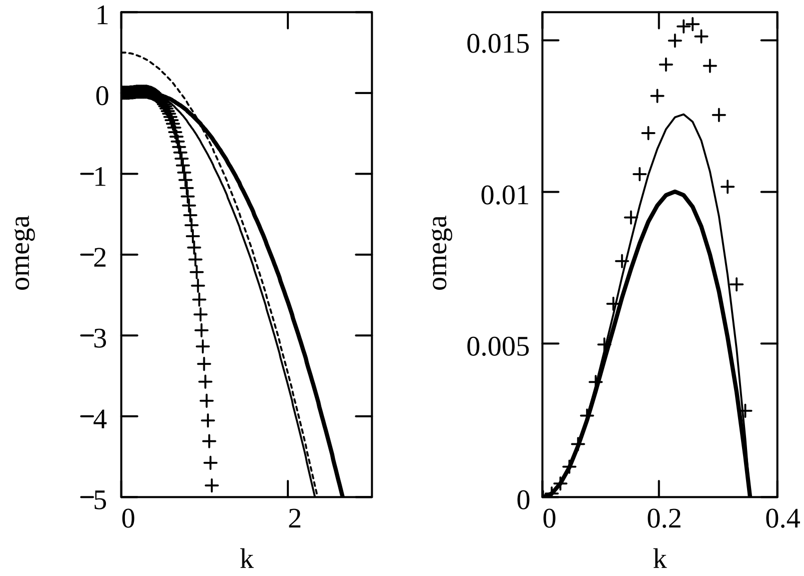

Figure 1: Dispersion relations for the KS (pluses), the QS (thin line), the BS (dots) and the original free-interface problem (thick line). A magnification of the long wave region is on the right.

as the size of the absorbing set and the dimension of the attractor; all the estimates lead to practically the same results for both. The reason being that it seems impossible to take advantage of the term with the (positive, bounded) operator A to improve the estimates.

At the same time it is this term that is responsible for the drastic difference in dynamics between the two. It should be noted that, unlike KS and QS, the dynamics generated by the BS equation is asymptotically trivial (see e.g. [BKS01, G94]) as all its solutions gravitate to a steady states dictated by the length of the interval and the parameter. This is obviously due to the fact that, in contrast to the KS and the QS equations, the most linearly unstable mode corresponds to the infinite wave length $k = 0$.

Regardless of the QS equation, if one carries out the estimates of the attractor for BS in the Sobolev spaces of the L-periodic functions using the rather refined techniques that were developed for the KS (and a number of other dissipative systems), one obtains an estimate for the Hausdorff dimension that is a certain power of the period L, with no indication whatsoever of the triviality of the dynamics on the attractor. In fact, strictly speaking, the bounds for BS are even somewhat worse than the ones for QS, although by a lower power of L. It is not surprising in view of the fact that the growth rates for the very long wave disturbances tend to zero for QS in contrast with BS, where they approach a positive constant.

This is a rather disturbing outcome especially in view of the considerable effort [NST85, G94, CEES93, GA04] to lower the estimates on the size of the absorbing set, and, consequently, dimension of the attractor based on the view that these characteristics measure the "amount of chaos" or "the strength of turbulence" in the solutions of KS. Therefore one should keep in mind that the estimates of the Hausdorff dimension of the attractor are just the upper bounds, and perhaps they do not provide a sufficiently sharp tool after all to be used for discerning the chaotic dynamics and the lack thereof, unless the exact value of the dimension could be calculated.

Nonetheless, even an upper bound should be considered an important step in investigation of the infinite-dimensional dynamical system generated by QS. Thus, the main results of the paper, the estimates on the size of the absorbing set, and on the Hausdorff dimension of the attractor are of a considerable interest.

We preface the main part by a brief summary of results of [BFHS05] in Sec. 2. The stability section represents a combination of the methods employed in [G94] and [CEES93], while the estimate of the dimension follows rather faithfully the book of Temam [Temam]. In view of the above comments we did not attempt to obtain necessarily the lowest estimate opting for the transparency of presentation. We should remark that the method suggested in [GA04] can also be easily customized for QS leading as it seems to lower estimates.

We conclude with the following notation convention: *unless otherwise specified, all the spatial integrals are assumed to be over the period*, i.e., we write $\int ...dx$ instead of $\int_{-L/2}^{L/2} ...dx$.

2 Mathematical setting and previous results

We consider (1.1), in the class of L-periodic functions in x. It is more convenient, and rather conventional (cf. [Temam]), however, to study the equation differentiated with respect to x

$$u_t + uu_x = u_{xx} + \alpha(I - A)u \tag{2.8}$$

for $u = \Phi_x$ in the class of periodic functions with zero average. We easily recover Φ from u, in view of

$$\frac{d}{dt}\int \Phi(x,t)dx = -\frac{1}{2}\int u(x,t)^2 dx.$$

However, it is (2.8) that we study below. In fact the discussion in the introduction also concerns rather the differentiated forms of the three equations QS, KS and BS.

For integer or arbitrary real s we denote by H^s the usual Sobolev spaces of L-periodic (generalized) functions *with zero average*, which we will represent as

$$H^s = \{w = \sum_{k=1}^{\infty} a_k w_k : \sum_{k=1}^{\infty} \lambda_k^s a_k^2 < \infty\},$$

with norm

$$||w||_s^2 = \sum_{k=1}^{\infty} \lambda_k^s a_k^2 \quad (\quad = |w^{(s)}|^2 = \int w^{(s)}(x)^2 dx \ \text{ if } \ \mathrm{s} = 0, 1, 2, \dots).$$

Here the w_k are a complete set of eigenfunctions of the operator $-\partial_x^2 : H^2 \to H^0$ corresponding to the positive eigenvalues $\lambda_k = (2k\pi/L)^2$. Note that these eigenfunctions are also eigenfunctions of the operator

$$A_\alpha = -\partial_x^2 - \alpha(I - A) : H^0 \ \to \ H^0,$$

Denoting the map $u \mapsto uu_x$ by F, the abstract form of (2.8) reads

$$u_t + A_\alpha u + F(u) = 0.$$

Clearly A_α defines a sectorial operator. Since any intermediate space H^s with $0 < s < 2$ is an interpolation space, local well-posedness for initial data in H^2 by analytic semigroup methods follows, provided we can choose s such that $F : H^s \to H^0$ is locally Lipschitz. With H^1 embedding in L^∞, we may choose $s = 1$.

The global existence is stated as follows

Theorem 2.1 *([BFHS05]) Let* $u_0 \in H^s$ *where* $s = 0, 1, 2, 3, \dots$. *Then (2.8) has a unique solution on any time interval* $[0, T]$ *with* $u(., 0) = u_0$. *The solution belongs to* $L^2(0, T; H^{s+1}) \cap C^0([0, T]; H^s)$, *and has* $u_t \in L^2(0, T; H^{s-1})$, *and the corresponding norms are bounded by a constant which depends only on* α, s, L, T *and* $||\Psi(0)||_s$. *Thus the solution is smooth for* $t > 0$.

In order to explain the asymptotic proximity result in [BFHS05] we note first that the form of KS that is habitually used in numerous papers (2.11) is (as opposed to (1.2)) parameter free, and was derived from the original free-interface problem via a certain rescaling of coordinates under the assumption $\varepsilon = \alpha - 1 \ll 1$. For this reason, in order to be able to compare the solutions of QS and KS, the former should also undergo the same rescaling of the coordinates.

Thus, we are interested in $\alpha = 1 + \varepsilon$, where ε is a (small) fixed positive number. We take the period to depend on ε, introducing a reference period $L_0 > 0$ and (reference) time interval $[0, T_0]$. The reference space and time variables are denoted by ξ and τ:

$$x = \xi/\sqrt{\varepsilon},\ t = \tau/\varepsilon^2,\ u = \varepsilon\varphi, \tag{2.9}$$

We consider (1.1), in the class of periodic functions with period $L_0/\sqrt{\varepsilon}$ in x, on the time interval $[0, T_0/\varepsilon^2]$ in t. The rescaled version of (1.1) is

$$(I - \varepsilon\,\partial_\xi^2)(\varphi_\tau + \frac{1}{2}\varphi_\xi^2) + \varphi_{\xi\xi\xi\xi} + \varphi_{\xi\xi} = 0. \tag{2.10}$$

The solution of (2.10) is sought as $\varphi = U + \varepsilon w$, so that for $\varepsilon = 0$ one recovers the (parameter-free) KS:

$$U_\tau + \frac{1}{2}U_\xi^2 + U_{\xi\xi\xi\xi} + U_{\xi\xi} = 0. \tag{2.11}$$

The equation for the perturbation w reads

$$(I - \varepsilon\,\partial_\xi^2)(w_\tau + \frac{1}{2}(2U_\xi w_\xi + \varepsilon w_\xi^2) + w_{\xi\xi\xi\xi} + w_{\xi\xi} = \partial_\xi^2(U_\tau + \frac{1}{2}U_\xi^2).$$

We assume for simplicity that both QS and KS start from exactly the same initial condition, i.e. $w(\xi, 0) = 0$. Then, one can perform an energy type estimate for w to obtain

Theorem 2.2 *Let $U_0 \in H^3$. There exists $\varepsilon_0 > 0$ such that, whenever $0 < \varepsilon < \varepsilon_0$, $|w(\xi,\tau)| \leq C$ for all (ξ,τ) in $[-L_0/2, L_0/2] \times [0, T_0]$. The number ε_0 and the bound C depend only on $U_0 \in H^3$, and T_0.*

Returning to the original problem, $\Phi = \varepsilon\varphi$, $\varphi = U + \varepsilon w$ leads the nearness estimate:

Corollary 2.3

$$\max |\Phi(x,t) - \varepsilon U(x\sqrt{\varepsilon}, t\varepsilon^2)| \leq C\,\varepsilon^2 \tag{2.12}$$

for $|x| \leq \dfrac{L_0}{2\sqrt{\varepsilon}}$, $0 \leq t \leq \dfrac{T_0}{\varepsilon^2}$.

3 Absorbing set

The essence of the dissipative nature of QS is expressed in uniform boundedness of its solutions. The well-posedness results of [BFHS05] do not provide a necessary bound. The purpose of this section is to establish it.

In the spirit of [G94, Proposition 1] we need the following Poincaré type result:

Lemma 3.1 *Define*

$$b(x) = \begin{cases} 0, |x| > \varepsilon \\ \dfrac{B}{\varepsilon}(1 - |x|/\varepsilon), |x| \leq \varepsilon \end{cases}$$

(i.e., the graph of b is a triangle of the area B with the base 2ε). Then for any $u \in H^1$

$$\int b(x)u^2(x)dx \leq \frac{4}{B}\langle ub\rangle^2 + \frac{1}{4}\int u'(x)^2 dx \tag{3.13}$$

where

$$\langle ub\rangle = \int b(y)u(y)dy$$

Proof On the interval $x, y \in (-\varepsilon, \varepsilon)$

$$u(x) = u(y) + \int_y^x u'(z)dz$$

We multiply this equation by $b(y)$ and integrate with respect to y

$$
\begin{aligned}
Bu(x) &= \langle ub\rangle + \int_{-\varepsilon}^{\varepsilon} b(y)dy \int_{y}^{x} u'(z)dz \\
&:= \langle ub\rangle + \int_{-\varepsilon}^{\varepsilon} K(x,z)u'(z)dz := \langle ub\rangle + \hat{K}u'
\end{aligned}
$$

where $\hat{K}$ is an integral operator on $L^2[-\varepsilon,\varepsilon]$ with the kernel

$$
K(x,z) = \begin{cases} \int_z^{\varepsilon} b(y)dy, & \text{if } z \geq x \\ \int_{-\varepsilon}^{z} b(y)dy, & \text{if } z \leq x \end{cases}
$$

Next we estimate the L_2 norm of u on $(-\varepsilon,\varepsilon)$

$$
\begin{aligned}
\int_{-\varepsilon}^{\varepsilon} u^2(x)dx &= \frac{1}{B^2}\int_{-\varepsilon}^{\varepsilon}\left[\langle ub\rangle^2 + 2\langle ub\rangle(\hat{K}u')(x) + (\hat{K}u')^2(x)\right]dx \\
\leq \frac{2}{B^2}\int_{-\varepsilon}^{\varepsilon}[\langle ub\rangle^2 + (\hat{K}u')^2(x)]dx &= \frac{2}{B^2}\left[2\varepsilon\langle ub\rangle^2 + \int_{-\varepsilon}^{\varepsilon}(\hat{K}u')^2(x)dx\right] \\
&\leq \frac{4\varepsilon}{B^2}\langle ub\rangle^2 + \frac{2}{B^2}\left\|\hat{K}\right\|^2\int_{-\varepsilon}^{\varepsilon}(u')^2(x)]dx
\end{aligned}
$$

It is clear that $|K(x,z)| \leq B$, hence the Hilbert-Schmidt norm

$$
\left\|\hat{K}\right\|^2 = \int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon}|K(x,z)|^2 dxdz \leq B^2\varepsilon^2
$$

Therefore

$$
\begin{aligned}
&\int b(x)u^2(x)dx \leq \sup b \int_{-\varepsilon}^{\varepsilon} u^2(x)dx \\
\leq \frac{B}{\varepsilon}&\left[\frac{4\varepsilon}{B^2}\langle ub\rangle^2 + \frac{2}{B^2}\left\|\hat{K}\right\|^2\int_{-\varepsilon}^{\varepsilon}(u')^2(x)]dx\right] = \frac{4}{B}\langle ub\rangle^2 + 2B\varepsilon\int u'(x)^2dx
\end{aligned}
$$

Finally we select $\varepsilon = 1/(8B)$ to obtain (3.13)

The main result of this section is the following

Theorem 3.2 *(Existence of an absorbing ball). Let $L \geq 1/(2\alpha)$; then for any solution u of (2.8)*

$$
\limsup_{t\to\infty} ||u(.,t)|| \leq C_a(\alpha L)^{3/2}.
$$

where C_a is a universal constant; $||.||$ is the H^0-norm.

Remark 3.3 *The assumption $L > 1/(2\alpha)$ covers the most interesting case, when unstable modes exist. Indeed, the smallest eigenvalue for the L-periodic problem with zero mean is $\lambda = (2\pi/L)^2$ From the neutral stability curve we get $L > 2\pi/\sqrt{\alpha - 1}$ for the unstable modes to appear, which is a stronger condition than $L > 1/(2\alpha)$. It is a rather simple matter to prove that all solutions decay exponentially in time in the opposite case.*

Proof of Theorem. Let s be a fixed periodic function with zero mean (to be selected later). Introduce $y(t)$ as a solution of the initial-value problem for the ode

$$\dot{y}(t) = \int u(x,t)s'(x + y(t)))dx, \quad y(0) = 0.$$

where $u(x,t)$ is a given solution. Introduce the function

$$\Phi(t) := \frac{1}{2}\int [u(x,t) - s(x + y(t)]^2 dx$$

(It is easy to see that for a shift y_0 for which the integral above is minimal $\int u(x)s'(x + y_0) = 0$)

We compute the derivative of Φ and substitute u_t from the equation (2.8) to obtain

$$\begin{aligned} \frac{d}{dt}\Phi(t) &= \int (u - s)\left[-\dot{y}s'(x + y(t)) - u_x u + u_{xx} + \alpha u - \alpha A u\right] \\ &= -\|u_x\|^2 + \alpha\|u\|^2 - \dot{y}\int us' - \alpha\int uAu \qquad (3.14) \\ &- \int su_{xx} + \int su_x u - \alpha\int su + \alpha\int sAu \end{aligned}$$

In the calculation above and everywhere in the sequel we keep the notation s for the shifted function $s(.+y(t))$; we also use s_0 for the unshifted $s(.+0)$. Taking into account the definition of $\dot{y}$, and noting again that $\int uAu \geq 0$, we continue the estimate:

$$\leq -\|u_x\|^2 + \alpha\|u\|^2 - \langle us'\rangle^2 \underbrace{+ \int su_x u - \int su_{xx} - \alpha\int su + \alpha\int sAu} \qquad (3.15)$$

Terms containing s are estimated as follows

$$\underbrace{\cdots} = -\frac{1}{2}\int s'u^2 + \int s'u_x - \alpha\int (s - As)u \qquad (3.16)$$

$$\leq -\frac{1}{2}\int s'u^2 + \frac{1}{2}\|s'\|^2 + \frac{1}{2}\|u_x\|^2 + \frac{\alpha}{2}\|s - As\|^2 + \frac{\alpha}{2}\|u\|^2$$

Then we have:

$$\frac{d}{dt}\Phi(t) = \frac{1}{2}\frac{d}{dt}\|u-s\|^2$$
$$\underset{(3.15)\text{-}(3.16)}{\leq} -\langle us'\rangle^2 - \frac{1}{2}\|u_x\|^2 + \int (\frac{3}{2}\alpha - \frac{1}{2}s')u^2 + \frac{1}{2}\|s'\|^2 + \frac{\alpha}{2}\|s-As\|^2$$
$$\leq -\langle us'\rangle^2 - \frac{1}{2}\|u_x\|^2 + \int u^2(2\alpha - \frac{1}{2}s') - \frac{\alpha}{2}\|u\|^2 + \frac{1}{2}(\|s'\|^2 + \alpha\|s-As\|^2)$$

So far s was arbitrary. Now we select s that satisfies

$$s' = 4\alpha - 2b(x)$$

where the particular choice of $b(x)$ is described in the lemma above.

To guarantee s being periodic we demand

$$\int s' = \int (4\alpha - 2b(x)) = 4\alpha L - 2B = 0,$$

which implies $B = 2\alpha L$. Note that

$$\int ub = \langle ub\rangle = \langle u(2\alpha - \frac{1}{2}s')\rangle = -\frac{1}{2}\langle us'\rangle$$

consequently $\langle ub\rangle^2$ in the Lemma can be replaced by $\langle us'\rangle^2/4$. Then

$$\frac{1}{2}\frac{d}{dt}\|u-s\|^2 \underset{(3.13)}{\leq}$$
$$\leq (-1 + \frac{1}{2\alpha L})\langle us'\rangle^2 - \frac{1}{4}\|u_x\|^2 - \frac{\alpha}{2}\|u\|^2 + \frac{1}{2}(\|s'\|^2 + \alpha\|s-As\|^2)$$
$$\underset{L>1/(2\alpha)}{\leq} -\frac{1}{4}\|u_x\|^2 - \frac{\alpha}{2}\|u\|^2 + \frac{1}{2}(\|s'\|^2 + \alpha\|s-As\|^2)$$
$$\leq -\frac{1}{4}\|u_x\|^2 - \frac{\alpha}{4}\|u-s\|^2 + \frac{\alpha}{2}\|s\|^2 + \frac{1}{2}(\|s'\|^2 + \alpha\|s-As\|^2)$$

where in the last step we used an elementary inequality

$$-\|u\|^2 \leq -\frac{1}{2}\|u-s\|^2 + \|s\|^2$$

that follows from the triangle inequality. Thus,

$$\frac{1}{2}\frac{d}{dt}\|u-s\|^2 \leq -\frac{\alpha}{4}\|u-s\|^2 + \underbrace{\frac{1}{2}(\|s'\|^2 + \alpha\|s\|^2 + \alpha\|s-As\|^2)}$$

The bound for the terms $\underbrace{\ldots}$ is easily obtained. First:

$$\frac{1}{2}\|s'\|^2 = -8\alpha^2 L + \frac{64}{3}B^3 \underset{B=2\alpha L}{\leq} C_1(\alpha L)^3$$

also

$$\frac{1}{2}(\|s\|^2 + \|s - As\|^2 \leq \frac{1}{2}(2 + \|A\|^2)\|s\|^2 \underset{\|A\|<1}{<} \frac{3}{2}\|s\|^2 := \frac{3}{2}C_s\alpha^2 L^3$$

Therefore $\underbrace{\ldots} \leq C(\alpha L)^3$ and consequently

$$\frac{1}{2}\frac{d}{dt}\|u - s\|^2 \leq -\frac{\alpha}{4}\|u - s\|^2 + C\alpha^3 L^3.$$

Finally, from the Gronwall Lemma we get

$$\|u - s\|^2 \leq \left(\|u_0 - s_0\|^2 - 2C\alpha^3 L^3\right)\exp(-\frac{\alpha}{2}t) + 2C\alpha^3 L^3,$$

showing the exponential approach to the absorbing set. This inequality yields

$$\|u\| \leq \|u - s\| + \|s\|$$
$$\leq (\|u_0 - s_0\|^2 - 2C\alpha^3 L^3)^{1/2}\exp(-\frac{\alpha}{4}t) + \sqrt{2C}(\alpha L)^{3/2} + \alpha\sqrt{C_s}(L)^{3/2}$$

where C_s is an absolute constant.

Remark 3.4 *In the above proof the specific form of the operator A is immaterial. The proof holds for any bounded operator A.*

Corollary 3.5 *Denote $R_a = C_a(\alpha L)^{3/2}$. Then for any ball $B_R = \{||u_0|| \leq R\}$,*
(i) for all $t > 0$, $||u(.,t)|| \leq R + 2R_a$;
(ii) there exist $t_0 = t_0(R)$ so that $||u(.,t)|| \leq 3R_a$ for any $t > t_0$

Proof If $\|u - s\| \geq 2\alpha(CL^3)^{1/2}$, or if $||u(.,t)|| > 2\alpha(CL^3)^{1/2} + \|s\|$ then

$$\frac{d}{dt}\int (u - s)^2 dx < -\alpha R_a^2$$

then $||u(.,t) - s||^2 \leq 2R_a^2$ for $t > t_0$ for some $t_0 > 0$. Therefore $||u(.,t)|| \leq (\sqrt{2} + 1)R_a < 3R_a$

We note that the constants in the proof above are not necessarily optimal.

4 Compact attractor

A compact attractor is obtained as an ω-limit set of the absorbing ball B_{3R_a}, $\mathcal{A} = \cap_{t_0 \geq 0} \overline{\Omega(t_0)}$. We have already proved uniform boundedness for the orbits. To demonstrate compactness via Rellich's theorem it is sufficient to obtain a uniform bound on the derivative:

Theorem 4.1 *For any t, and any $r > 0$,*

$$\|u_x(.,t+r)\|^2 \leq (\alpha + \frac{1}{r})R^2 \exp\{(\alpha r + 1)\frac{LR^2}{2} + 2\alpha\}$$

where $R = \|u\,(.,0)\| + 2R_a$.

$$\frac{1}{t}\int_0^t \|u_x\|^2 \leq \alpha R^2 + \frac{1}{t}R^2 = (\alpha + \frac{1}{t})C\alpha^2 L^3 \tag{4.17}$$

Proof To obtain an estimate on $\|u_x\|^2$ we shall need a variant of the energy estimate for the derivative. We multiply the equation by u_{xx} and integrate by parts

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_x\|^2 &= -\|u_{xx}\|^2 + \alpha\|u_x\|^2 - \alpha\int u_x A u_x - \int u u_x u_{xx} \\ &\leq -\|u_{xx}\|^2 + \alpha\|u_x\|^2 + |\int u u_x u_{xx}|\end{aligned}$$

The cubic term is routinely handled as follows

$$\begin{aligned}|\int u u_x u_{xx}| \leq \max|u(x)|\int|u_x u_{xx}| &\underset{\text{Poincaré'}}{\leq} L^{1/2}\|u_x\|\int|u_x u_{xx}| \\ &\leq \|u_{xx}\|^2 + \frac{1}{4}L\|u_x\|^4\end{aligned}$$

Therefore

$$\frac{1}{2}\frac{d}{dt}\|u_x\|^2 \leq \|u_x\|^2(\alpha + \frac{1}{4}L\|u_x\|^2)$$

On the other hand, integrating the 1st energy identity from t to $t+r$, we see that

$$\int_t^{t+r}\|u_x\|^2 = \alpha\int_t^{t+r}\|u\|^2 - \alpha\int_t^{t+r}\int uAu - \frac{1}{2}\|u\|^2]_t^{t+r} \leq \alpha r R^2 + R^2 \tag{4.18}$$

where $R = 3R_a$ is a uniform bound for $\|u\|$, see Corollary 3.5.

Finally applying the Uniform Gronwall Lemma ([Temam, p.91]) with $y = \|u_x\|^2$, $g = 2\alpha + \frac{1}{2}L\|u_x\|^2$, and $h = 0$) yields

$$\|u_x(.,t+r)\|^2 \leq (\alpha + \frac{1}{r})R^2 \exp\{(\alpha r + 1)\frac{LR^2}{2} + 2\alpha\} \tag{4.19}$$

Corollary 4.2 *If $u(.,t)$ belongs to the attractor then*

$$\|u_x(.,t)\|^2 \leq (\alpha + 1)R_a^2 \exp\{\frac{\alpha+1}{2}R_a^2 L + 2\alpha\} \tag{4.20}$$

The estimate in (4.19) together with the uniform bounds on $\|u(.,t)\|$ allow us to conclude that the orbit $\cup_{t>t_0} S(t)B_R$ is precompact in H^0. Thus we conclude that the ω-limit set of B_R is a maximal, connected, compact attractor in H^0.

5 Dimension of attractor

In order to obtain the estimate on the Hausdorff dimension of the attractor we study evolution of the infinitesimal volume along the orbits in the attractor. We demonstrate that for sufficiently large m "the m-dimensional volume" decays exponentially. This property combined with compactness of the attractor and differentiability of the semigroup yield that the Hausdorff dimension of the attractor is no larger than m. In the argument regarding the Hausdorff dimension of the attractor we follow quite closely the ideas outlined in [Temam].

Let v be a solution of (2.8) in the attractor, and $u = v + \varepsilon w$, its perturbation, then the linearization is given by

$$z_t = -vz_x - zv_x + z_{xx} + \alpha z - \alpha(I - \partial_x^2)^{-1}z =: \mathcal{L}[v]z \tag{5.21}$$

Since v is smooth, clearly the linearized problem is well-posed in H^0.

Now we will estimate the evolution of the volume element. To this end we need to estimate the trace of the finite-dimensional projections of the generator of the linear semigroup. Let ξ_j, $j = 1, ..., m$ be elements of H^0 and let z_j be the solution of the linearized problem with initial data ξ_j. Then the volume element spanned by $\{\xi_1, \ldots, \xi_m\}$ evolves accordingly to the formula

$$|z_1(t) \wedge \ldots \wedge z_m(t)| = |\xi_1 \wedge \ldots \wedge \xi_m)| \exp \int_0^t Tr\,[\mathcal{L}[v(\tau)]) \circ Q_m(\tau)]d\tau,$$

where $Q_m(\tau)$ is the projector in H^0 onto the space spanned by $\Xi(\tau) = \{z_1(\tau), \ldots, z_m(\tau)\}$.

In order to calculate the trace we choose a basis $\{\phi_1, \ldots, \phi_m\}$ in $\Xi(\tau)$ orthonormal in H^0. Then for the corresponding diagonal entry of the matrix of the m-dimensional projection we have

$$\langle \mathcal{L}\phi_j, \phi_j\rangle = -\left\|\partial\phi_j\right\|^2 + \alpha||\phi_j||^2 + \int v\phi_j\partial\phi_j - \alpha\int\phi_j A\phi_j,$$

The term $\int v\phi_j\partial\phi_j$ is estimated as follows

$$\begin{aligned}
\left|-\int v\phi_j\partial\phi_j\right| &\leq \sup|v|\int|\phi_j||\partial\phi_j| \underset{\text{Swartz}}{\leq} \sup|v|\,\|\phi_j\|\;\;\|\partial\phi_j\| \\
\underset{\text{Young}}{\leq} \frac{1}{2}\left(\sup|v|\,\|\phi_j\|\right)^2 + \frac{1}{2}\left\|\partial\phi_j\right\|^2 &\underset{\text{Poincaré}}{\leq} \frac{1}{2}||\sqrt{L}v_x||^2\left\|\phi_j\right\|^2 + \frac{1}{2}\left\|\partial\phi_j\right\|^2 \\
&= \frac{1}{2}L\left\|v_x\right\|^2 + \frac{1}{2}\left\|\partial\phi_j\right\|^2
\end{aligned}$$

For the trace we get

$$\sum_{j=1}^{m}\langle\mathcal{L}\phi_j, \phi_j\rangle \leq -\frac{1}{2}\sum_{j=1}^{m}\left\|\partial\phi_j\right\|^2 + \alpha m + \frac{m}{2}L\left\|v_x\right\|^2$$

Since the eigenvalues of $\partial^2/\partial x^2$ are given by $(2\pi j/L)^2$, $j = 1, 2, \ldots$, then for some absolute constant c,

$$\sum_{j=1}^{m}\left\|\partial\phi_j\right\|^2 \geq cm^3/L^2$$

As a result we get the estimate for the trace

$$\begin{aligned}
\frac{1}{t}\int_0^t\sum_{j=1}^{m}\langle\mathcal{L}\phi_j, \phi_j\rangle\,dt &\leq -\frac{1}{2}cm^3/L^2 + \alpha m + \frac{m}{2}L\left(\left(\frac{1}{t}\int_0^t\left\|v_x\right\|^2 d\tau\right)dt\right) \\
&\underset{(4.17)}{\leq} -\frac{1}{2}cm^3/L^2 + \alpha m + c_1 m\alpha^3L^4
\end{aligned}$$

Thus, the trace becomes negative when

$$m \sim \alpha^{3/2}L^3$$

Remark 5.1 *A substantially more involved argument, based on a more precise estimate of the cubic terms (due to Lieb and Thirring) yields a better estimate for the dimension:* $m \sim cL^{29/18}$. *However, in view of the remarks on the relation between dynamics and dimension made in the introduction, we present a more trasparent estimate given above.*

6 Differentiability of the semigroup

To utilize the trace estimate developed in the previous section we need to demonstrate that the nonlinear evolution of the volume is well approximated by its linear counterpart. This will be ensured by the differentiability of the semigroup solving the problem with respect to the initial conditions, cf. [Temam].

Theorem 6.1 *Let* U *and* W *be two orbits in the attractor* $U = S(t)U_0$, $W = S(t)W_0$, *and* $z(t)$ *be a solution of the linearized problem (5.21) with the initial condition* $z(0) = U_0 - W_0$. *Then for any* t, $0 \le t \le t_0$,

$$\|U(t) - W(t) - z(t)\| \le C \;\; \|U_0 - W_0\|^2$$

where the constant C *depends only on* t_0.

In this case the Frechét differential of $S(t)$ at the point U_0 is the mapping $z(0) \to z(t)$.

Proof The difference $w = U - W$ solves the following problem

$$w_t = w_{xx} + \alpha w - \alpha(I - \partial_x^2)^{-1} w - wU_x - Ww_x$$
$$w(x, 0) = U_0(x) - W_0(x).$$

As usual we perform an energy estimate:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w\|^2 &= -\|w_x\|^2 + \alpha\|w\|^2 - \underbrace{\alpha\int w(I - \partial_x^2)^{-1} w}_{\ge 0} + \int (2U - W) w_x w \\
&\le -\|w_x\|^2 + \alpha\|w\|^2 + \underbrace{\sqrt{L}(2\|U_x\| + \|W_x\|)}_{\text{Poincaré}}\left(\frac{1}{2b}\|w\|^2 + \frac{b}{2}\|w_x\|^2\right) \\
&\le -\|w_x\|^2 + \alpha\|w\|^2 + 3C\sqrt{L}\left(\frac{1}{2b}\|w\|^2 + \frac{b}{2}\|w_x\|^2\right) \\
&\le -\frac{1}{2}\|w_x\|^2 + (\alpha + c)\|w\|^2,
\end{aligned}$$

where C is the uniform estimate on $\|U_x\|, \|W_x\|$ (cf. (4.20)); the constant b is selected so that $3bC\sqrt{L} = 1$. Therefore

$$\|w(.,t)\|^2 \le \|w(.,0)\|^2 \exp(2(\alpha+c)t)$$

In addition

$$\begin{aligned}\int_0^t \|w_x\|^2 &\le (\alpha+c)\int_0^t \|w\|^2 + \|w(.,t)\|^2 - \|w(.,0)\|^2 \\ &\le \|w(.,0)\|^2 \left[\frac{\exp(2(\alpha+c)t)-1}{2(\alpha+c)} + \exp(2(\alpha+c)t) - 1\right] \\ &:= C_0 \|U_0 - W_0\|^2\end{aligned} \tag{6.22}$$

For the difference $y = w - z$ we have the following problem

$$y_t = y_{xx} + \alpha y - \alpha(I - \partial_x^2)^{-1} y - U_x y - U y_x + w w_x, \tag{6.23}$$

We multiply the equation in (6.23) by y and integrate by parts to obtain the following identity for the norm:

$$\frac{1}{2}\frac{d}{dt}\|y\|^2 = -\|y_x\|^2 + \alpha\|y\|^2 - \alpha\int y(I-\partial_x^2)^{-1}y + \int U y y_x + \int w w_x y \tag{6.24}$$

In the right hand side we drop the positive quadratic term $\alpha \int y(I-\partial_x^2)^{-1}y$ and use

$$\int w w_x y \underset{\text{Poincaré}}{\le} \sqrt{L}\|w_x\| \int w_x y \underset{\text{Schwartz}}{\le} \sqrt{L}\|w_x\|^2 \|y\|$$

to estimate the right hand side of (6.24) as follows:

$$\le -\|y_x\|^2 + \alpha\|y\|^2 + \sqrt{L}\|U_x\|\left(\frac{1}{2b}\|y\|^2 + \frac{b}{2}\|y_x\|^2\right) + \sqrt{L}\|w_x\|^2\|y\|$$

We select b so that

$$\sqrt{L}\|U_x\|\frac{b}{2} < 1$$

which yields

$$\frac{1}{2}\frac{d}{dt}\|y\|^2 \le C\|y\|^2 + \sqrt{L}\|w_x\|^2\|y\|$$

or, consequently,

$$\frac{d}{dt}\|y\| \le C\|y\| + \sqrt{L}\|w_x\|^2.$$

Then from the Gronwall lemma we get

$$\|y(.,t)\| \le e^{Ct}\int_0^t \sqrt{L}\|w_x\|^2 e^{-C\tau}d\tau \underset{(6.22)}{\le} \sqrt{L}e^{Ct}C_0\|U_0 - W_0\|^2$$

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