

STABILITY OF THE TRAVELLING WAVE IN A 2D WEAKLY NONLINEAR STEFAN PROBLEM

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This paper is dedicated to the memory of Basil Nicolaenko

ABSTRACT. We investigate the stability of the travelling wave (TW) solution in a 2D Stefan problem, a simplified version of a solid-liquid interface model. It is intended as a paradigm problem to present our method based on: (i) definition of a suitable linear one dimensional operator, (ii) projection with respect to the x coordinate only; (iii) Lyapunov-Schmidt method. The main issue is that we are able to derive a parabolic equation for the corrugated front φ near the TW as a solvability condition. This equation involves two linear pseudo-differential operators, one acting on φ , the other on $(\varphi_y)^2$ and clearly appears as a generalization of the Kuramoto-Sivashinsky equation in combustion theory. A large part of the paper is devoted to study the properties of these operators in the context of functional spaces in the y and x, y coordinates with periodic boundary conditions. Technical results are deferred to the appendices.

1. Introduction. In this paper, we consider a scalar two dimensional Stefan-like free boundary problem (in short FBP), a simplified version of a solid-liquid interface model introduced by Frankel [9, 10]. For simplicity we deal directly with the non-dimensional problem. The solidification front is represented by $x = \xi(t, y)$. The (supercooled) liquid phase is for $x < \xi(t, y)$, the solid phase is for $x > \xi(t, y)$. The dynamics of heat is described by the heat conduction equation

$$T_t = \Delta T, \quad x \neq \xi(t, y), \quad (1.1)$$

$x, y \in \mathbb{R}$. It will be convenient to assume periodicity in y with period ℓ , and restrict attention to $y \in [-\ell/2, \ell/2]$. At $x = -\infty$, the temperature of the liquid is

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normalized to 0. At the front $x = \xi(t, y)$ we have two conditions. First, the balance of energy at the interface is given by the jump

$$\left[\frac{\partial T}{\partial n} \right] = V_n = \frac{\xi_t}{\sqrt{1 + \xi_y^2}}, \quad (1.2)$$

where V_n is the normal velocity. Second, according to the Gibbs-Thompson law, the non-equilibrium interface temperature is defined by

$$T = 1 - \gamma\kappa + r(V_n), \quad (1.3)$$

where the melting temperature has been normalized to 1, κ is the interface curvature and the positive constant γ represents the solid-liquid surface tension. The function r is increasing and such that $r(-1) = 0$, $r'(-1) = 1$ (see [9, 10]). However, we assume throughout this paper that $r - 1$ is linear, hence (1.3) becomes:

$$T = 1 - \gamma\kappa + 1 + V_n. \quad (1.4)$$

It can be easily seen that the system (1.1), (1.2), coupled with (1.3) or (1.4) admits a one-phase planar travelling wave (in short TW) solution $\hat{T}$, corresponding to a planar front, which satisfies

$$\hat{T}_x = \hat{T}_{xx}, \quad x \neq 0.$$

At the front $x = 0$, $[\hat{T}_x] = -1$, $\hat{T} = 1$, hence $\hat{T}(x) = e^x$ for $x \leq 0$, and $\hat{T}(x) = 1$ for $x > 0$.

We are interested in two kinds of results:

(i) first, we are interested in the stability of the TW. We anticipate the following

Main Theorem. *There exists a $\gamma_c < 1$ such that, for $\gamma > \gamma_c$ the TW is orbitally stable (with asymptotic phase). For $0 < \gamma < \gamma_c$ it is unstable. As $\ell \rightarrow +\infty$, $\gamma_c \rightarrow 1$.*

(ii) Second, in the spirit of the Kuramoto-Sivashinsky (K-S) equation in combustion theory [14]

$$\psi_t + \nu\psi_{yyyy} + \psi_{yy} + \frac{1}{2}(\psi_y)^2 = 0, \quad \nu > 0,$$

we want to derive a 1D equation for the corrugated front, at least near the planar front. To achieve this goal, we use a method based on a suitable projection with respect to the x coordinate only, that we already used in [5, 11] to derive a Burgers equation.

Because of the complexity of the calculations, we present our analysis on a paradigm model, a weakly nonlinear, quasi-steady version of (1.1), (1.2) and (1.4), whose physical relevancy will not be discussed here. It is obtained by assuming a slightly distorted planar front propagating along the y -axis, $\xi(t, y) = -t + \varphi(t, y)$, therefore we reformulate the problem for φ and the perturbation of the temperature $u = T - \hat{T}$; we replace the curvature κ by the second-order derivative and V_n by $-1 + \varphi_t + \frac{1}{2}(\varphi_y)^2$. Hence, (1.4) reads:

$$u|_{x=0} = -\gamma\varphi_{yy} + \varphi_t + \frac{1}{2}(\varphi_y)^2. \quad (1.5)$$

Equation (1.1) for the temperature yields:

$$u_t + (1 - \varphi_t)u_x - \Delta_\varphi u - \varphi_t \hat{T}_x = (\Delta_\varphi - \Delta)\hat{T}, \quad (1.6)$$

where $\Delta_\varphi = (1 + (\varphi_y)^2)D_{xx} + D_{yy} - \varphi_{yy}D_x - 2\varphi_y D_{xy}$.

Beside, it has been observed in similar problems [2, 3] that, at least not far from the instability threshold, the time derivative in the temperature equation has a relatively small effect on the solution. Based on this observation one can define a *quasi-steady* model which reads, keeping only the linear terms for simplicity:

$$u_x - \Delta u - \varphi_t \hat{T}_x = ((\varphi_y)^2 - \varphi_{yy}) e^x \chi_{(-\infty, 0)} = ((\varphi_y)^2 - \varphi_{yy}) \hat{T}_x. \quad (1.7)$$

The front is now fixed at $x = 0$. The first condition (1.2) is equivalent to:

$$\sqrt{1 + (\varphi_y)^2} \left[\frac{\partial T}{\partial x} \right] = \frac{-1 + \varphi_t}{\sqrt{1 + (\varphi_y)^2}}.$$

Hence, the equations for the velocity at the front read up to second-order:

$$(a) \quad \varphi_t = \left[\frac{\partial u}{\partial x} \right] - (\varphi_y)^2, \quad (b) \quad \varphi_t = u|_{x=0} + \gamma \varphi_{yy} - \frac{1}{2} (\varphi_y)^2. \quad (1.8)$$

Our final problem is then (1.7), (1.8). Moreover, we look for solutions u , which are continuous at $x = 0$. Periodicity conditions are assumed at $y = \pm \ell/2$ for formulating the rigorous results. Exponential decay holds at $x \rightarrow -\infty$, whereas algebraic growth is allowed as $x \rightarrow +\infty$. In this respect weights will be introduced in the mathematical setting.

Our method is based on three main steps: (i) definition of a suitable linear one dimensional operator $\mathcal{L}$ in the x variable only, with kernel spanned by U ; (ii) projection with respect to the x coordinate only, the front equation appearing as a natural solvability condition; (iii) lifting of the condition (1.8a) and a Lyapunov-Schmidt method via (1.8b).

A large part of our work is devoted to the study of a suitable realization L of the operator $\mathcal{L}$ and the computation of its resolvent. Several functional analysis tools are needed in this respect. For the convenience of the reader, we defer this material to the appendices, see also [4].

We eventually get to a self-consistent parabolic equation for the front φ :

$$\frac{d\varphi}{dt} + G((\varphi_y)^2) = \Omega\varphi, \quad (1.9)$$

where both Ω and G are linear pseudo-differential operators. Introducing a complete set of eigenfunctions $\{w_k\}$ of the operator D_{yy} with ℓ -periodic boundary conditions, corresponding to the non-positive eigenvalues (counted with multiplicity) $-\lambda_k$, we define naturally through the discrete Fourier transform:

$$\varphi = \sum_{k=0}^{+\infty} \hat{\varphi}(k) w_k, \quad \Omega\varphi = \sum_{k=0}^{+\infty} \omega_k \hat{\varphi}(k) w_k,$$

where the symbol or Fourier multipliers ω_k is explicitly the set of

$$\omega_k = -\lambda_k \frac{\gamma X_k^2 + \gamma X_k - 2}{X_k^2 + 2X_k - 1}, \quad X_k = \sqrt{1 + 4\lambda_k}.$$

Likewise, for G acting on φ_y^2 ,

$$G\varphi = \sum_{k=0}^{+\infty} g_k \hat{\varphi}(k) w_k, \quad g_k = \frac{1}{2} \frac{X_k^2 + 3X_k - 2}{X_k^2 + 2X_k - 1},$$

where

$$g_k \geq \frac{1}{2}; \quad g_k \rightarrow \frac{1}{2} \quad (X_k \rightarrow 1); \quad g_k \rightarrow \frac{1}{2} \quad (X_k \rightarrow +\infty),$$

is reminiscent of $g_k \equiv \frac{1}{2}$ in K-S.

We show that the operator Ω is sectorial in the framework of spaces consisting of periodic functions with zero mean value and compute its spectrum. These will be the main tools to prove our main theorem and compute γ_c explicitly.

Finally, it is interesting to think of our results in the whole space. With a slight abuse of notation, using the wave number k now as a continuous Fourier variable for y , the growth rate $\omega(k)$ reads

$$\omega(k) = -k^2 \frac{\gamma X_k^2 + \gamma X_k - 2}{X_k^2 + 2X_k - 1}, \quad X_k = \sqrt{1 + 4k^2},$$

which is the dispersion relation of our simplified FBP. It is not difficult to see that the growth rate expands as

$$\omega(k) = (1 - \gamma)k^2 + (\gamma - 4)k^4 + \dots \quad (k \rightarrow 0); \quad \omega(k) \sim -\gamma k^2 \quad (k \rightarrow +\infty),$$

with exchange of stability at $\gamma_c = 1$ as predicted in the above theorem. Note that in the short waves, $\omega(k)$ decays as $-k^2$ in contrast to K-S, see the discussion in [1].

We will extend our method to the full problem (1.1), (1.2), (1.4), and other FBP's, in a series of forthcoming papers. Note that the general formulation of the front equation is a 2d-order in time, 4th-order in space equation as it has been already observed in [1]. In the paper [4], we use Sivashinsky's scaled variables

$$y = \frac{\eta}{\sqrt{\varepsilon}}, \quad t = \frac{\tau}{\varepsilon^2}, \quad \varphi = \varepsilon\psi,$$

for a rigorous asymptotical derivation of K-S equation.

2. Some mathematical setting. In this section we introduce some notation, the functional spaces and operators we will use below. Moreover, we introduce the main technical results deferring their proofs to the appendices.

2.1. Discrete Fourier transform and functional spaces in y . We will mainly use the discrete Fourier transform with respect to the variable y . For this purpose, given a function $f : (-\ell/2, \ell/2) \rightarrow \mathbb{C}$, we denote by $\hat{f}(k)$ its k -th Fourier coefficient, that is, we write

$$f(y) = \sum_{k=0}^{+\infty} \hat{f}(k) w_k(y), \quad y \in (-\ell/2, \ell/2),$$

where $\{w_k\}$ is a complete set of eigenfunctions of the operator

$$A = D_{yy} : D(A) = H^2(-\ell/2, \ell/2) \rightarrow L^2(-\ell/2, \ell/2),$$

with ℓ -periodic boundary conditions, corresponding to the non-negative eigenvalues

$$0, -\frac{4\pi^2}{\ell^2}, -\frac{4\pi^2}{\ell^2}, -\frac{16\pi^2}{\ell^2}, -\frac{16\pi^2}{\ell^2}, -\frac{36\pi^2}{\ell^2}, \dots$$

We shall find it convenient to label this sequence as

$$0 = -\lambda_0 > -\lambda_1 = -\lambda_2 > -\lambda_3 = -\lambda_4 > \dots \quad (2.1)$$

When f depends also on t and/or x , by $\hat{f}(\cdot, k)$ we denote the k -th Fourier coefficient of f with respect to y . For instance, for fixed t and x , $\hat{f}(t, x, k)$ will denote the k -th Fourier coefficient of the function $f(t, x, \cdot)$.

For integer or arbitrary real s we denote by $\dot{H}^s$ the usual Sobolev spaces of ℓ -periodic (generalized) functions with zero average, which we will conveniently represent as

$$\dot{H}^s = \left\{ w = \sum_{k=1}^{+\infty} a_k w_k : \sum_{k=1}^{+\infty} \lambda_k^s a_k^2 < +\infty \right\},$$

with norm

$$\|w\|_s^2 = \sum_{k=1}^{+\infty} |\lambda_k|^s a_k^2.$$

Note that the eigenvalue zero does not occur in this setting because nonzero constant functions are excluded from our Sobolev spaces. This can be rephrased saying that $\dot{H}^s$ is the set of all the functions $u \in H^s(-\ell/2, \ell/2)$ such that $\mathcal{M}u = 0$, where

$$\mathcal{M}u = \frac{1}{\ell} \int_{-\ell/2}^{\ell/2} u(y) dy. \quad (2.2)$$

Of course, the functions w_k ($k = 1, 2, \dots$) are also eigenfunctions of the “Kuramoto-Sivashinsky” operator $\nu A^2 + A$, with eigenvalues $\nu \lambda_k^2 - \lambda_k$.

Next, for any $\beta \geq 0$, we denote by $\dot{C}^\beta$ the space defined as follows:

(i) if $\beta \in \mathbb{N}$, then

$$\dot{C}^\beta = \{f \in C^\beta([-\ell/2, \ell/2]) : f^{(j)}(-\ell/2) = f^{(j)}(\ell/2), \ j \leq \beta - 1, \ \mathcal{M}f = 0\},$$

(ii) if $\beta \notin \mathbb{N}$, then

$$\dot{C}^\beta = \{f \in C^\beta([-\ell/2, \ell/2]) : f^{(j)}(-\ell/2) = f^{(j)}(\ell/2), \ j \leq [\beta], \ \mathcal{M}f = 0\}.$$

In particular, $\dot{C} = (I - \mathcal{M})(C([-\ell/2, \ell/2]))$. All the spaces $\dot{C}^\beta$ are endowed with the natural norm of the corresponding spaces $C^\beta([-\ell/2, \ell/2])$.

2.2. Functional spaces in x and in x, y . We denote by I , I_- and I_+ , respectively, the sets

$$I = \mathbb{R} \times [-\ell/2, \ell/2], \quad I_- = (-\infty, 0] \times [-\ell/2, \ell/2], \quad I_+ = [0, +\infty) \times [-\ell/2, \ell/2].$$

Moreover, we use the bold notation to denote functions which belong to the space $C((-\infty, 0] \times C([0, +\infty)))$ (resp. to $C(I_-) \times C(I_+)$), or, equivalently, functions which are continuous in the $\mathbb{R} \setminus \{0\}$ (resp. I) and admit finite left- and right-limits at $x = 0$. Given a function $\mathbf{u}$ in the previous space, we denote, respectively, by u_1 and u_2 its components. We write $D_x^{(i)} \mathbf{u}$ (resp. $D_y^{(i)} \mathbf{u}$) ($i = 1, 2, \dots$) to denote the function whose components are $D_x^{(i)} u_1$ and $D_x^{(i)} u_2$ (resp. $D_y^{(i)} u_1$ and $D_y^{(i)} u_2$).

By $\mathbf{T}$ and $\mathbf{T}'$ we denote the functions defined, respectively, by

$$\begin{cases} T_1(x) = e^x, & x \leq 0, \\ T_2(x) = 1, & x \geq 0, \end{cases} \quad \begin{cases} T'_1(x) = e^x, & x \leq 0, \\ T'_2(x) = 0, & x > 0. \end{cases}$$

We also define the functions $\mathbf{U}$ and $\mathbf{V}$ by setting

$$\begin{cases} U_1(x) = \frac{1-x}{3} e^x, & x \leq 0, \\ U_2(x) = \frac{1}{3}, & x \geq 0, \end{cases} \quad \begin{cases} V_1(x) = \left(1 - \frac{2}{3}x + \frac{x^2}{6}\right) e^x, & x \leq 0, \\ V_2(x) = 1 + \frac{x}{3}, & x \geq 0. \end{cases} \quad (2.3)$$

Given a function $\mathbf{u} \in C((-\infty, 0]) \times C([0, +\infty))$, we denote by $\tilde{\mathbf{u}}$ the function defined by

$$\tilde{u}_1(x) = e^{-\frac{x}{2}} u_1(x), \quad x \leq 0, \quad \tilde{u}_2(x) = e^{-\frac{x}{2}} u_2(x), \quad x \geq 0.$$

Next, we introduce the space $\mathcal{X}$ defined as follows:

$$\mathcal{X} = \left\{ \mathbf{f} = (f_1, f_2) \in C(I_-) \times C(I_+) : \tilde{f}_1 \in C_b(I_-), \tilde{f}_2 \in C_b(I_+) \right\},$$

where “ b ” stands for bounded. It is a Banach space when endowed with the norm

$$\|\mathbf{f}\|_{\mathcal{X}} = \|\tilde{f}_1\|_{C_b(I_-)} + \|\tilde{f}_2\|_{C_b(I_+)}.$$

2.3. The linear operator $\mathcal{L}$ and its realization L . The formal differential operator

$$\mathcal{L} : u \mapsto D_{xx}u - D_xu + u|_{x=0} \hat{T}',$$

and equations of the form $\mathcal{L}u = f$ will appear several times in this paper. We shall be solving equations of the form $\mathcal{L}u = f$ with two jump conditions, namely continuity of u across $x = 0$ and a jump of D_xu , written, in terms of $\mathbf{u} = (u_1, u_2)$, as $u_2(0, \cdot) = u_1(0, \cdot)$ and $D_xu_2(0, \cdot) - D_xu_1(0, \cdot) = g$. The realization L of $\mathcal{L}$ is given by

$$\begin{cases} D(L) = \left\{ \mathbf{u} \in C^{2,0}(I_-) \times C^{2,0}(I_+) : \mathbf{u}, L\mathbf{u} \in \mathcal{X} \right. \\ \quad \left. D_x^{(j)}u_1(0, \cdot) = D_x^{(j)}u_2(0, \cdot), \quad j = 0, 1 \right\}, \\ L\mathbf{u} = \begin{cases} D_{xx}u_1 - D_xu_1 + u_1(0, \cdot)e^x, & (x, y) \in I_-, \\ D_{xx}u_2 - D_xu_2, & (x, y) \in I_+. \end{cases} \end{cases}$$

Note that the condition $D_xu_2(0, \cdot) - D_xu_1(0, \cdot) = g$ prohibits $\mathbf{u}$ to be in the domain of L . For such $\mathbf{u}$ we shall write $\mathcal{L}\mathbf{u}$, regarding $x \leq 0$ and $x \geq 0$ separately.

Theorem 2.1. *The following properties are met:*

- (a) *the operator L is sectorial and, hence, it generates an analytic semigroup in $\mathcal{X}$;*
- (b) *the spectrum of the operator L consists of 0 and the halfline $(-\infty, -1/4]$;*
- (c) *the spectral projection on the kernel of L is the operator $\mathcal{P}$ defined by*

$$\mathcal{P}\mathbf{f} = \left(\int_{-\infty}^0 f_1(x, \cdot) dx + \int_0^{+\infty} e^{-x} f_2(x, \cdot) dx \right) \mathbf{U} := Q(\mathbf{f})\mathbf{U}, \quad \mathbf{f} \in \mathcal{X};$$

- (d) *let $\mathbf{f} \in \mathcal{X}$. Then, the equation $L\mathbf{u} = \mathbf{f}$ has a solution $\mathbf{u} \in D(L)$ if and only if $\mathcal{P}\mathbf{f} = 0$.*

Next we consider the operator $(L + A)$ defined by

$$\begin{cases} D(L + A) = \left\{ \mathbf{u} \in \tilde{u}_1 \in C^{2,0}(I_-) \cap C^{0,2}(I_-) \times C^{2,0}(I_+) \cap C^{0,2}(I_+) : \right. \\ \quad \mathbf{u}, \mathbf{u}_{yy}, \mathcal{L}\mathbf{u} \in \mathcal{X}, \quad D_x^{(j)}u_1(0, \cdot) = D_x^{(j)}u_2(0, \cdot), \quad j = 0, 1, \\ \quad \left. D_y^{(j)}u_i(\cdot, -\ell/2) = D_y^{(j)}u_i(\cdot, \ell/2), \quad i = 1, 2, \quad j = 0, 1 \right\}, \\ (L + A)\mathbf{u} = \begin{cases} D_{xx}u_1 + D_{yy}u_1 - D_xu_1 + u_1(0, \cdot)e^x, & (x, y) \in I_-, \\ D_{xx}u_2 + D_{yy}u_2 - D_xu_2, & (x, y) \in I_+. \end{cases} \end{cases} \quad (2.4)$$

Theorem 2.2. *The following properties are met.*

- (a) The operator $L + A$ is closable and its closure L_1 is sectorial;
- (b) the restriction of L_1 to $(I - \mathcal{P})(\mathcal{X})$ is sectorial and its spectrum is confined in the halfline $(-\infty, -1/4]$.

Remark 2.3. In the sense of distributions we could alternatively say that we are in fact solving $\mathcal{L}u = f + g\delta$ where δ is the Dirac distribution in x , without separating $x \leq 0$ and $x \geq 0$. We should then write $\mathcal{L}_d u$, subscript d for distributional to indicate the different viewpoint, For instance, $\mathcal{L}_d \hat{T} = \hat{T}' - \delta$. Letting $\mathcal{L}_d$ act on (continuous) functions $u(x)$ in the sense of distributions as

$$\mathcal{L}_d : u \mapsto u'' - u' + u(0)\hat{T}',$$

its formal adjoint is easily computed as

$$\mathcal{L}_d^* : u \mapsto u'' + u' + \left(\int_{\mathbb{R}} u(x)T'(x)dx \right) \delta,$$

The kernels of $\mathcal{L}_d$ and $\mathcal{L}_d^*$ are respectively spanned by U and U^* , where U is as in (2.3) and

$$U^*(x) = 1, \quad x \leq 0, \quad U^*(x) = e^{-x}, \quad x \geq 0.$$

The (spectral) projection on the kernel of $\mathcal{L}_d$ is given by

$$f \mapsto \left(\int_{\mathbb{R}} f(x)U^*(x)dx \right) U = \langle f, U^* \rangle U,$$

the latter also being valid for f which contain a Dirac.

2.4. A lifting operator. Let us introduce a suitable lifting operator $\mathcal{N} : \mathbb{R} \rightarrow \mathcal{X}$ which will play a fundamental role in what follows. It is defined by

$$\mathcal{N}g = g(\mathbf{V} - \mathbf{T}), \quad g \in \mathbb{R}. \quad (2.5)$$

This enlightens the role of the function $\mathbf{V}$ we defined in (2.3) in such a way that $[D_x \mathbf{V}] = 0$ and $\mathcal{L}\mathbf{V} = \mathbf{T}' - \mathbf{U}$. Note that $\mathcal{L}(\mathbf{V} - \mathbf{T}) = -\mathbf{U}$ while, with the notations of Remark 2.3, $\mathcal{L}_d(V - \hat{T}) = \delta - U$. So, $\mathcal{N}g$ takes care of the jump condition $[D_x \mathbf{u}] = g$ for the solution of $\mathcal{L}\mathbf{u} = \mathbf{f}$.

Taking advantage of the properties of the functions $\mathbf{V}$ and $\mathbf{T}$ here mentioned, it is immediate to check that the operator $\mathcal{N}$ enjoys the properties stated in the following proposition.

Proposition 2.4. *For any $g \in \mathbb{R}$, let us identify in the trivial way the function $\mathcal{N}(g)$ with a real valued function defined in $\mathbb{R} \setminus \{0\}$, then this function can be extended by continuity at zero setting $\mathcal{N}(0) = 0$ and it belongs to $C_b^\alpha(\mathbb{R}) \cap C_b^2((-\infty, 0]) \cap C_b^2([0, +\infty))$ for any $g \in \mathbb{R}$ and any $\alpha \in (0, 1)$. Further,*

$$Q(\mathcal{N}(1)) = \frac{4}{3}, \quad [D_x \mathcal{N}(1)] = 1, \quad Q(\mathcal{L}\mathcal{N}(1)) = -1. \quad (2.6)$$

3. An equivalent problem to (1.7), (1.8). The aim of this section consists in transforming the problem (1.7), (1.8) into an equivalent one. In deriving the equivalent problem, we assume that the pair (u, φ) is a solution to problem (1.7), (1.8) in the time domain $[0, T]$ (with possibly $T = +\infty$) with the following properties:

- (i) the function φ is continuously differentiable in $[0, T] \times [-\ell/2, \ell/2]$ once with respect to time and four times with respect to the spatial variables;
- (ii) φ_t is twice continuously differentiable with respect to y in $[0, T] \times [-\ell/2, \ell/2]$;

- (iii) u is twice continuously differentiable with respect to the spatial variables in $[0, T] \times I_-$ and in $[0, T] \times I_+$;
- (iv) the functions $(t, x, y) \mapsto e^{-x/2} D_x^{(i)} u(t, x, y)$ and $(t, x, y) \mapsto e^{-x/2} D_y^{(i)} u(t, x, y)$ are bounded in $[0, T] \times I_-$ and in $[0, T] \times I_+$ for any $i = 0, 1, 2$.

3.1. Elimination of φ_t . First we eliminate φ_t in (1.7) thanks to (1.8b)

$$u_x - \Delta u - \left(u(\cdot, 0, \cdot) + \gamma \varphi_{yy} - \frac{1}{2} (\varphi_y)^2 \right) \hat{T}_x = (\Delta_\varphi - \Delta) \hat{T},$$

or, equivalently

$$u_x - \Delta u - u(\cdot, 0, \cdot) \hat{T}_x = \left(\frac{1}{2} (\varphi_y)^2 - (1 - \gamma) \varphi_{yy} \right) \hat{T}_x. \quad (3.1)$$

Setting $\mathbf{u}(t, x, y) := (u(t, x, y) \chi_{(-\infty, 0](x)}, u(t, x, y) \chi_{[0, +\infty)}(x))$ and

$$\mathbf{F}_0 = \left((1 - \gamma) \varphi_{yy} - \frac{1}{2} (\varphi_y)^2 \right) \mathbf{T}', \quad g = \varphi_t + (\varphi_y)^2,$$

from (1.8) and (3.1), we easily see that the function $\mathbf{u}$ solves the problem

$$\begin{cases} \mathcal{L} \mathbf{u} = \mathbf{F}_0 - \mathbf{u}_{yy}, \\ u_2(\cdot, 0, \cdot) - u_1(\cdot, 0, \cdot) = 0, \\ D_x u_2(\cdot, 0, \cdot) - D_x u_1(\cdot, 0, \cdot) = g. \end{cases} \quad (3.2)$$

Remark 3.1. Obviously (3.2) is not equivalent to (1.6), (1.8) since we are missing condition (1.8a).

3.2. Lifting. Now we are going to use the first part of (1.8). Introducing the new unknown $\mathbf{v} = \mathbf{u} - \mathcal{N}(g)$, where $\mathcal{N}$ is the lifting operator in Proposition 2.4, with a straightforward computation, we see that the function $\mathbf{v}$ turns out to solve the problem

$$\begin{cases} \mathcal{L} \mathbf{v} = \mathbf{F}_0 - \mathbf{v}_{yy} - g_{yy} \mathcal{N}(1) - g \mathcal{L} \mathcal{N}(1), \\ v_2(\cdot, 0, \cdot) - v_1(\cdot, 0, \cdot) = 0, \quad D_x v_2(\cdot, 0, \cdot) - D_x v_1(\cdot, 0, \cdot) = 0, \end{cases} \quad (3.3)$$

Since by the property (iv) at the very beginning of the section, $\mathbf{u}(t, \cdot), \mathcal{L} \mathbf{u}(t, \cdot) \in \mathcal{X}$ for any $t \in [0, T]$, then taking Proposition 2.4 into account, it follows immediately that the function $\mathbf{v}$ belongs to $\mathcal{X}$ and, hence, to $D(L)$. Thus, (3.3) may be rewritten as

$$L \mathbf{v} = \mathbf{F}_0 - \mathbf{v}_{yy} - g_{yy} \mathcal{N}(1) - g \mathcal{L} \mathcal{N}(1),$$

where L is defined in Section 2.3.

3.3. Projection $\mathcal{P}$ with respect to the x coordinate. We are going to project the differential equation in (3.3), using $\mathcal{P}$, to derive an equation for the front φ . For this purpose, we observe that the function $\mathbf{v}$ is as smooth as the function $\mathbf{u}$. Moreover, it satisfies the boundary conditions $v_2(0, \cdot) = v_1(0, \cdot)$ and $D_x v_2(0, \cdot) = D_x v_1(0, \cdot)$. Hence, taking Theorem 2.1(d) into account, it follows that

$$0 = Q(\mathbf{F}_0 - \mathbf{v}_{yy} - g_{yy} \mathcal{N}(1) - g \mathcal{L} \mathcal{N}(1)). \quad (3.4)$$

We compute, using (2.5):

$$Q(\mathbf{F}_0) = (1 - \gamma) \varphi_{yy} - \frac{1}{2} (\varphi_y)^2,$$

$$\begin{aligned} Q(g_{yy}\mathcal{N}(1)) &= \frac{4}{3}g_{yy} = \frac{4}{3}(\varphi_{t yy} + ((\varphi_y)^2)_{yy}), \\ Q(g\mathcal{L}\mathcal{N}(1)) &= -g = -\varphi_t - (\varphi_y)^2. \end{aligned}$$

Hence, we can rewrite equation (3.4) as follows:

$$(1 - \gamma)\varphi_{yy} - \frac{1}{2}(\varphi_y)^2 - \frac{4}{3}(\varphi_{t yy} + (\varphi_y)_{yy}^2) + \varphi_t + (\varphi_y)^2 = Q(\mathbf{v}_{yy}),$$

or, equivalently,

$$\varphi_t - \frac{4}{3}\varphi_{t yy} + \frac{1}{2}(\varphi_y)^2 + (1 - \gamma)\varphi_{yy} - \frac{4}{3}(\varphi_y)_{yy}^2 = Q(\mathbf{v}_{yy}). \quad (3.5)$$

We now split $\mathbf{v}(t, \cdot)$ ($t \in [0, T]$) along $\mathcal{P}(\mathcal{X})$ and $(I - \mathcal{P})(\mathcal{X})$. So, writing $\mathbf{v} = a\mathbf{U} + \mathbf{w}$ and observing that our assumptions on u guarantee that the function $\mathbf{w}_{yy}$ belongs to $(I - \mathcal{P})(\mathcal{X})$, we get

$$Q(\mathbf{v}_{yy}) = Q(a_{yy}\mathbf{U} + \mathbf{w}_{yy}) = a_{yy}.$$

Let us compute a and its derivatives. For this purpose, we use the second relation in (1.8) to obtain

$$\frac{1}{3}a + w_1(\cdot, 0, \cdot) = -\gamma\varphi_{yy} + \varphi_t + \frac{1}{2}(\varphi_y)^2. \quad (3.6)$$

Thus,

$$\begin{aligned} a &= -3w_1(\cdot, 0, \cdot) - 3\gamma\varphi_{yy} + 3\varphi_t + \frac{3}{2}(\varphi_y)^2, \\ a_{yy} &= -3D_{yy}w_1(\cdot, 0, \cdot) - 3\gamma\varphi_{yy yy} + 3\varphi_{t yy} + \frac{3}{2}((\varphi_y)^2)_{yy}. \end{aligned}$$

It follows that

$$Q(\mathbf{v}_{yy}) = -3D_{yy}w_1(\cdot, 0, \cdot) - 3\gamma\varphi_{yy yy} + 3\varphi_{t yy} + \frac{3}{2}((\varphi_y)^2)_{yy}.$$

Replacing into (3.5) we get the following final equation for φ :

$$\varphi_t - \frac{13}{3}\varphi_{t yy} + 3\gamma\varphi_{yy yy} + (1 - \gamma)\varphi_{yy} + \frac{1}{2}(\varphi_y)^2 + 3D_{yy}w_1(\cdot, 0, \cdot) = \frac{17}{6}((\varphi_y)^2)_{yy}. \quad (3.7)$$

Remark 3.2. Equation (3.7) is the generalization of the Kuramoto-Sivashinsky equation for the front we were looking for. The next point is to compute $D_{yy}w_1(\cdot, 0, \cdot)$

3.4. Projection $I - \mathcal{P}$. We observe that the function $\mathbf{w} \in D(L)$ solves the equation

$$L\mathbf{w} = (I - \mathcal{P})(\mathbf{F}_0) - \mathbf{w}_{yy} - g_{yy}(I - \mathcal{P})\mathcal{N}(1) - g(I - \mathcal{P})\mathcal{L}\mathcal{N}(1). \quad (3.8)$$

From the computations in the previous subsections, we know that

$$\begin{aligned} (I - \mathcal{P})(\mathbf{F}_0) &= \left((1 - \gamma)\varphi_{yy} - \frac{1}{2}(\varphi_y)^2 \right) (\mathbf{T}' - \mathbf{U}), \\ g_{yy}(I - \mathcal{P})\mathcal{N}(1) &= (\varphi_{t yy} + ((\varphi_y)^2)_{yy}) \left(\mathbf{V} - \mathbf{T} - \frac{4}{3}\mathbf{U} \right), \\ g(I - \mathcal{P})\mathcal{L}\mathcal{N}(1) &= 0, \end{aligned}$$

so that, we can rewrite equation (3.8) as

$$L_1\mathbf{w} = \left((1 - \gamma)\varphi_{yy} - \frac{1}{2}(\varphi_y)^2 \right) (\mathbf{T}' - \mathbf{U}) - (\varphi_t + ((\varphi_y)^2)_{yy})_{yy} \left(\mathbf{V} - \mathbf{T} - \frac{4}{3}\mathbf{U} \right), \quad (3.9)$$

where $L_1 = L + A$.

3.5. Final system. We invert (3.9) using $R(0, L_1) = (-L_1)^{-1}$, collecting linear and nonlinear terms in φ :

$$\begin{aligned} \mathbf{w} = & R(0, L_1) \left(-(1 - \gamma) \varphi_{yy}(\mathbf{T}' - \mathbf{U}) + \varphi_{tyy} \left(\mathbf{V} - \mathbf{T} - \frac{4}{3} \mathbf{U} \right) \right) \\ & + R(0, L_1) \left(\frac{1}{2} (\varphi_y)^2 (\mathbf{T}' - \mathbf{U}) + ((\varphi_y)^2)_{yy} \left(\mathbf{V} - \mathbf{T} - \frac{4}{3} \mathbf{U} \right) \right), \end{aligned} \quad (3.10)$$

hence,

$$\begin{aligned} w_1(\cdot, 0, \cdot) = & -(1 - \gamma) (R(0, L_1) [\varphi_{yy}(\mathbf{T}' - \mathbf{U})]) (\cdot, 0, \cdot) \\ & + \left(R(0, L_1) \left[\varphi_{tyy} \left(\mathbf{V} - \mathbf{T} - \frac{4}{3} \mathbf{U} \right) \right] \right) (\cdot, 0, \cdot) \\ & + R(0, L_1) \left(\frac{1}{2} (\varphi_y)^2 (\mathbf{T}' - \mathbf{U}) + ((\varphi_y)^2)_{yy} \left(\mathbf{V} - \mathbf{T} - \frac{4}{3} \mathbf{U} \right) \right) (\cdot, 0, \cdot). \end{aligned}$$

Equation (3.7) eventually reads

$$\begin{aligned} & \frac{\partial}{\partial t} \left(\varphi - \frac{13}{3} \varphi_{yy} + 3D_{yy} \left(R(0, L_1) \left[\varphi_{yy} \left(\mathbf{V} - \mathbf{T} - \frac{4}{3} \mathbf{U} \right) \right] \right) (\cdot, 0, \cdot) \right) \\ & - 3(1 - \gamma) D_{yy} (R(0, L_1) [\varphi_{yy}(\mathbf{T}' - \mathbf{U})]) (t, 0, y) + 3\gamma \varphi_{yyyy} + (1 - \gamma) \varphi_{yy} \\ = & F((\varphi_y)^2), \end{aligned} \quad (3.11)$$

where

$$F(v) = \frac{17}{6} v_{yy} - \frac{1}{2} v - 3D_{yy} R(0, L_1) \left(\frac{1}{2} v(\mathbf{T}' - \mathbf{U}) + v_{yy} \left(\mathbf{V} - \mathbf{T} - \frac{4}{3} \mathbf{U} \right) \right) (\cdot, 0, \cdot), \quad (3.12)$$

and is of the form:

$$\frac{\partial}{\partial t} \mathcal{B}\varphi = \mathcal{S}\varphi + F((\varphi_y)^2), \quad (3.13)$$

with

$$\mathcal{S}\varphi = -3\gamma \varphi_{yyyy} - (1 - \gamma) \varphi_{yy} + 3(1 - \gamma) D_{yy} (R(0, L_1) [\varphi_{yy}(\mathbf{T}' - \mathbf{U})]) (\cdot, 0, \cdot), \quad (3.14)$$

$$\mathcal{B}\varphi = \varphi - \frac{13}{3} \varphi_{yy} + 3D_{yy} \left(R(0, L_1) \left[\varphi_{yy} \left(\mathbf{V} - \mathbf{T} - \frac{4}{3} \mathbf{U} \right) \right] \right) (\cdot, 0, \cdot). \quad (3.15)$$

The following theorem states the equivalence of problems (1.7), (1.8) and (3.9), (3.13).

Theorem 3.3. *The problems (1.7), (1.8) and (3.9), (3.13) are equivalent. More precisely, to any solution (u, φ) of problem (1.7), (1.8), with the properties stated at the very beginning of the section, there corresponds a solution $(\mathbf{w}, \varphi)$ to problem (3.9), (3.13), with the following properties*

- (i) *the function φ is continuously differentiable in $[0, T] \times [-\ell/2, \ell/2]$ once with respect to time and four times with respect to the spatial variables;*
- (ii) *φ_t is twice continuously differentiable with respect to y in $[0, T] \times [-\ell/2, \ell/2]$;*
- (iii) *the function $\mathbf{w}(t, \cdot) \in (I - \mathcal{P})(\mathcal{X})$ for any $t \in [0, T]$;*
- (iv) *w_1 and w_2 are continuously differentiable, twice with respect to x and twice with respect to y , in $[0, T] \times I_-$ and in $[0, T] \times I_+$, respectively;*
- (v) *the functions $D_x^{(i)} \mathbf{w}$ and $D_y^{(i)} \mathbf{w}$ belong to $\mathcal{X}$ for $i = 0, 1, 2$;*
- (vi) *$w_1(t, 0, y) = w_2(t, 0, y)$ and $D_x u_1(t, 0, y) = D_x u_2(t, 0, y)$ for any $t \in [0, T]$ and any $y \in [-\ell/2, \ell/2]$,*

and vice versa.

Proof. In view of the arguments in Subsections 3.1-3.5 we just need to show that to any solution $(\mathbf{w}, \varphi)$ to problem (3.9), (3.13), with the properties claimed in the statement of the theorem, there corresponds a solution to problem (1.7), (1.8) with the properties stated at the very beginning of the section. Of course, we just need to determine the function u or, equivalently, the function $\mathbf{u} := (u_1, u_2)$ where $u_1(t, x, y) = u(t, x, y)\chi_{(-\infty, 0]}(x)$ for any $(t, x, y) \in [0, T] \times I_-$ and $u_2(t, x, y) = u(t, x, y)\chi_{[0, +\infty)}(x)$ for any $(t, x, y) \in [0, T] \times I_+$. Clearly, $\mathbf{w}$ will represent the component along $(I - \mathcal{P})(\mathcal{X})$ of the function $\mathbf{u} - \mathcal{N}(\varphi_t + (\varphi_y)^2)$. Hence, the computations in Subsections 3.1-3.5 suggest to set $\mathbf{u} := a\mathbf{U} + \mathbf{w} + \mathcal{N}(\varphi_t + (\varphi_y)^2)$, where a is defined by (3.6). Now, it is an easy task to check that the function $\mathbf{u}$ solves problem (1.7), (1.8) and has the wished regularity properties. $\square$

4. Study of the symbols of pseudo differential operators. In (3.13) we have exhibited three operators, respectively $\mathcal{B}$, $\mathcal{S}$ and G . We have in mind to prove that the realization B of $\mathcal{B}$ is invertible in order to transform (3.13) in (1.9). We think it quite fruitful to give some insight on this class of pseudo differential operators within the context of discrete Fourier transform in the next theorem. They are characterized by their symbol or Fourier multipliers. This technical section is organized as follows: in a series of propositions and lemmas, we compute the symbols of $\mathcal{B}$, $\mathcal{S}$ and G . Theorem 4.8 is eventually devoted to $\hat{\Omega}$, the restriction to zero average functions of $\Omega = B^{-1}S$. One of the main features is that $\hat{\Omega}$ appears as a perturbation of A .

Let us define a pseudo-differential operator $\mathcal{R}$ acting on a function $f = f(y)$ by

$$\mathcal{R}(f) = \sum_{k=k_0}^{+\infty} r(\lambda_k) \hat{f}(k) w_k, \quad k_0 = 0, 1,$$

where the real numbers $r(\lambda_k)$'s are the symbol or Fourier multipliers of the operator $\mathcal{R}$.

Theorem 4.1. *The following properties are met:*

- (i) *assume that $k_0 = 0$ and $r(\lambda_k) = O(\lambda_k^s)$ (for some $s \geq 0$) as $k \rightarrow +\infty$, then the operator $\mathcal{R}$ has a realization R in $L^2(-\ell/2, \ell/2)$ which is a bounded operator from $H^s(-\ell/2, \ell/2)$ into $L^2(-\ell/2, \ell/2)$. Moreover, the spectrum of R consists of the eigenvalues $r(\lambda_k)$ $k = 0, 1, \dots$;*
- (ii) *if $k_0 = 1$ and*

$$r(\lambda_k) = -a\lambda_k + r_1(\lambda_k),$$

where $r_1(\lambda_k) = O(\lambda_k^s)$, for some $0 < s \leq 1/2$ and a is a positive constant, then the realization R of the operator $\mathcal{R}$ in $\dot{C}$ defines a sectorial operator with domain $\dot{C}^2$. Moreover, its spectrum consists of the isolated eigenvalues $r(\lambda_k)$, $k = 1, 2, \dots$

Proof. (i): the proof of the first statement is immediate due to the characterization of the spaces $H^s(-\ell/2, \ell/2)$ as the set of all functions $f \in L^2(-\ell/2, \ell/2)$ such that $\|f\|_{H^s(-\ell/2, \ell/2)}^2 := \sum_{k=0}^{+\infty} \lambda_k^s |\hat{f}(k)|^2 < +\infty$.

Let us now characterize the spectrum of R . For this purpose, we fix $f \in H^s(-\ell/2, \ell/2)$ and consider the resolvent equation

$$\mu u - Ru = f,$$

which can be rewritten in Fourier coordinates as follows:

$$\mu \hat{u}(k) - r(\lambda_k) \hat{u}(k) = \hat{f}(k), \quad k = 0, 1, \dots$$

Hence, if $\mu \neq r(\lambda_k)$ for any k , then the Fourier coefficients of u are uniquely determined by f through the formula

$$\hat{u}(k) = \frac{\hat{f}(k)}{\mu - r(\lambda_k)}, \quad k = 0, 1, \dots$$

By the properties of the sequence $r(\lambda_k)$, it follows easily that $\hat{u}(k)$ ($k = 1, 2, \dots$) are the Fourier coefficients of a function in $H^s(-\ell/2, \ell/2)$. This shows that such a μ is in the resolvent set of R . Showing that $r(\lambda_k)$ are eigenvalues of the operator R is immediate. Hence, it follows immediately that the spectrum of R consists of the sequence of eigenvalues $\{r(\lambda_k)\}$.

(ii): let us consider the pseudo differential operator T formally defined by

$$Tf = \sum_{k=1}^{+\infty} r_1(\lambda_k) \hat{f}(k) w_k.$$

We observe that, by assumptions, there exists a positive constant C , independent of k such that $|r(\lambda_k)| \leq C\sqrt{\lambda_k}$ for any $k = 1, 2, \dots$. Since the eigenfunctions w_k ($k = 1, 2, \dots$) are continuous functions and $\|w_k\|_\infty = 1$ for any k , for any $f \in \dot{C}$, we can estimate

$$|(Tf)(y)| \leq C \sum_{k=1}^{+\infty} \sqrt{\lambda_k} |\hat{f}(k)| \leq \left(\sum_{k=1}^{+\infty} \lambda_k^\beta |\hat{f}(k)|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{+\infty} \lambda_k^{1-\beta} \right)^{\frac{1}{2}}, \quad (4.1)$$

for any $y \in [-\ell/2, \ell/2]$. Therefore, if we take $\beta \in (3/2, 2)$, we easily deduce that the series defining Tf converges uniformly in $[-\ell/2, \ell/2]$ and, observing that $\dot{C}^\beta$ is continuously embedded into $\dot{H}^\beta$, we can conclude that T is a bounded operator mapping $\dot{C}^\beta$ into $\dot{C}$.

Summing up, we have proved that $R = aD_{yy} + T$. Since the realization of aD_{yy} in $\dot{C}$ (with domain $\dot{C}^2$) is a sectorial operator and T_1 is defined in an intermediate space between $\dot{C}$ and $\dot{C}^2$, from [13, Proposition 2.4.1], R , with domain $\dot{C}^2$, is a sectorial operator.

To conclude the proof, let us show that $\sigma(R) = \{r(\lambda_k) : k = 1, 2, \dots\}$. For this purpose, we fix $f \in \dot{C}$ and consider the resolvent equation

$$\mu u - Ru = f. \quad (4.2)$$

Adapting the same arguments as in the proof of property (i), we can easily show that, for $\mu \neq r(\lambda_k)$ for any k , the resolvent equation (4.2) has a unique solution in $\dot{H}^2$. Since $\dot{H}^2$ is continuously embedded in $\dot{H}^\beta$, from the estimate (4.1), it follows that $Tu \in \dot{C}$. Therefore, by difference $u_{yy} \in \dot{C}$, so that $u \in C^2([-\ell/2, \ell/2])$. In particular, the Fourier series defining u_y converges uniformly and, hence, $u'(-\ell/2) = u'(\ell/2)$, implying that $u \in \dot{C}^2$. We have so proved that $\sigma(R) \subset \{r(\lambda_k) : k = 1, 2, \dots\}$. The other inclusion being trivial, this completes the proof. $\square$

Throughout the section, for notation convenience, we set

$$X_k = \sqrt{1 + 4\lambda_k}, \quad k = 0, 1, \dots$$

up to Proposition 4.7. In Theorem 4.8, the eigenvalue 0 is excluded, therefore, $k \geq 1$.

Proposition 4.2. *The k -th Fourier multiplier b_k of $\mathcal{B}$ is given by*

$$b_k = \frac{3}{4} \frac{X_k + 1}{X_k + 2} (X_k^2 + 2X_k - 1) \sim 3\lambda_k \quad (k \rightarrow +\infty).$$

In particular, the operator $\mathcal{B}$ has a realization B in $\dot{H}^0$ with domain $\dot{H}^2$ and B is invertible.

In the proof of Proposition 4.2 we will take advantage of the following lemma.

Lemma 4.3. *For any continuous ℓ -periodic function φ , set*

$$\mathbf{u} = \left\{ R(0, L_1) \left((\mathbf{V} - \mathbf{T} - \frac{4}{3}\mathbf{U}) \varphi \right) \right\} (0, \cdot).$$

Then,

$$\hat{u}_1(0, k) = -\frac{4}{9} \frac{4X_k + 7}{(X_k + 1)^2(X_k + 2)} \hat{\varphi}(k), \quad k = 0, 1, \dots$$

Proof. Let us first assume that φ is smooth enough. Then, from Proposition A.4(ii), it follows that $\mathbf{u} \in D(L) \cap D(A)$, so that $L\mathbf{u} + A\mathbf{u} = -(\mathbf{V} - \mathbf{T} - \frac{4}{3}\mathbf{U})\varphi$. Moreover, the function $\hat{\mathbf{u}}(\cdot, k)$ belongs to $(I - \mathcal{P})D(\mathcal{L})$ and solves the equation $(\lambda_k - L)\hat{\mathbf{u}}(\cdot, k) = (\mathbf{V} - \mathbf{T} - \frac{4}{3}\mathbf{U})\hat{\varphi}(k)$ for any $k = 0, 1, \dots$. Since λ_k is in the resolvent set of the operator L by Theorem 2.1 for any $k = 0, 1, \dots$, it follows that

$$\hat{\mathbf{u}}(\cdot, k) = R(\lambda_k, L) \left(\mathbf{V} - \mathbf{T} - \frac{4}{3}\mathbf{U} \right) \hat{\varphi}(k), \quad k = 0, 1, \dots \quad (4.3)$$

Formula (4.3) can be extended to any periodic function $\varphi \in C([- \ell/2, \ell/2])$ by a straightforward approximation argument.

Using formula (A.6) and the very definition of the functions $\mathbf{V}$, $\mathbf{T}$ and $\mathbf{U}$ we can compute

$$\begin{aligned} & \left\{ R(\lambda, L) \left(\mathbf{V} - \mathbf{T} - \frac{4}{3}\mathbf{U} \right) \right\}_1(0) \\ &= \frac{2\lambda\sqrt{1+4\lambda}}{1 + (2\lambda - 1)\sqrt{1+4\lambda}} \left\{ (\lambda + D_x - D_x^2)^{-1} \left(\mathbf{V} - \mathbf{T} - \frac{4}{3}\mathbf{U} \right) \right\}_1(0), \end{aligned}$$

for any $\lambda > 0$ where $\{\cdot\}_1$ denotes the first component of the vector in brackets. Let us compute $\{(\lambda_k + D - D^2)^{-1} (\mathbf{V} - \mathbf{T} - \frac{4}{3}\mathbf{U})\}_1(0)$. For this purpose, we recall that $\mathbf{V} - \mathbf{T} - \frac{4}{3}\mathbf{U}$ is given by

$$\left(\frac{1}{6}x^2 - \frac{2}{9}x - \frac{4}{9} \right) e^x \quad \text{and} \quad \frac{1}{3}x - \frac{4}{9},$$

for $x \leq 0$ and $x \geq 0$ respectively. Hence, from formula (A.1) we get

$$\begin{aligned} & \left\{ (\lambda + D_x - D_x^2)^{-1} \left(\mathbf{V} - \mathbf{T} - \frac{4}{3}\mathbf{U} \right) \right\}_1(0) \\ &= \frac{1}{\sqrt{1+4\lambda}} \left\{ \int_{-\infty}^0 e^{\nu_2 t} \left(\frac{t^2}{6} - \frac{2t}{9} - \frac{4}{9} \right) dt + \int_0^{+\infty} e^{-\nu_2 t} \left(\frac{t}{3} - \frac{4}{9} \right) dt \right\} \\ &= -\frac{8\nu_2^2 - 5\nu_2 - 3}{9\nu_2^3\sqrt{1+4\lambda}} = -\frac{4(4X+7)(X-1)}{9(X+1)^3} \frac{1}{\sqrt{1+4\lambda}}, \end{aligned}$$

where $\nu_1 = \frac{1}{2} - \frac{1}{2}\sqrt{1+4\lambda}$, $\nu_2 = \frac{1}{2} + \frac{1}{2}\sqrt{1+4\lambda}$ and $X = \sqrt{1+4\lambda}$. Since 0 is in the resolvent set of the restriction of L to $(I - \mathcal{P})(\mathcal{X})$, we can extend the previous

formula, by continuity, to $\lambda = 0$. Thus,

$$\begin{aligned}\hat{u}_1(0, k) &= -\frac{4}{9} \frac{2\lambda_k}{1 + (2\lambda_k - 1)\sqrt{1 + 4\lambda_k}} \frac{(4X_k + 7)(X_k - 1)}{(X_k + 1)^3} \hat{\varphi}(k) \\ &= \frac{4}{9} \frac{X_k + 1}{X_k + 2} \frac{(4X_k + 7)(X_k - 1)}{(X_k + 1)^3},\end{aligned}$$

for any $k = 0, 1, \dots$, and the assertion follows. $\square$

Proof of Proposition 4.2. We now use the lemma to evaluate the right hand side of (3.15). By considering its action on the eigenfunctions w_k it easily follows that the k -th Fourier coefficient of $\mathcal{B}\varphi$ is given by

$$\widehat{\mathcal{B}\varphi}(k) = \left(1 + 13\lambda_k - \frac{4}{3}\lambda_k^2 \frac{4X_k + 7}{(X_k + 1)^2(X_k + 2)}\right) \hat{\varphi}(k) = b_k \hat{\varphi}(k),$$

which evaluates as the formula to prove for b_k . Thus the multipliers b_k are positive, bounded away from zero. The assertion now follows from Theorem 4.1. $\square$

Proposition 4.4. *The k -th Fourier multiplier s_k of $\mathcal{S}$ is given by*

$$s_k = -\frac{3}{4}\lambda_k \frac{X_k + 1}{X_k + 2} (\gamma X_k^2 + \gamma X_k - 2) \sim -3\gamma\lambda_k^2 \quad (k \rightarrow +\infty).$$

In the proof of Proposition 4.4 we take advantage of another lemma which is proved along similar lines as Lemma 4.3 above. The proof is omitted.

Lemma 4.5. *For any continuous ℓ -periodic function φ , set*

$$\mathbf{u} = (R(0, L_1)((\mathbf{T}' - \mathbf{U})\varphi))(0, \cdot).$$

Then,

$$\hat{u}_1(0, k) = \frac{2}{3} \frac{1}{(X_k + 1)(X_k + 2)} \hat{\varphi}(k), \quad k = 0, 1, \dots$$

Proof of Proposition 4.4. We use the lemma to evaluate the right hand side of (3.14). By considering its action on w_k it now easily follows that the k -th Fourier coefficient of $\mathcal{S}\varphi$ is given by

$$\widehat{\mathcal{S}\varphi}(k) = \left(-3\gamma\lambda_k^2 + (1 - \gamma)\lambda_k - (\gamma - 1)\lambda_k^2 \frac{2}{(X_k + 1)(X_k + 2)}\right) \hat{\varphi}(k) = s_k \hat{\varphi}(k).$$

which evaluates as the formula to prove for s_k . $\square$

The following proposition can be proved as Propositions 4.2 and 4.4, taking Lemmas 4.3 and 4.5 into account. Hence, its proof is skipped.

Proposition 4.6. *The k -th Fourier multiplier of $F\varphi$ is given by $f_k \hat{\varphi}(k)$, where*

$$f_k = -\frac{3}{8} \frac{X_k + 1}{X_k + 2} (X_k^2 + 3X_k - 2).$$

As a consequence of Propositions 4.2, 4.4 and 4.6, dividing out the common (non-zero) factor in the multipliers b_k , s_k and f_k , equation (3.13) can be expressed under the abstract form

$$\varphi_t + G((\varphi_y)^2) = \Omega\varphi, \quad (4.4)$$

where G and Ω are determined by their Fourier multipliers. We supplement (4.4) with the initial condition

$$\varphi(0) = \varphi_0. \quad (4.5)$$

We eventually project (4.4) on the set of zero mean value by means of the projection $\mathcal{M}$ defined in (2.2) on integrable functions $u : (-\ell/2, \ell/2) \rightarrow \mathbb{C}$. Applying $I - \mathcal{M}$ to (4.4), it comes

$$((I - \mathcal{M})\varphi)_t = (I - \mathcal{M})\Omega\varphi - (I - \mathcal{M})G((\varphi_y)^2).$$

We set $\varphi := \psi + \bar{\varphi}$ and $\dot{\Omega} := (I - \mathcal{M})\Omega$. Observing that $G((\psi_y)^2) = G((\varphi_y)^2)$ and that $I - \mathcal{M}$ commutes with Ω , the system (4.4)(4.5) eventually becomes:

$$\begin{cases} \psi_t = \dot{\Omega}\psi - (I - \mathcal{M})G((\psi_y)^2), \\ \bar{\varphi}_t = \mathcal{M}G((\psi_y)^2), \\ \psi(0) = \psi_0 = \varphi_0 - \overline{\varphi_0}, \\ \bar{\varphi}(0) = \overline{\varphi_0}. \end{cases} \quad (4.6)$$

We now analyze the operators G and $\dot{\Omega}$.

Proposition 4.7. *G is determined by its Fourier multipliers:*

$$g_k = \frac{1}{2} \frac{X_k^2 + 3X_k - 2}{X_k^2 + 2X_k - 1}, \quad k = 0, 1, \dots \quad (4.7)$$

Proof. It follows immediately combining Propositions 4.2 and 4.6. $\square$

Theorem 4.8. *The operator $\dot{\Omega}$ is determined by its Fourier multipliers*

$$\omega_k = -\lambda_k \frac{\gamma X_k^2 + \gamma X_k - 2}{X_k^2 + 2X_k - 1}, \quad k = 1, 2, \dots, \quad (4.8)$$

In particular, the operator $\dot{\Omega}$ extends to a sectorial operator in the space $\dot{C}$ with domain $\dot{C}^2$. The spectrum $\sigma(\dot{\Omega})$ of $\dot{\Omega}$ consists of the numbers ω_k . Finally, $D_{\dot{\Omega}}(\alpha, \infty) = \dot{C}^{2\alpha}$, for any $\alpha \in (0, 2) \setminus \{1/2, 1\}$, with equivalence of the corresponding norms.

Proof. Combining Propositions 4.2 and 4.4, it is immediate to check that the symbol of Ω is given by (4.8). It is immediate to check that ω_k can be rewritten in the following way:

$$\omega_k = \lambda_k \frac{\gamma X_k^2 + \gamma X_k - 2}{X_k^2 + 2X_k - 1} = -\gamma \lambda_k - \frac{\gamma}{2} \sqrt{\lambda_k} + M_k, \quad k = 1, 2, \dots,$$

where the sequence $\{M_k\}$ is bounded. Hence, the other statements in the theorem, but the characterization of the interpolation spaces, follow immediately from Theorem 4.1. On the other hand, the characterization of the interpolation spaces $D_{\dot{\Omega}}(\alpha, \infty)$ follows immediately observing that $D_{\dot{\Omega}}(\alpha, \infty) = (\dot{C}, \dot{C}^2)_{\alpha, \infty}$, if $\alpha \in (0, 1)$, and $D_{\dot{\Omega}}(\alpha, \infty) = (\dot{C}^2, \dot{C}^4)_{\alpha-1, \infty}$, if $\alpha \in (1, 2)$. $\square$

5. Proof of the main theorem. The following proposition plays a fundamental tool in the proof of the main theorem of this paper. It characterizes the position of the spectrum of the operator $\dot{\Omega}$ with respect to the imaginary axis depending on the value of the parameter γ .

Proposition 5.1. *Let*

$$\gamma_c = \frac{2\ell^2}{\ell\sqrt{\ell^2 + 16\pi^2} + \ell^2 + 16\pi^2}.$$

Then,

$$\sup \sigma(\hat{\Omega}) < 0 \text{ for } \gamma > \gamma_c, \quad \sup \sigma(\hat{\Omega}) > 0 \text{ for } 0 < \gamma < \gamma_c.$$

Moreover, as $\ell \rightarrow +\infty$, one has $\gamma_c \rightarrow 1^-$. More precisely,

$$\gamma_c \sim 1 - 3\lambda_1.$$

Proof. Let $\omega_k = \omega_k(\gamma)$ denote the Fourier multipliers of the operator $\hat{\Omega}$ defined in (4.8) as a function of parameter γ . It is immediate to infer that $\omega_k(\gamma) > 0$ if and only if $\gamma < \gamma_{c,k}$, where

$$\gamma_{c,k} = \frac{2}{X_k(X_k + 1)} = \frac{2}{1 + 4\lambda_k + \sqrt{1 + 4\lambda_k}}.$$

As it is immediately seen, the sequence $\gamma_{c,k}$ is decreasing to zero. Therefore, $\omega_k(\gamma)$ are negative for any $k = 1, 2, \dots$ if and only if $\gamma > \gamma_{c,1} := \gamma_c$. The first assertion now follows taking the definition of λ_k (see (2.1)) into account, namely $\lambda_1 = 4\pi^2/\ell^2$.

To prove the other statement of the lemma, it suffices to observe that

$$\gamma_c = 1 - 3\lambda_1 + \dots$$

as $\lambda_1 \rightarrow 0$. $\square$

The proof of our main result will be achieved in some steps. First, we consider the problem (4.6) and prove the following result.

Theorem 5.2. *The following properties are met:*

- (i) *for $0 < \gamma < \gamma_c$, the equilibrium $(0, 0)$ of problem (4.6) is unstable with respect to smooth and sufficiently small perturbations;*
- (ii) *for $\gamma > \gamma_c$, the equilibrium $(0, 0)$ is stable (with asymptotic phase). More precisely, for any $\alpha \in (0, 1/2)$, there exists $\rho_0 > 0$ such that if $\varphi_0 \in \dot{C}^{2+\alpha}$ satisfies $\|(I - \mathcal{M})\varphi_0\|_{\dot{C}^{2+\alpha}} \leq \rho_0$, problem (4.6) admits a classical solution $\varphi := \psi + \bar{\varphi}$ defined in $[0, +\infty) \times [-\ell/2, \ell/2]$. Further, $\varphi \in C^{1+\alpha/2, 2+\alpha}([0, +\infty) \times [-\ell/2, \ell/2])$ and*

$$\lim_{t \rightarrow +\infty} \|\varphi(t, \cdot) - c_\infty\|_\infty = 0, \quad \lim_{t \rightarrow +\infty} e^{-\xi t} \|\varphi_y(t, \cdot)\|_{\dot{C}^{1+\alpha}} = 0,$$

for any $\xi \in (\sup \sigma(\hat{\Omega}), 0)$, where

$$c_\infty := \frac{1}{\ell} \int_{-\ell/2}^{\ell/2} \varphi_0(y) dy + \frac{1}{\ell} \int_0^{+\infty} dt \int_{-\ell/2}^{\ell/2} (G((\psi_y(t, \cdot))^2))(y) dy.$$

Proof. As a first step, we observe that we can rewrite the symbol of the function G (see (4.7)) as follows:

$$\frac{1}{2} \frac{X_k^2 + 3X_k - 2}{X_k^2 + 2X_k - 1} = \frac{1}{2} + \frac{1}{2} \frac{X_k - 1}{X_k^2 + 2X_k - 1}, \quad k = 1, 2, \dots$$

Let us consider the pseudo differential operator $\mathcal{R}$ whose symbol is

$$r_k = \frac{1}{2} \frac{X_k - 1}{X_k^2 + 2X_k - 1}, \quad k = 0, 1, \dots$$

Since r_k behaves like $\lambda_k^{-1/2}$ as $k \rightarrow +\infty$, a simple computation shows that

$$\sum_{k=0}^{+\infty} \left| \frac{X_k - 1}{X_k^2 + 2X_k - 1} \right| |\hat{h}(k)| \leq K_1 \sum_{k=1}^{+\infty} \lambda_k^{-1/2} |\hat{h}(k)|$$

$$\begin{aligned}
&\leq K_2 \left(\sum_{k=1}^{+\infty} |\hat{h}(k)|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{+\infty} \lambda_k^{-1} \right)^{\frac{1}{2}} \\
&\leq K_3 \|h\|_0 \leq K_4 \|h\|_\infty,
\end{aligned}$$

for any $h \in C_{\text{per}} := \{h \in C([- \ell/2, \ell/2]) : h(-\ell/2) = h(\ell/2)\}$ and some positive constants K_j ($j = 1, \dots, 4$), independent of h . Recalling that the functions w_k are continuous in $[-\ell/2, \ell/2]$ and $\|w_k\|_\infty = 1$ for any $k = 1, 2, \dots$, we have so proved that $\mathcal{R}$ admits a realization R which defines a bounded operator in C_{per} (endowed with the sup-norm). With a completely similar argument, one can easily show that R is also bounded from $C_{\text{per}}^k := \{h \in C^k([- \ell/2, \ell/2]) : h^{(j)}(-\ell/2) = h^{(j)}(\ell/2), j = 0, \dots, k-1\}$. As a byproduct, we deduce, first, that $(I - \mathcal{M})R$ is bounded from $\dot{C}^j$ into itself for any $j = 0, 1, \dots$, and, then, by interpolation, that it is bounded from $\dot{C}^\theta$ into itself for any $\theta \geq 0$. Since

$$(I - \mathcal{M})G(\psi) = \frac{1}{2}(I - \mathcal{M})\psi + (I - \mathcal{M})R(\psi), \quad \psi \in \dot{C},$$

it follows that the nonlinear term in the right-hand side of (4.6) is well defined in $\dot{C}^1$ and there exists a positive constant K_6 , depending only on θ , such that

$$\|(I - \mathcal{M})G((\psi_y)^2)\|_{\dot{C}^\theta} \leq K_6 \|(\psi_y)^2\|_{\dot{C}^\theta} \leq 2K_6 \|\psi\|_{\dot{C}^{1+\theta}}^2, \quad (5.1)$$

for any $\psi \in \dot{C}^{1+\theta}$ and any $\theta \geq 0$. Similarly, since F is linear, taking advantage of (5.1) we can also estimate we can estimate

$$\begin{aligned}
&\|(I - \mathcal{M})G((\psi_y)^2) - (I - \mathcal{M})G((\varphi_y)^2)\|_{\dot{C}^\theta} \\
&\leq K_7 (\|\varphi\|_{\dot{C}^{1+\theta}} + \|\psi\|_{\dot{C}^{1+\theta}}) \|\psi - \varphi\|_{\dot{C}^{1+\theta}},
\end{aligned} \quad (5.2)$$

for any $\psi, \varphi \in \dot{C}^{1+\theta}$ and some positive constant K_7 , independent of φ and ψ .

Estimates (5.1) and (5.2), combined with Theorem 4.8 and Proposition 5.1, are all one needs to apply well established arguments (see e.g., [13, Chapter 9], [5, 6, 7, 11, 12]) which guarantee that:

- (a) if $\gamma > \gamma_c$, there exists $\rho_0 > 0$ such that, if $\psi_0 \in \dot{C}^{2+\alpha}$ satisfies $\|\psi_0\|_{\dot{C}^{2+\alpha}} \leq \rho_0$ for some $\alpha \in (0, 1)$, then the Cauchy problem

$$\begin{cases} \psi_t = \dot{\Omega}\psi + (I - \mathcal{M})G((\psi_y)^2), \\ \psi(0) = \psi_0, \end{cases} \quad (5.3)$$

admits a unique classical solution ψ defined in $[0, +\infty) \times [-\ell/2, \ell/2]$ (i.e., ψ is continuously differentiable in $[0, +\infty) \times [-\ell/2, \ell/2]$, once with respect to t and twice with respect to y). Moreover, $\psi \in C^{1+\alpha/2, 2+\alpha}([0, +\infty) \times [-\ell/2, \ell/2])$ and the $\dot{C}^{2+\alpha}$ -norm of the function $\psi(t, \cdot)$ decays exponentially to 0 as $t \rightarrow +\infty$. More precisely, for any $\varepsilon > 0$, there exists a positive constant K_ε , such that

$$\|\psi(t, \cdot)\|_{C^{2+\alpha}([- \ell/2, \ell/2])} \leq K_\varepsilon e^{(\mu_1 + \varepsilon)t}, \quad t > 0;$$

- (b) if $0 < \gamma < \gamma_c$, then the null solution to problem (5.3) is unstable.

From these two remarks the assertion of the theorem follows immediately. $\square$

Corollary 5.3. *Suppose that $\gamma > \gamma_c$. Further, assume that $\varphi_0 \in \dot{C}^{4+2\alpha}$ for some $\alpha \in (0, 1/2)$ and $\|(I - \mathcal{M})\varphi_0\|_{\dot{C}^{2+\alpha}} \leq \rho_0$, where ρ_0 is the same as in the statement of Theorem 5.2. Then, the solution φ of problem (4.6) is such that φ_t and φ_{yy} are bounded and continuous in $[0, +\infty)$, with values in $\dot{C}^{2+\alpha}$. Moreover, for any*

$\xi \in (\sup \sigma(\dot{\Omega}), 0)$, $e^{-\xi t} \|\varphi_t(t, \cdot)\|_{\dot{C}^{2+\alpha}}$ and $e^{-\xi t} \|\varphi_{yy}(t, \cdot)\|_{\dot{C}^{2+\alpha}}$ tend to 0 as $t \rightarrow +\infty$. As a byproduct, the functions φ_t and φ_{yy} are twice continuously differentiable in $[0, +\infty) \times [-\ell/2, \ell/2]$ with respect to the variable y .

Proof. To prove the assertion we use a bootstrap argument. Of course, we can limit ourselves to dealing with the function $\psi = (I - \mathcal{M})\varphi$. Since ψ is a bounded function with values in $\dot{C}^{2+\alpha}$, and its $\dot{C}^{2+\alpha}$ -norm decreases exponentially to 0 as $t \rightarrow +\infty$ (faster than $e^{\xi t}$ for any $\xi \in (\sup \sigma(\dot{\Omega}), 0)$), the function $\eta := (I - \mathcal{M})G((\psi_y)^2)$ turns out to be bounded with values in $\dot{C}^{1+\alpha} = D_{\dot{\Omega}}((1+\alpha)/2, \infty)$ and the $\dot{C}^{1+\alpha}$ -norm of $\eta(t, \cdot)$ decreases to 0, as $t \rightarrow +\infty$, faster than $e^{2\xi t}$. Indeed, we can estimate $\|\eta(t, \cdot)\|_{\dot{C}^{1+\alpha}} \leq K_1 \|\psi_y(t, \cdot)\|_{\dot{C}^{2+\alpha}}^2$ for any $t > 0$ and some positive constant K_1 , independent of ψ (see the proof of Theorem 5.2).

Recalling that ψ is given by the variation of constants formula

$$\psi(t, \cdot) = e^{t\dot{\Omega}}\psi_0 + \int_0^t e^{(t-s)\dot{\Omega}}\eta(s, \cdot)ds, \quad t \geq 0, \quad (5.4)$$

we can write

$$e^{-\xi t}\psi(t, \cdot) = e^{t(\dot{\Omega}-\xi)}\psi_0 + \int_0^t e^{(t-s)(\dot{\Omega}-\xi)}e^{-\xi s}\eta(s, \cdot)ds, \quad t \geq 0.$$

Hence, from (the proof of) [13, Theorem 4.3.8 & Corollary 4.3.9], it follows that the function $t \mapsto e^{-\xi t}\psi(t, \cdot)$ is bounded in $[0, +\infty)$ with values in $\dot{C}^{3+\alpha}$. This implies that the function $t \mapsto e^{-2\xi t}\eta(t, \cdot)$ is bounded in $[0, +\infty)$ with values in $\dot{C}^{2+\alpha} = D_{\dot{\Omega}}(1+\alpha/2, \infty) \subset D(\dot{\Omega})$. Therefore, from formula (5.4) it follows that

$$e^{-\xi t}\dot{\Omega}\psi(t, \cdot) = e^{t(\dot{\Omega}-\xi)}\dot{\Omega}\psi_0 + \int_0^t e^{(t-s)(\dot{\Omega}-\xi)}e^{-\xi s}\dot{\Omega}\eta(s, \cdot)ds, \quad t \geq 0.$$

Since the function $t \mapsto e^{-2\xi t}\dot{\Omega}\eta(t, \cdot)$ is bounded in $[0, +\infty)$ with values in $\dot{C}^{2\alpha} = D_{\dot{\Omega}}(\alpha/2, \infty)$, again from (the proof of) [13, Corollary 4.3.9], we deduce that the function $t \mapsto e^{-\xi t}\dot{\Omega}^2\psi(t, \cdot)$ is bounded in $[0, +\infty)$ with values in $\dot{C}^{2\alpha}$. Since $\psi_t = \dot{\Omega}\psi + \eta$, it follows immediately that the function $t \mapsto e^{-\xi t}\psi_t(t, \cdot)$ is bounded in $[0, +\infty)$ with values in $\dot{C}^{2+\alpha}$. This accomplishes the proof. $\square$

We are now in a position to prove the following result.

Theorem 5.4. *Let $\gamma > \gamma_c$ and φ_0 be as in the statement of Corollary 5.3. Then, problem (3.9), (3.13) admits a unique solution $(\rho, \mathbf{w})$ satisfying the properties in the statement of Theorem 3.3. Moreover,*

- (a) $\varphi \in C^{1+\alpha/2, 2+\alpha}([0, +\infty) \times [-\ell/2, \ell/2])$;
- (b) φ_t and φ_{yy} are bounded in $[0, +\infty)$ with values in $\dot{C}^{2+\alpha}$;
- (c) for any $\xi \in (\sup \sigma(\dot{\Omega}), 0)$,

$$\lim_{t \rightarrow +\infty} \|\varphi(t, \cdot) - c_\infty\|_\infty = \lim_{t \rightarrow +\infty} e^{-\xi t} \|\varphi_y\|_{\dot{C}^{3+2\alpha}} = 0$$

- (d) the functions $t \mapsto e^{-\xi t} \|D_x^{(i)} \mathbf{w}(t, \cdot)\|_{\mathcal{X}}$ and $t \mapsto e^{-\xi t} \|D_y^{(i)} \mathbf{w}(t, \cdot)\|_{\mathcal{X}}$ ($i = 0, 1, 2$) decrease to 0 exponentially as $t \rightarrow +\infty$.

Proof. As a first step, let us show that the function φ , provided by Corollary 5.3, solves the equation (3.13). For this purpose, we observe that, the regularity of the function φ implies that the left- and right-hand side of (3.13) define two functions in $C_b([0, +\infty) \times [-\ell/2, \ell/2])$. Taking all the results in Section 4 into account, we

deduce that the symbols of the left- and the right-hand side of (3.13), coincides since φ solves (4.6). It follows that φ satisfies (3.13), as well as properties (a)-(c).

Let us now consider the function $\mathbf{w}$ defined by (3.10). Since the functions φ_{yy} , $\varphi_{t yy}$, $(\varphi_y)^2$ and $((\varphi_y)^2)_{yy}$ are in $D_A(\alpha, \infty) := \{f \in C^{2\alpha}([-\ell/2, \ell/2]) : u(-\ell/2) = u(\ell/2)\}$, from Proposition A.4 it follows that the function $\mathbf{w}$ solves the equation (3.9) and satisfies the properties (iii)-(vi) in the statement of Theorem 3.3 as well as property (d). This concludes the proof. $\square$

Now, the proof of the Main Theorem follows from Theorems 3.3 and (5.4).

Appendix A. Proofs of Theorems 2.1 and 2.2. In this section we will study the operators L (see (2.4)) and L_1 .

A.1. Proof of Theorem 2.1. (a) & (b): as a first step, we consider the realization of the operator $\lambda + D_x - D_x^2$ in $\mathcal{X}$ with domain $D(L)$. A direct computation shows that, for any $\lambda \notin (-\infty, -\frac{1}{4}]$, this operator is invertible and, for any $\mathbf{h} \in \mathcal{X}$, $(\lambda + D_x - D_x^2)^{-1}\mathbf{h} =: \mathbf{v}$, where

$$\begin{aligned} v_1(x, y) &= \frac{e^{\nu_1 x}}{\sqrt{1+4\lambda}} \left(\int_{-\infty}^x e^{-\nu_1 t} h_1(t, y) dt \right) + \frac{e^{\nu_2 x}}{\sqrt{1+4\lambda}} \left(\int_x^0 e^{-\nu_2 t} h_1(t, y) dt \right) \\ &\quad + \frac{e^{\nu_2 x}}{\sqrt{1+4\lambda}} \left(\int_0^{+\infty} e^{-\nu_2 t} h_2(t, y) dt \right), \end{aligned} \quad (\text{A.1})$$

for any $(x, y) \in I_-$ and

$$\begin{aligned} v_2(x, y) &= \frac{e^{\nu_1 x}}{\sqrt{1+4\lambda}} \left(\int_{-\infty}^0 e^{-\nu_1 t} h_1(t, y) dt + \int_0^x e^{-\nu_1 t} h_2(t, y) dt \right) \\ &\quad + \frac{e^{\nu_2 x}}{\sqrt{1+4\lambda}} \left(\int_x^{+\infty} e^{-\nu_2 t} h_2(t, y) dt \right), \end{aligned} \quad (\text{A.2})$$

for any $(x, y) \in I_+$, where

$$\nu_1 = \frac{1}{2} - \frac{1}{2}\sqrt{1+4\lambda}, \quad \nu_2 = \frac{1}{2} + \frac{1}{2}\sqrt{1+4\lambda},$$

and by $\sqrt{1+4\lambda}$ we denote the root of $1+4\lambda$ with nonnegative real part.

Observe now that, for any $\lambda \notin (-\infty, -\frac{1}{4}]$ and any $\mathbf{f} \in \mathcal{X}$, the equation $\lambda \mathbf{u} - L\mathbf{u} = \mathbf{f}$ can be rewritten as

$$(\lambda + D_x - D_x^2)\mathbf{u} = u_1(0, \cdot)\mathbf{T}' + \mathbf{f}, \quad (\text{A.3})$$

or equivalently as

$$\mathbf{u} = u_1(0, \cdot)(\lambda + D_x - D_x^2)^{-1}\mathbf{T}' + (\lambda + D_x - D_x^2)^{-1}\mathbf{f}. \quad (\text{A.4})$$

Since $(\lambda + D_x - D_x^2)^{-1}\mathbf{T}' = \mathbf{g}$, where

$$g_1(x) = \frac{1}{\lambda}e^x + \frac{1}{\sqrt{1+4\lambda}}\frac{1}{\nu_1}e^{\nu_2 x}, \quad x < 0, \quad g_2(x) = \frac{1}{\sqrt{1+4\lambda}}\frac{1}{\nu_2}e^{\nu_1 x}, \quad x > 0, \quad (\text{A.5})$$

we can determine $u_1(0, \cdot)$ by substituting $x = 0$ in the first component of (A.3). This gives

$$u_1(0, \cdot) = -\frac{\nu_1}{\lambda\sqrt{1+4\lambda}}u_1(0, \cdot) + ((\lambda + D_x - D_x^2)^{-1}\mathbf{f})_1(0, \cdot),$$

whence

$$u_1(0, \cdot) = \frac{2\lambda\sqrt{1+4\lambda}}{1 + (2\lambda-1)\sqrt{1+4\lambda}} ((\lambda + D_x - D_x^2)^{-1}\mathbf{f})_1(0, \cdot), \quad (\text{A.6})$$

provided that $1 + (2\lambda-1)\sqrt{1+4\lambda} \neq 0$. Let us observe that $1 + (2\lambda-1)\sqrt{1+4\lambda} = m(\sqrt{1+4\lambda})$, where $m(x) = x^3 - 3x + 2$. Since $x = 1$ and $x = -2$ are the unique solutions to the previous equation, it follows that $1 + (2\lambda-1)\sqrt{1+4\lambda} \neq 0$ if and only if $\lambda = 0$. Hence, $\lambda \in \sigma(L)$. Now, if $\lambda \neq 0$, replacing (A.1), (A.2) and (A.6) into (A.4) leads us to the following representation formulas for the solution to the equation $\lambda \mathbf{u} - L\mathbf{u} = \mathbf{f}$:

$$\begin{aligned} u_1(x, y) &= \frac{e^{\nu_1 x}}{\sqrt{1+4\lambda}} \left(\int_{-\infty}^x e^{-\nu_1 t} f_1(t, y) dt \right) + \frac{e^{\nu_2 x}}{\sqrt{1+4\lambda}} \left(\int_x^0 e^{-\nu_2 t} f_1(t, y) dt \right) \\ &+ \frac{e^{\nu_2 x}}{\sqrt{1+4\lambda}} \left(\int_0^{+\infty} e^{-\nu_2 t} f_2(t, y) dt \right) \\ &+ \frac{2\lambda g_1(x)}{1 + (2\lambda-1)\sqrt{1+4\lambda}} \left(\int_{-\infty}^0 e^{-\nu_1 t} f_1(t, y) dt + \int_0^{+\infty} e^{-\nu_2 t} f_2(t, y) dt \right), \end{aligned} \quad (\text{A.7})$$

for any $(x, y) \in I_-$ and

$$\begin{aligned} u_2(x, y) &= \frac{e^{\nu_2 x}}{\sqrt{1+4\lambda}} \left(\int_x^{+\infty} e^{-\nu_2 t} f_2(t, y) dt \right) + \frac{e^{\nu_1 x}}{\sqrt{1+4\lambda}} \left(\int_0^x e^{-\nu_1 t} f_2(t, y) dt \right) \\ &+ \frac{e^{\nu_1 x}}{\sqrt{1+4\lambda}} \left(\int_{-\infty}^0 e^{-\nu_1 t} f_1(t, y) dt \right) \\ &+ \frac{2\lambda g_2(x)}{1 + (2\lambda-1)\sqrt{1+4\lambda}} \left(\int_{-\infty}^0 e^{-\nu_1 t} f_1(t, y) dt + \int_0^{+\infty} e^{-\nu_2 t} f_2(t, y) dt \right), \end{aligned} \quad (\text{A.8})$$

for any $(x, y) \in I_+$.

Let us now estimate the $\mathcal{X}$ -norm of the function in (A.7)-(A.8). We limit ourselves to dealing with the function u_1 , since the same arguments may then be applied to estimate the function u_2 . We denote by $\mathcal{J}_i$ ($i = 1, 2, 3, 4$) the four terms in the definition of the function u_1 . Moreover, in the rest of the proof, we denote by C positive constants which may vary from line to line but are independent of $\mathbf{f}$ and λ .

We begin by estimating $\mathcal{J}_1$. For this purpose, we observe that

$$\begin{aligned} |e^{-\frac{x}{2}} \mathcal{J}_1(x, y)| &\leq \frac{1}{|1+4\lambda|^{1/2}} \int_{-\infty}^0 e^{\frac{1}{2}t \operatorname{Re} \sqrt{1+4\lambda}} |\tilde{f}_1(t+x, y)| dt \\ &\leq \frac{1}{|1+4\lambda|^{1/2}} \|\tilde{f}_1\|_{C_b(I_-)} \int_{-\infty}^0 e^{\frac{1}{2}t \operatorname{Re} \sqrt{1+4\lambda}} dt \\ &\leq \frac{1}{|1+4\lambda|^{1/2}} \frac{1}{\operatorname{Re} \sqrt{1+4\lambda}} \|\mathbf{f}\|_{\mathcal{X}}, \end{aligned} \quad (\text{A.9})$$

for any $(x, y) \in I_-$.

Introducing the function $f = f_1\chi_{(-\infty, 0)} + f_2\chi_{(0, +\infty)}$, allows us to rewrite $\mathcal{J}_2 + \mathcal{J}_3$ in a more compact form as follows:

$$\mathcal{J}_2(x, y) + \mathcal{J}_3(x, y) = \frac{1}{\sqrt{1+4\lambda}} \int_0^{+\infty} e^{-\nu_2 t} f(t+x, y) dt, \quad (x, y) \in I_-.$$

Hence, arguing as in the estimate of $\mathcal{J}_1$ yields us to an estimate completely similar to (A.9).

Finally,

$$\begin{aligned}
|e^{-\frac{\pi}{2}} \mathcal{J}_4(x, y)| &\leq \left| \frac{2\lambda}{1 + (2\lambda - 1)\sqrt{1 + 4\lambda}} \right| e^{-\frac{\pi}{2}} |g_1(x)| \\
&\quad \times \left(\int_{-\infty}^0 e^{\frac{1}{2}t\sqrt{1+4\lambda}} |\tilde{f}_1(t, y)| dt + \int_0^{+\infty} e^{-\frac{1}{2}t\sqrt{1+4\lambda}} |\tilde{f}_2(t, y)| dt \right) \\
&\leq 2 \left| \frac{2\lambda}{1 + (2\lambda - 1)\sqrt{1 + 4\lambda}} \right| \frac{1}{\operatorname{Re} \sqrt{1 + 4\lambda}} \|\mathbf{f}\|_{\mathcal{X}} \\
&\quad \times \left(\frac{1}{|\lambda|} + \frac{1}{\sqrt{|1 + 4\lambda|}} \frac{2}{|1 - \sqrt{1 + 4\lambda}|} \right) \\
&\leq \frac{C}{|\lambda|^3} \|\mathbf{f}\|_{\mathcal{X}}, \tag{A.10}
\end{aligned}$$

for $|\lambda|$ sufficiently large and any $(x, y) \in I_-$.

Summing up, from (A.9) and (A.10), it follows that there exist two positive constants C and M such that

$$\|\tilde{u}_1\|_{C_b(I_-)} \leq \frac{C}{|\lambda|} \|f\|_{\mathcal{X}},$$

for any $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \geq M$. Since, as it has been already claimed, a similar estimate is satisfied by the function u_2 , it follows that L is sectorial in $\mathcal{X}$ by [13, Proposition 2.1.11].

(c): let us introduce the space $\mathcal{X}_0$ of all the functions $f \in \mathcal{X}$, which are independent of y , and consider the realization of the operator $\mathcal{L}$ in $\mathcal{X}_0$. We denote it by L_0 and observe that $D(L_0) = D(L) \cap \mathcal{X}_0$. It is clear that L_0 is sectorial, its spectrum is given by $(-\infty, -1/4] \cup \{0\}$ and $R(\lambda, L_0)\mathbf{f}$ is given by formulas (A.7) and (A.8) for any $\mathbf{f} \in \mathcal{X}_0$.

Let us prove that 0 is a simple eigenvalue of L_0 . Observing that, we can write

$$\frac{2\lambda}{1 + (2\lambda - 1)\sqrt{1 + 4\lambda}} = \frac{4\lambda}{(\sqrt{1 + 4\lambda} + 2)(\sqrt{1 + 4\lambda} - 1)^2} = \frac{(\sqrt{1 + 4\lambda} + 1)^2}{4\lambda},$$

from formulas (A.7) and (A.7) it is immediate to deduce that the function $R(\cdot, L_0)$ has a simple pole at $\lambda = 0$. [13, Proposition A.2.2] now implies that $\lambda = 0$ is a semisimple eigenvalue of L_0 .

To conclude that $\lambda = 0$ is simple it suffices to observe that the more general solution to the equation $L_0 \mathbf{u}$ in $D(L)$ is given by $\mathbf{u} = a\mathbf{U}$, a being an arbitrary constant. This tells us that the kernel of L_0 is one dimensional and is spanned by $\mathbf{U}$. According to [13, Lemma A.2.8] the spectral projection associated with the eigenvalue 0 of the operator L_0 is the only projection on the kernel of L_0 which commutes with L_0 . Of course, the operator $\mathcal{P}$ is a projection. Hence, its restriction to $\mathcal{X}_0$ is a projection as well. Moreover, a straightforward computation shows that $\mathcal{P}$ commutes with L_0 . Hence,

$$\mathcal{P}\mathbf{f} = \frac{1}{2\pi i} \int_{\partial B(0, \varepsilon)} R(\lambda, L_0)\mathbf{f} d\lambda,$$

for any $\mathbf{f} \in \mathcal{X}_0$, where the boundary of $B(0, \varepsilon)$ is oriented counterclockwise and $\sigma(L) \cap \overline{B(0, \varepsilon)} = \{0\}$. Since, $(R(\lambda, L)\mathbf{f})(\cdot, y) = R(\lambda, L_0)\mathbf{f}(\cdot, y)$ for any $y \in [-\ell/2, \ell/2]$,

we easily conclude that $\mathcal{P}$ is the spectral projection associated with the eigenvalue $\lambda = 0$ of the operator L .

(d): it follows immediately from (c). $\square$

Remark A.1. We stress that $\lambda = 0$ is not a simple eigenvalue of the operator L . Indeed, the kernel of operator L is not one dimensional, since any function $\mathbf{u} = a\mathbf{U}$ with $a \in C([-\ell, \ell])$ is in the kernel of L .

We now completely characterize the interpolation spaces between $\mathcal{X}$ and $D(L)$.

Proposition A.2. *For any $\alpha \in (0, 1) \setminus \{1/2\}$ it holds that*

$$D_L(\alpha, \infty) = \left\{ \mathbf{f} \in \mathcal{X} : \tilde{f}_1 \in C_b^{2\alpha, 0}(I_-), \tilde{f}_2 \in C_b^{2\alpha, 0}(I_+), \right. \\ \left. D_x^{(j)} f_1(0, \cdot) = D_x^{(j)} f_2(0, \cdot), j = 0, \dots, [\alpha] \right\}, \quad (\text{A.11})$$

with a continuous embedding.

Proof. As a first step, we show that $D(L)$ is continuously embedded in the space $\mathcal{X}_2 := \{\mathbf{u} \in \mathcal{X} : \tilde{\mathbf{u}} := (\tilde{u}_1, \tilde{u}_2) \in C_b^{2, 0}(I_-) \times C_b^{2, 0}(I_+)\}$, endowed with the norm $\|\mathbf{u}\|_{\mathcal{X}_2} = \|\tilde{u}_1\|_{C_b^{2, 0}(I_-)} + \|\tilde{u}_2\|_{C_b^{2, 0}(I_+)}$. Clearly, we have only to show that $\|\mathbf{u}\|_{\mathcal{X}_2} \leq C\|\mathbf{u}\|_{D(L)}$ for any $\mathbf{u} \in D(L)$ and some positive constant C , independent of $\mathbf{u}$. Since any function $\mathbf{u} \in D(L)$ can be represented through formulas (A.7) and (A.8), with $\lambda = 1$ and $\mathbf{f} := \mathbf{u} - L\mathbf{u}$, the wished inequality follows immediately multiplying (A.7) and (A.8) by $e^{-x/2}$ and differentiating the so obtained formulas twice with respect to x .

Let us now introduce the spaces $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{X}}_2$ defined as follows:

$$\tilde{\mathcal{X}} = C_b(I_-) \times C_b(I_+), \\ \tilde{\mathcal{X}}_2 = \{\mathbf{f} \in C_b^{2, 0}(I_-) \times C_b^{2, 0}(I_+) : D_x^{(j)} f_1(0, \cdot) = D_x^{(j)} f_2(0, \cdot), j = 0, 1\},$$

and the operator $\mathcal{T}$ formally defined by $\mathcal{T}\mathbf{f} = \tilde{\mathbf{f}}$ for any $\mathbf{f} \in C(I_-) \times C(I_+)$. As it is immediately seen, $\mathcal{T}$ is an isomorphism between $\mathcal{X}$ and $\tilde{\mathcal{X}}$ and between $\mathcal{X}_2$ and $\tilde{\mathcal{X}}_2$. By [13, Proposition 1.2.6], it follows that $(\mathcal{X}, D(L)) \subset (\mathcal{X}, \mathcal{X}_2)_{\alpha, \infty} = \mathcal{T}^{-1}(\tilde{\mathcal{X}}, \tilde{\mathcal{X}}_2)_{\alpha, \infty}$. Let us characterize the interpolation space $(\tilde{\mathcal{X}}, \tilde{\mathcal{X}}_2)_{\alpha, \infty}$. It is well known that

$$(C_b(I_-) \times C_b(I_+), C_b^{2\alpha, 0}(I_-) \times C_b^{2\alpha, 0}(I_+))_{\alpha, \infty} = C_b^{2\alpha, 0}(I_-) \times C_b^{2\alpha, 0}(I_+)$$

with equivalence of the corresponding norm. Hence, $(\tilde{\mathcal{X}}, \tilde{\mathcal{X}}_2)_{\alpha, \infty} \subset C_b^{2\alpha, 0}(I_-) \times C_b^{2\alpha, 0}(I_+)$, with a continuous embedding. Since $(\mathcal{X}, D(L))_{\alpha, \infty}$ is continuously embedded into $\overline{D(L)}$ (see [13, Proposition 1.2.3]), $f_1(0, \cdot) = f_2(0, \cdot)$ for any $\mathbf{f} \in (\mathcal{X}, D(L))_{\alpha, \infty}$. The same proposition implies that $(\mathcal{X}, D(L))_{\alpha, \infty}$ is continuously embedded in $(\mathcal{X}, D(L))_\theta$ for any $\theta < \alpha$. By [13, Proposition 1.2.12], $(\mathcal{X}, D(L))_\theta$ is the closure of $D(L)$ in $(\mathcal{X}, D(L))_{\theta, \infty}$. Hence, if $\alpha > 1/2$, any function $\mathbf{f} \in (\mathcal{X}, D(L))$ satisfies the condition $D_x f_1(0, \cdot) = D_x f_2(0, \cdot)$. The inclusion “ $\subset$ ” in (A.11) now follows.

Let us prove the inclusion “ $\supset$ ”. For this purpose, we observe that

$$\mathcal{X}^\sharp := \{\mathbf{f} \in C_b(I_-) \times C_b(I_+) : f_1(0, \cdot) = f_2(0, \cdot)\} \subset \tilde{\mathcal{X}}, \\ \mathcal{X}_2^\sharp := \{\mathbf{f} \in C_b^{2, 0}(I_-) \times C_b^{2, 0}(I_+) : D_x^{(j)} f_1(0, \cdot) = D_x^{(j)} f_2(0, \cdot), j = 0, 1, 2\} \subset \tilde{\mathcal{X}}_2,$$

with continuous embeddings. Hence, $(\mathcal{X}^\sharp, \mathcal{X}_2^\sharp)_{\alpha, \infty} \subset (\tilde{\mathcal{X}}, \tilde{\mathcal{X}}_2)_{\alpha, \infty}$ with a continuous embedding. Since $\mathcal{X}^\sharp$ and $\mathcal{X}_2^\sharp$ are isomorphic to the spaces $C_b(I)$ and $C_b^{2, 0}(I)$,

respectively, and $(C_b(I), C_b^{2,0}(I))_{\alpha, \infty} = C_b^{2\alpha, 0}(I)$, with equivalence of the respective norms, the inclusion “ $\supset$ ” in (A.11) follows at once. $\square$

A.2. The operator A (in different settings). In this subsection, we introduce three different sectorial operators. With a slight abuse of notation, we denote all of them by the same letter A . This abuse of notation is motivated by the fact that they all are suitable realizations of the differential operator D_{yy} in spaces of functions of one or two variables. It is clear from the context what realization of the operator D_{yy} we are referring at.

We first consider the one dimensional setting and the realization of D_{yy} in $L^2(-\ell/2, \ell/2)$, defined by

$$\begin{cases} D(A) = H^2(-\ell/2, \ell/2), \\ Au = u_{yy}, \quad u \in D(A). \end{cases} \quad (\text{A.12})$$

It is well known (and it can be checked by elementary computations) that A is sectorial. Hence, it generates an analytic semigroup. Moreover, its spectrum is given by

$$\sigma(A) = \left\{ -\frac{4k^2\pi^2}{\ell^2} : k = 0, 1, \dots \right\}. \quad (\text{A.13})$$

Similarly, also the realization of D_{yy} in $C([- \ell/2, \ell/2])$ (endowed with the sup-norm), with the set of all functions $f \in C^2([- \ell/2, \ell/2])$ such that $f^{(j)}(-\ell/2) = f^{(j)}(\ell/2)$ for $j = 0, 1$ as domain, is sectorial and its spectrum is still given by (A.13).

Finally, we consider the realization of the operator D_{yy} in $\mathcal{X}$, defined by

$$\begin{cases} D(A) = \left\{ \mathbf{u} \in C^{0,2}(I_-) \times C^{0,2}(I_+) : \mathbf{u}, D_{yy}\mathbf{u} := (D_{yy}u_1, D_{yy}u_2) \in \mathcal{X}, \right. \\ \quad \left. D^{(j)}u_i(\cdot, -\ell/2) = D^{(j)}u_i(\cdot, \ell/2), i = 1, 2, j = 0, 1 \right\} \\ Au = D_{yy}\mathbf{u}, \quad \mathbf{u} \in D(A). \end{cases}$$

The following result is now a straightforward consequence of the previous remarks.

Theorem A.3. *The operator A generates an analytic semigroup in $\mathcal{X}$. Moreover, $\sigma(A) = \left\{ -\frac{4k^2\pi^2}{\ell^2} : k = 0, 1, \dots \right\}$. Finally, if, for any $f \in \mathcal{X}$ and any $\lambda \notin \sigma(A)$, we set $\mathbf{u} := R(\lambda, A)\mathbf{f}$, it holds that*

$$\begin{cases} u_1(x, y) = (R(\lambda, A)f_1(x, \cdot))(y), & x \leq 0, y \in [-\ell/2, \ell/2], \\ u_2(x, y) = (R(\lambda, A)f_2(x, \cdot))(y), & x \geq 0, y \in [-\ell/2, \ell/2]. \end{cases}$$

A.3. Proof of Theorem 2.2 and other remarkable properties of the operator $L + A$. In this section we prove that the operator $L + A$ is closable and its closure L_1 is sectorial. Moreover, we exploit some remarkable properties of L_1 .

A.3.1. Proof of Theorem 2.2. (a): note that the operator $L + A$ coincides with the algebraic sum of the operators L and A . Since these two operators commute in the resolvent sense, acting on different variables, and are sectorial (see Theorems 2.1 and A.3), taking advantage of [8] it immediately follows that the closure of the operator $L + A$ generates an analytic semigroup $\{e^{tL_1}\}$ in $\mathcal{X}$ and

$$e^{tL_1} = e^{tL}e^{tA} = e^{tA}e^{tL}, \quad t > 0.$$

(b): as a first step, we claim that e^{tL_1} maps $(I - \mathcal{P})(\mathcal{X})$ into itself for any $t > 0$. For this purpose, we recall that, being the spectral projection associated with the eigenvalue 0 of the operator L , $I - \mathcal{P}$ commutes with e^{tL} for any $t > 0$. Hence,

$$(I - \mathcal{P})(e^{tL_1}\mathbf{f}) = (I - \mathcal{P})(e^{tL}e^{tA}\mathbf{f}) = e^{tL}(I - \mathcal{P})(e^{tA}\mathbf{f}) = e^{tL}e^{tA}(I - \mathcal{P})(\mathbf{f}),$$

for any $t > 0$ and any $\mathbf{f} \in \mathcal{X}$, where the last equality follows from the fact that observe that $\mathcal{P}$ and $\{e^{tA}\}$ act on different variables. Thus, the restriction of $\{e^{tL_1}\}$ to $(I - \mathcal{P})(\mathcal{X})$ gives rise to an analytic semigroup, which decays exponentially to 0 as t tends to $+\infty$. Indeed, the classical maximum principle implies that $\{e^{tA}\}$ is a contraction semigroup in $\mathcal{X}$. Moreover, $\{e^{tL}(I - \mathcal{P})\}$ decays exponentially to 0 as t tends to $+\infty$, since the spectrum of the part of L in $(I - \mathcal{P})(\mathcal{X})$ is contained in the left-half plane and there is a gap with the imaginary axis. Hence,

$$\|e^{tL_1}\|_{L((I-\mathcal{P})(\mathcal{X}))} \leq \|e^{tL}\|_{L((I-\mathcal{P})(\mathcal{X}))} \|e^{tA}\|_{L((I-\mathcal{P})(\mathcal{X}))} \leq M_\omega e^{-\omega t}, \quad t > 0,$$

for any $\omega < 1/4$ and some positive constant M_ω . This implies that 0 is not in the spectrum of the restriction of L_1 to $(I - \mathcal{P})(\mathcal{X})$, by [13, Proposition 2.3.1]. $\square$

A.4. Some remarkable properties of the semigroup e^{tL_1} . This subsection is devoted to prove the following proposition.

Proposition A.4. *The following properties are met:*

(i) *for any $\omega < 1/4$, there exists a positive constant M_ω such that*

$$\|e^{tL_1}\|_{L((I-\mathcal{P})(\mathcal{X}))} \leq M_\omega e^{-\omega t}, \quad t > 0; \quad (\text{A.14})$$

(ii) *let $\mathbf{f} = \mathbf{h}\varphi$ for some $\mathbf{h} \in (I - \mathcal{P})(\mathcal{X})$, independent of y , and some $\varphi \in D_A(\alpha, \infty)$ ($\alpha \in (0, 1)$). Then, the function $R(0, L_1)\mathbf{f}$ belongs to $D(L) \cap D(A)$. Moreover, there exists a positive constant C , independent of $\mathbf{h}$ and φ , such that*

$$\|D_x^{(i)} R(0, L_1)\mathbf{f}\|_{\mathcal{X}} + \|D_y^{(i)} R(0, L_1)\mathbf{f}\|_{\mathcal{X}} \leq C_1 \|\mathbf{h}\|_{\mathcal{X}} \|\varphi\|_{D_A(\alpha, \infty)}, \quad (\text{A.15})$$

for $i = 0, 1, 2$.

Proof. (i): it follows immediately from [13, Proposition 2.3.1] and Proposition 2.2(b).

(ii): we first show that the function $\mathbf{u} := R(0, L)\mathbf{f}$ belongs to $D(A)$. For this purpose, we observe that, from the very definition of the semigroup $\{e^{tL_1}\}$, it is immediate to check that

$$R(0, L_1)\mathbf{f} = \int_0^{+\infty} e^{tL_1}\mathbf{f} dt = \int_0^{+\infty} e^{tL}\mathbf{h} e^{tA}\varphi dt.$$

It is well known that, for any $\alpha \in (0, 1) \setminus \{1/2\}$,

$$D_A(\alpha, \infty) = \{\varphi \in C^{2\alpha}([-l/2, l/2]) : D^{(j)}\varphi(-l/2) = D^{(j)}\varphi(l/2), j \leq [\alpha]\}.$$

Moreover, $\|Ae^{tA}h\|_\infty \leq C_\varepsilon t^{-1+\alpha/2} \|h\|_{D_A(\alpha, \infty)} e^{\varepsilon t}$, for any $t > 0$, any $\varepsilon > 0$ and some positive constant C_ε , independent of t . Hence,

$$\|e^{tL}\mathbf{h} D_{yy} e^{tA}\varphi\|_{\mathcal{X}} \leq \|e^{tL}\mathbf{h}\|_{\mathcal{X}} \|Ae^{tA}\varphi\|_{\mathcal{X}} \leq C_{\omega, \varepsilon} e^{-(\omega - \varepsilon)t} t^{-1+\frac{\alpha}{2}} \|\mathbf{h}\|_{\mathcal{X}} \|\varphi\|_{D_A(\alpha, \infty)},$$

for any $t > 0$ any $\omega \in (0, 1/4)$, any $\varepsilon > 0$ and any $\alpha \in (0, 1)$. Choosing ω and ε such that $-\omega + \varepsilon < 0$, we can conclude that the right-hand side of the previous chain of inequalities belongs to $L^1(0, +\infty)$. This shows that the function $\mathbf{u}$ belongs to $D(A)$. Moreover,

$$\|D_{yy}\mathbf{u}\|_{\mathcal{X}} \leq C_1 \|\mathbf{h}\|_{\mathcal{X}} \|\varphi\|_{D_A(\alpha, \infty)},$$

for some positive constant C_1 , independent of $\mathbf{h}$ and φ . Clearly from this estimate, we immediately deduce that

$$\|D_{yy}\mathbf{u}\|_{\mathcal{X}} \leq C_2 \|\mathbf{h}\|_{\mathcal{X}} \|\varphi\|_{D_A(\alpha, \infty)},$$

with C_2 being independent of $\mathbf{h}$ and φ .

Now, by difference, we can conclude that the function $\mathcal{L}\mathbf{u} := \mathbf{f} - D_{yy}\mathbf{u}$ belongs to $\mathcal{X}$. Moreover, since $D_L(\alpha, \infty) \subset D(L_1)$ for any $\alpha \in (0, 1)$, Proposition A.2 implies that $\tilde{u}_1$ and $\tilde{u}_2$ are in $C_b^1(I_-)$ and $C_b^1(I_+)$, respectively. Moreover, $u_1(0, \cdot) = u_2(0, \cdot)$ and $D_x u_1(0, \cdot) = D_x u_2(0, \cdot)$. Hence, the function $\mathbf{u}_{yy}$ belongs to $\mathcal{X}$, i.e., $\mathbf{u} \in D(L)$. The estimates for the functions $D_x \mathbf{u}$ and $D_{xx} \mathbf{u}$ in (A.15) now follow from the arguments in the first part of the proof of Proposition A.2. $\square$

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REFERENCES

- [1] C.-M. Brauner, M.L. Frankel, J. Hulshof, A. Lunardi, G.I. Sivashinsky, *On the $\kappa - \theta$ model of cellular flames: existence in the large and asymptotics*, Discr. Cont. Dyn. Syst. S **1** (2007), no. 1, pp. 27-39.
- [2] C.-M. Brauner, M.L. Frankel, J. Hulshof, V. Roytburd, *Stability and attractors for quasi-steady model of cellular flames*, Interfaces Free Bound **8** (2006), pp. 301-316.
- [3] C.-M. Brauner, M.L. Frankel, J. Hulshof, G.I. Sivashinsky, *Weakly nonlinear asymptotic of the $\kappa - \theta$ model of cellular flames: the Q-S equation*, Interfaces Free Bound **7** (2005), pp. 131-146.
- [4] C.-M. Brauner, J. Hulshof, L. Lorenzi, *Rigorous derivation of the Kuramoto-Sivashinsky equation in a 2D weakly nonlinear quasi-steady Stefan problem* (in preparation).
- [5] C.-M. Brauner, J. Hulshof, A. Lunardi, *A critical case of stability in a Free Boundary Problem*, J. Evolution Equations, **1** (2001), pp. 85-113.
- [6] C.-M. Brauner, J. Hulshof, A. Lunardi, *A general approach to stability in free boundary problems*, J. Differential Equations **164** (2000), no. 1, pp. 16-48.
- [7] C.-M. Brauner, A. Lunardi, *Instabilities in a two-dimensional combustion model with free boundary*, Arch. Ration. Mech. Anal. **154** (2000), no. 2, pp. 157-182.
- [8] G. Da Prato, P. Grisvard, *Sommes d'opérateurs linéaires et équations différentielles opérationnelles*, J. Math pures et appl. **54** (1975), pp. 305-387.
- [9] M.L. Frankel, *On the weakly nonlinear evolution of a perturbed solid-liquid interface*, Physica **27D** (1987), pp. 260-266.
- [10] M.L. Frankel, V. Roytburd, G. Sivashinsky, *Complex dynamics generated by a sharp interface model of self-propagating high temperature synthesis*, Combust. Theory Modelling **2** (1998), pp. 1-18.
- [11] J. Hulshof, L. Lorenzi, *Stability in a two-dimensional combustion model*, Interfaces Free Bound. **6** (2004), no. 1, pp. 1-29.
- [12] L. Lorenzi, *A free boundary problem stemmed from combustion theory. II. Stability, instability and bifurcation results*, J. Math. Anal. Appl. **275** (2002), no. 1, pp. 131-160.
- [13] A. Lunardi, "Analytic semigroups and optimal regularity in parabolic problems", Birkhäuser, Basel, 1995.
- [14] G.I. Sivashinsky, *On flame propagation under conditions of stoichiometry*, SIAM J. Appl. Math. **39** (1980), pp. 67-82.

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