Formal asymptotics of bubbling in the harmonic map heat flow

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Abstract

The harmonic map heat flow is a model for nematic liquid crystals and also has origins in geometry. We present an analysis of the asymptotic behaviour of singularities arising in this flow for a special class of solutions which generalises a known (radially symmetric) reduction. Specifically, the rate at which blowup occurs is investigated in settings with certain symmetries using the method of matched asymptotic expansions. We identify a range of blowup scenarios in both finite and infinite time, including degenerate cases.

1 Introduction

We consider the equation

$$\theta_t = \theta_{rr} + \frac{1}{r}\theta_r - \frac{\sin 2\theta}{2r^2}, \qquad 0 < r < 1, \tag{1}$$

with boundary conditions $\theta(t,0) \in \pi\mathbb{Z}$ and $\theta(t,1) = \theta_1 \in \mathbb{R}$; the reason for this type of boundary condition at $r = 0$ will become clear shortly. Solutions of (1) may develop a singularity. In this paper we analyse this blowup behaviour using formal matched asymptotics.

Equation (1) is a special case of the harmonic map heat flow

$$\frac{\partial u}{\partial t} = \Delta u + |\nabla u|^2 u, \tag{2}$$

where $u(t,\cdot) : \Omega \to S^2$, i.e. $u(t,x)$ denotes a unit vector in $\mathbb{R}^3$, $\Omega \subset \mathbb{R}^N$ (in most physical models $N = 3$) and $|\nabla u|^2 = \sum_{j=1}^{N} \sum_{i=1}^{3} (\frac{\partial u_i}{\partial x_j})^2$. Stationary solutions of (2) are harmonic maps from Ω to S^2.

Observe that we are dealing with blowup of the derivative ∇u while u remains bounded (in fact $|u(t,x)| = 1$ for all t and x), and similarly θ_r blows up while θ remains bounded (as we shall see, θ can make finite jumps). This in contrast to many widely studied blowup problems, such as the reaction-diffusion equation $u_t = \Delta u + u^p$ with $p > 1$, where u itself blows up (see e.g. [8] and references therein).

Equation (2) may be reduced to (1) if Ω is a disk in $\mathbb{R}^2$: assuming the solution to be radially symmetric and using polar coordinates (r,ϕ) on the unit disk, a special type of solutions of (2) is given by

$$u(t,\cdot) : (r,\phi) \to \begin{pmatrix} \cos\phi \, \sin\theta(t,r) \\ \sin\phi \, \sin\theta(t,r) \\ \cos\theta(t,r) \end{pmatrix}, \tag{3}$$

where $\theta(t,r)$ satisfies Equation (1). Similarly, when Ω is a cylinder then the problem of finding solutions of (2) which are both radially symmetric and uniform in the axial direction may be reduced to (1). This is the configuration studied in [11] as a model for aligned nematic liquid crystals, with the motivation coming from applications in fibre spinning. Beside the context of liquid crystals (see e.g. [14]) another application in which (2) appears is in the theory of ferromagnetic materials (e.g. [6, 3]). In geometry, Equation (2) is studied in the construction harmonic maps of certain homotopy types (see e.g. [15]), where Ω is generally (a subset of) an N-manifold (the target manifold may also differ from S^2, but if it is not a sphere Equation (2) is altered). The formation of singularities in the flow of (2) has been extensively studied; we refer to [15, 16, 9]. Singularities occur due to topological obstructions, a situation which is comparable to closely related problems in the Ginzburg-Landau equation (see e.g. [4]). In the present context the issue is that while all solutions eventually converge to equilibria we can choose initial data in a topological class that does not contain any equilibria. The solution must then "jump" to another topological class. The different topological classes are (for fixed $\theta(t,1) = \theta_1$) characterised by the value of θ at the origin, which has to be a multiple of π for the solution to have finite energy (see below).

We recall some important results, where in view of (1) we concentrate on domains $\Omega \subset \mathbb{R}^2$. Equation (2) is the gradient flow associated with the energy $\mathcal{E} = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx$. It is well known that a weak solution of (2) exists globally. There may be many weak solutions, but there is a unique one in the class of energy-decreasing solutions, see [7, 15]. Weak solutions are in $H^1(\Omega, S^2)$ for almost all $t > 0$ and, when the initial data are smooth, the solution is locally a classical solution. In fact, the solution is smooth everywhere except for at most a finite number of space-time points [15]. Moreover, there are smooth initial conditions for which a singularity occurs in finite time [5, 2]. As $t \to \infty$ the solution converges weakly to a stationary solution and does so smoothly away from at most a finite number of points. At a singularity, either in finite time or at $t = \infty$, a sphere is said to bubble off: an appropriate blowup near the singularity converges to a harmonic map on the sphere $S^2 \cong \overline{\mathbb{R}^2}$ [15]. In this paper we investigate the rate at which these spheres bubble off in the symmetric setting of (1). In the context of liquid crystals this bubbling means that quanta of energy (i.e. a multiple of 4π) are stored in a singularity (a region smaller than that captured by the model).

Let us now concentrate on the implications for Equation (1). The (weak) solution $\theta(t,\cdot)$ is continuous on $[0,1]$ and $\theta(t,0) \in \pi\mathbb{Z}$ for all $t > 0$. The requirement that $\theta(t,0) \in \pi\mathbb{Z}$ is necessary for solutions to have finite energy. Singularities can develop only at the origin. At a singularity the energy

$$\mathcal{E}(t) = \pi \int_0^1 \Big(r\,\theta_r^2 + \frac{\sin^2\theta}{r} \Big)\,dr \tag{4}$$

decreases (jumps) by 4π or a multiple thereof (of course, away from such singularities the energy $\mathcal{E}(t)$ decreases continuously throughout the evolution since Equation (1) is the gradient flow associated to (4)). Notice that the stationary solutions of (1) with finite energy $\mathcal{E}$ are given by

$$\theta(r) = m\pi + 2\arctan qr \qquad \text{for any } q \in \mathbb{R} \text{ and } m \in \mathbb{Z},$$

and their energy tends to 4π as $q \to \infty$. The solutions $\theta(r) = (m + \frac{1}{2})\pi$, $m \in \mathbb{Z}$ have infinite energy and can be disregarded.

Whether or not singularities occur will depend on the initial and boundary data (see also Figure 1). Consider initial conditions such that $|\theta(0,0) - \theta_1| < \pi$. Then the solution may converge to one of the stationary states as $t \to \infty$ without forming a singularity. This is indeed what happens when $\theta(0,r) \in C^1$ and $\|\theta(0,r) - \theta(0,0)\|_\infty \le \pi$. On the other hand, it has been proved in [2] that blowup may occur for initial data with $|\theta(0,0) - \theta_1| < \pi$ but $|\theta(0,0) - \theta(0,r)| > \pi$ for some $r \in (0,1)$. A more dramatic situation occurs when $|\theta(0,0) - \theta_1| \ge \pi$: no stationary solution is available that

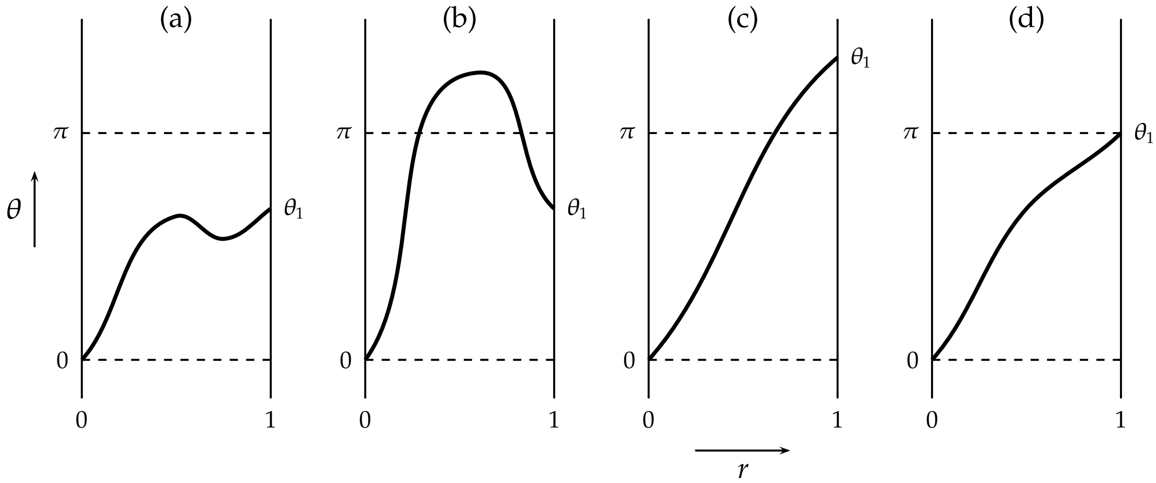

Figure 1: Several initial conditions $\theta(0, r)$: (a) no blowup will occur; (b) blowup may occur; (c) blowup must occur; (d) the degenerate case $\theta_1 = \pi$, which leads to infinite time blowup, as opposed to finite time blowup for $\theta_1 > \pi$.

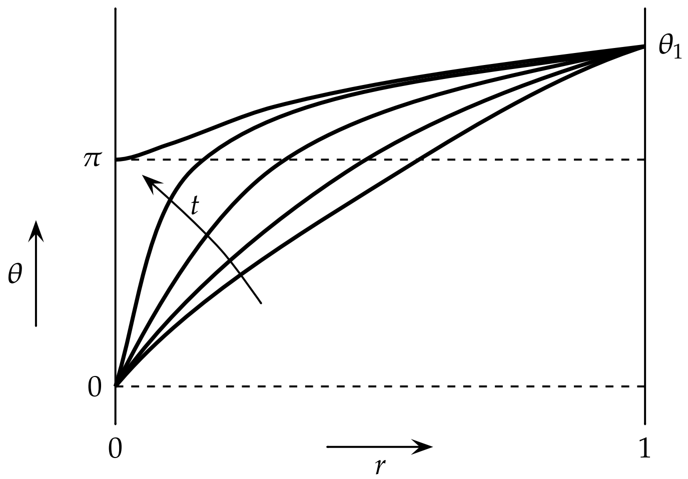

Figure 2: The profile at several times leading up to blowup at $t = T$. The slope $\theta_r(0)$ goes to infinity and the solution jumps from 0 to π at the origin.

obeys both boundary conditions (i.e. $\theta(0) = \theta(0, 0)$ and $\theta(1) = \theta_1$). Therefore, for the solution to approach any of the stationary solutions a jump at the origin must necessarily occur. This is depicted in Figure 2. We focus on this case and after shifting θ by a multiple of π we may restrict our attention to initial/boundary data $\theta(0, 0) = 0$ and $\theta(t, 1) = \theta_1 \geq \pi$, and without loss of generality we will analyse the first instance of blowup.

As mentioned before, when blowup occurs then appropriately zooming in on the singularity will reveal a harmonic map, i.e. one of the stationary solutions. We first focus on boundary data $\theta_1 > \pi$ and consider the special case $\theta_1 = \pi$ later. Assuming the jump of $\theta(t, 0)$ to be upwards (without loss of generality), we select a zooming function $R(t) > 0$ by requiring that

$$R(t)\theta_r(t, 0) = 2 \qquad \text{for all } t \text{ up to the blowup time } T \in (0, \infty].$$

We choose the constant in the right hand side to be 2 in order to keep the subsequent algebra as simple as possible. When blowup occurs at $t = T$ we will thus have

$$\lim_{t \uparrow T} \theta(t, \rho R(t)) = 2 \arctan \rho \qquad \text{for all fixed } \rho > 0.$$

The main objective of this paper is to determine the asymptotic form of $R(t)$.

At first sight one might think that the blowup rate is simply the one corresponding to self-similar variables, i.e. $R(t) \sim \kappa\sqrt{T-t}$. However, no suitable self-similar solution exists (we postpone justification of this statement until the end of this section). Indeed we find that the blowup rate is not the self-similar one. Using formal asymptotics we match an inner layer near $r = 0$ to an outer region near $\theta = \pi$ (where the solution is approximately self-similar), which in turn matches into the remote region where $r = O(1)$. We find that generically

$$R(t) \sim \kappa \frac{T-t}{|\ln(T-t)|^2} \qquad \text{as } t \uparrow T \tag{5}$$

for some blowup time $T > 0$ and some constant $\kappa > 0$. That there is an unknown constant κ is a consequence of the fact that the profile in the remote region plays a subdued role. Therefore, in spite of the finiteness of the domain, the scaling invariance $(t, r) \mapsto (\mu^2 t, \mu r)$ with $\mu > 0$ of (1) causes an indeterminacy.

The point $S(t)$, the smallest intersection of $\theta(t, r)$ with π, behaves as

$$S(t) \sim 2\sqrt{\frac{T-t}{|\ln(T-t)|}} \qquad \text{as } t \uparrow T.$$

In particular, in this scenario the solution always intersects π close to blowup. There is no undetermined constant in this asymptotic expression for $S(t)$ because it is (to leading order) invariant under the scaling invariance. The limit profile at $t = T$ for small r becomes

$$\theta(T, r) \sim \pi + \tfrac{1}{4}\kappa \frac{r}{|\ln r|} \qquad \text{for small } r,$$

with the same constant $\kappa > 0$ as in (5). We remark that there may be additional blowup times $T' > T$, for example when $|\theta(0, 0) - \theta_1| > 2\pi$, and that $\theta(t, 0)$ can jump only by $\pm\pi$ at a time (and thus the energy by 4π), see [12].

A non-generic case arises when $\theta_1 = \pi$ and, for example, $-\pi < \theta(0, r) < \pi$ for $r \in (0, 1)$. In that case the asymptotics indicate blowup in infinite time:

$$R(t) \sim e^{-2\sqrt{t}-5/4} \qquad \text{as } t \to \infty.$$

Notice that the blowup is now in infinite time as opposed to finite time for $\theta_1 > \pi$. Besides, there is no undetermined constant in the leading order term for $R(t)$; the length of the interval has a direct bearing on the analysis so that the scaling invariance is lifted. We remark that when $\theta(0, r) > \pi$ for some $r \in (0, 1)$ then it can (but does not necessarily) happen that blowup occurs in finite time via the scenario described before. In that case the solution intersects π just before blowup. On the other hand, for initial profiles with $-\pi < \theta(0, r) < \pi$ a comparison argument shows that this is not possible.

A different situation in which we can easily see that non-generic behaviour must occur is when $\theta_1 < \pi$ and the initial data are roughly as depicted in Figure 1b. When the initial profile has a sufficiently large bump above π then the solution will blow up in finite time via the scenario described above. On the other hand, when the bump is small (e.g. stays below π) no blowup occurs. In between these generic (co-dimension 0) possibilities there needs to be at least one borderline (non-generic) scenario and such degenerate cases are also discussed below. Regarding the large time behaviour of these solutions, in the latter generic case (when no blowup occurs) the limit profile as $t \to \infty$ is $\theta_\infty(r) = 2\arctan\big(r \tan\frac{\theta_1}{2}\big)$, while in the former one the stationary state $\theta_\infty(r) = \pi - 2\arctan\big(r \tan\frac{\pi-\theta_1}{2}\big)$ is selected (provided no additional jump back to 0 occurs) and it

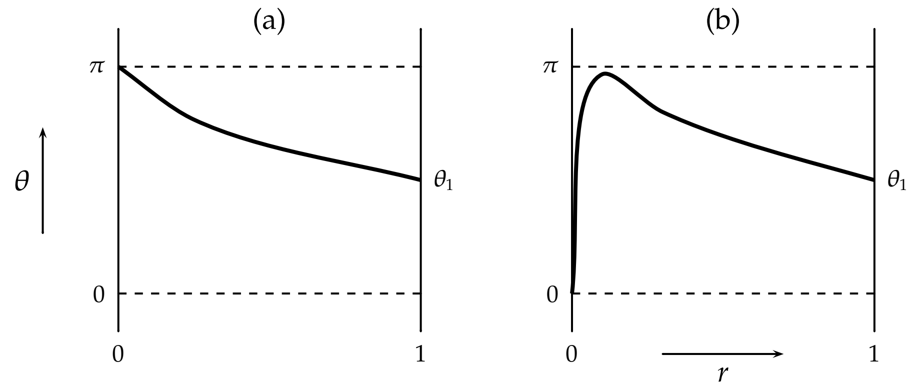

Figure 3: In a reverse jump the energy of the solution increases: (a) just before t_0; (b) just after t_0.

turns out this last one is also the limit profile in the degenerate scenario (if no additional jumps occur).

There is another issue related to these singularities. As explained in [2, 18, 5] weak solutions have the possibility to release the energy formerly lost in a singularity, thereby causing a sudden *increase* in the energy $\mathcal{E}(t)$. The physical interpretation is that this released energy was stored in a region of smaller scale than that captured by the model. We consider the situation where θ makes such a 'reverse' jump at $t = t_0$, see also Figure 3. When θ jumps from π to 0 at $t = t_0$, we define as before $R(t) = \frac{2}{\theta_r(t,0)}$ for $t > t_0$. One finds that generically

$$R(t) \sim \kappa \frac{t - t_0}{|\ln(t - t_0)|} \qquad \text{as } t \downarrow t_0,$$

for some arbitrary constant $\kappa > 0$. Notice the slight difference with (5).

Reexamining the reduction of (2) to (1), the physical meaning of the Ansatz (3) is that the direction field $u(t, \cdot)$ at the boundary of the cylinder is axisymmetric and the in-plane component points in the radial direction. In fact the solution class defined by (3) belongs to a family of solution classes given by

$$u(t, \cdot) : (r, \phi) \to \begin{pmatrix} \cos n\phi \, \sin \theta(t, r) \\ \sin n\phi \, \sin \theta(t, r) \\ \cos \theta(t, r) \end{pmatrix}. \tag{6}$$

The equation for θ now becomes

$$\theta_t = \theta_{rr} + \frac{1}{r}\theta_r - n^2 \frac{\sin 2\theta}{2r^2}, \qquad 0 \le r \le 1. \tag{7}$$

From a mathematical point of view the constant $n > 0$ (ignoring the trivial case $n = 0$ throughout) can be considered as a continuous parameter in Equation (7), and it can be used to unravel the delicate analysis of blowup for $n = 1$ (which is a borderline case, so that the asymptotic analysis is particularly delicate). From a physical point of view, only the values $n = 1, 2, 3, \ldots$ make sense. In Figure 4 the configurations for $n = 1$, 2 and 3 are depicted. We note that for $n = \frac{1}{2}$ (and odd multiples of $\frac{1}{2}$) the view from the top (see Figure 4) gives the impression of smoothness (because the molecules in a nematic liquid crystal are invariant under inversion, or in other words, in Equation (2) the function $u(t, \cdot)$ maps from Ω to the projective plane instead of to the sphere). However, on closer inspection one observes that in fact a line singularity with infinite energy is unavoidable, hence such cases fall outside the scope of the present paper.

The stationary solutions of (7) with finite energy are

$$\theta(r) = m\pi + 2\arctan(qr^n), \qquad m \in \mathbb{Z},\ q \in \mathbb{R}.$$

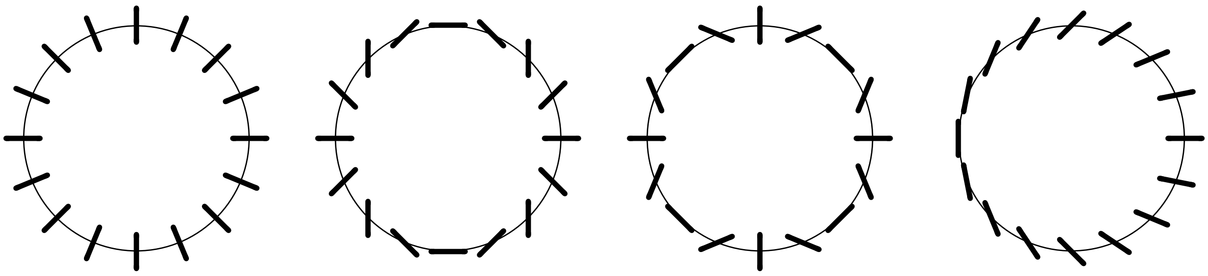

Figure 4: Top view of the behaviour of the vector u at the boundary of the domain for $n = 1$, $n = 2$, $n = 3$ and $n = \frac{1}{2}$, the last one necessarily leading to configurations with infinite energy.

We define $R(t)$ such that

$$R(t)^n \theta(t, r) \sim 2r^n \quad \text{as } r \downarrow 0 \qquad \text{for all } t \text{ up to the blowup time.}$$

After rescaling with this blowup rate the profile tends to a harmonic map (a stationary state) as t approaches to the blowup time T:

$$\lim_{t \uparrow T} \theta(t, \rho R(t)) = 2 \arctan \rho^n \qquad \text{for all fixed } \rho > 0.$$

The results of our asymptotic analysis for $R(t)$ give as the generic blowup behaviour:

$$\begin{aligned}
n < 1: &\quad R \sim \kappa (T-t)^{1/n} &&\text{as } t \uparrow T \\
n = 1: &\quad R \sim \kappa \frac{T-t}{|\ln(T-t)|^2} &&\text{as } t \uparrow T \\
1 < n < 2: &\quad R \sim \kappa (T-t)^{1/(2-n)} &&\text{as } t \uparrow T \\
n = 2: &\quad R \sim \kappa e^{-\frac{\alpha_0}{E_2} t} &&\text{as } t \to \infty \\
n > 2: &\quad R \sim \left(\tfrac{(n-2)\alpha_0}{E_n} t\right)^{-1/(n-2)} &&\text{as } t \to \infty.
\end{aligned}$$

Here $\kappa > 0$ is an arbitrary constant, $E_n = \frac{\pi}{2n^2 \sin(\frac{\pi}{n})}$, and $\alpha_0 = \tan(\frac{\theta_1 - \pi}{2})$ for $\theta_1 \in (\pi, 2\pi)$. The above represent the generic behaviour (e.g. $\theta_1 = \pi$ needs to be considered separately), and for $n \geq 2$ it does not apply to boundary conditions with $\theta_1 \geq 2\pi$; the analysis is more involved in that case, see Section 3.6. Notice that the blowup is in finite time for $n < 2$ versus infinite time blowup for $n \geq 2$. Furthermore, it is remarkable that the borderline case $n = 1$ has the fastest blowup rate. Finally, there is no unknown constant for $n > 2$ since the boundary condition on the right has direct influence on the asymptotics (and thus there is no scaling invariance). This is equally true for $n = 2$, and the translation invariance in time rather than the scaling invariance can be considered responsible for the indeterminacy here. On the other hand, $n = 2$ marks the transition from finite to infinite time blowup and subtle behaviour can be expected at such a critical value.

There is an additional symmetry which needs to be noted. In the right hand side of (6) one may replace ϕ by $\phi + \phi_0$, which again leads to (7). For $n = 1$ this presents us with a family of geometrically different solutions, while for $n \neq 1$ all these solutions are equivalent by rotation of the domain. In Figure 5 we have depicted the situation occurring for $\phi_0 = \frac{\pi}{4}$ and $\phi_0 = \frac{\pi}{2}$ (and $n = 1$) which may be compared to $\phi_0 = 0$ to see the difference in geometry. All these cases are covered by Equation (7).

In order to prevent cumbersome bookkeeping and to be able to clarify the crucial points, we will first analyse the special case $n = 1$ in Section 2. Degenerate cases, including the special boundary condition $\theta_1 = \pi$, and reverse jumps are treated in Sections 2.5 to 2.7. In Section 3 we analyse the general case (7), and the special role of $n = 1$ will become apparent. We also discuss in

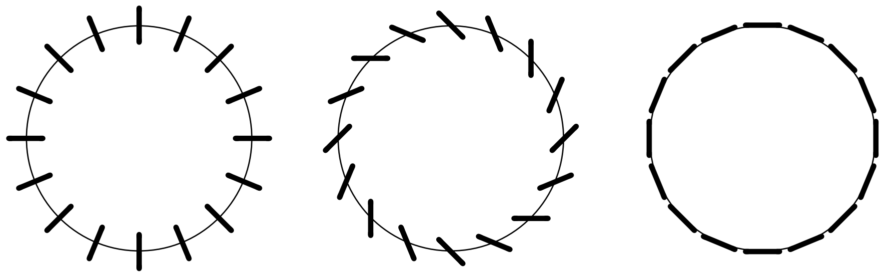

Figure 5: Top view of the behaviour of the vector u at the boundary of the domain for $n = 1$ with $\phi_0 = 0$, $\phi_0 = \frac{\pi}{4}$ and $\phi_0 = \frac{\pi}{2}$.

Section 3.6 the multi-scale blowup associated with boundary conditions $\theta_1 \geq 2\pi$ for $n \geq 2$; in Section 3.8 we deal with the case of an unbounded domain. Finally, we present an overview of our results and draw conclusions in Section 4.

It remains a challenge to find proofs for the formal asymptotic results in this paper. We refer to [1] for some tentative results in which the comparison principle and lap number theorem for parabolic equations are exploited. These methods circumvent non-degeneracy conditions so that, while avoiding the problems associated with degenerate cases, they fail to uncover the full range of the generic behaviour. Another open problem is what happens in a non-symmetric situation, both in two and three dimensions, and what role is played by symmetric solutions in that context. The fact that $n = 1$ is such an exceptional case may suggest that it plays a special role.

To finish this section we show why blowup is not governed by self-similar variables (this was also observed in [1]). A self-similar solution for finite time blowup is of the form $\theta(t,r) = \Theta(r/\sqrt{T-t})$. For the function $\Theta(y)$ one obtains the equation

$$\Theta_{yy} + \left(\frac{1}{y} - \frac{y}{2}\right)\Theta_y - n^2\frac{\sin 2\Theta}{2y^2} = 0.$$

Since $\theta(t,0)$ jumps from 0 to π the boundary conditions are $\Theta(0) = 0$ and $\lim_{y\to\infty}\Theta(y) = \pi$. We now change coordinates to $z = \ln y$ and obtain for $\tilde{\Theta}(z) = \Theta(y)$

$$\tilde{\Theta}_{zz} - \tfrac{1}{2}e^{2z}\tilde{\Theta}_z - \tfrac{n^2}{2}\sin 2\tilde{\Theta} = 0, \tag{8}$$

with boundary conditions $\lim_{z\to-\infty}\tilde{\Theta}(z) = 0$ and $\lim_{z\to\infty}\tilde{\Theta}(z) = \pi$. There is no solution to this problem since on the one hand $G(z) \stackrel{\text{def}}{=} \frac{1}{8}\tilde{\Theta}_z^2 + n^2\cos 2\tilde{\Theta}$ is monotonically increasing, while on the other hand $\lim_{z\to-\infty}G(z) = \lim_{z\to+\infty}G(z) = n^2$. This contradiction shows that there is no such solution and hence this self-similar scenario for blowup cannot occur. In fact, $\tilde{\Theta} = \pi$ is the only solution satisfying the condition as $z \to \infty$; this in part explains why $\theta \sim \pi$ necessarily holds in the outer region described below. In this argument it is crucial that $\int_0^\pi \sin 2\theta\, d\theta = 0$. For nonlinearities that do not have zero average a self-similar blowup rate may be expected. It could thus be interesting to study a problem in which the average of the nonlinearity approaches zero as a parameter is varied.

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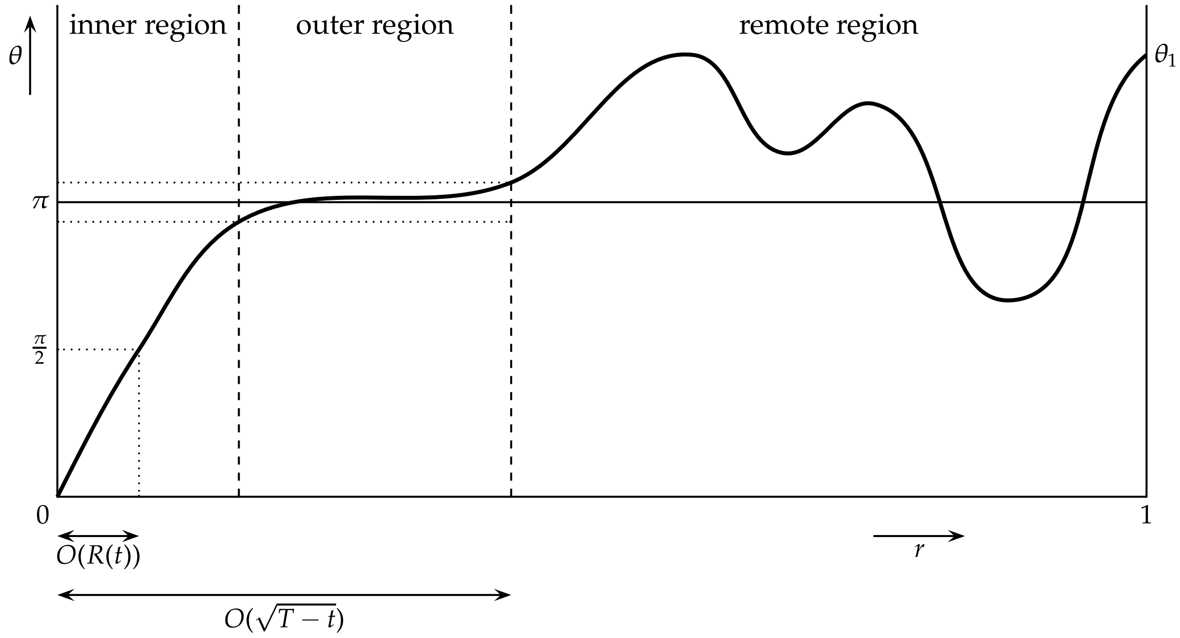

Figure 6: The three different scales.

2 The case $n = 1$

2.1 Preamble

There will be three scales: the inner, the outer and the remote region (see Figure 6). The inner is a small region near the origin in which the blowup is concentrated. It is of order $r = O(R(t))$ where $R(t)$ is an unknown function (in fact the main goal is to determine the asymptotic behaviour of $R(t)$), and in this region the profile near blowup is $2\arctan(r/R(t))$. The outer region is a region with θ near π where the equation can be linearised. The typical scale in this region is $r = O(\sqrt{T-t})$, where T is the blowup time. An obvious requirement for self-consistency is that $R(t) \ll \sqrt{T-t}$. The remote region is the region where $r = O(1)$, and at the time of blowup the profile in this region is unknown, but the limit profile as the origin is approached will come out of the matching procedure.

Throughout the paper the constants C, $\tilde{C}$ and C_i ($i \in \mathbb{N}$) will vary from subsection to subsection.

2.2 The inner approximation

We analyse the boundary layer near $r = 0$. As explained in Section 1 it is known on general grounds that, when we zoom in appropriately, we should see $2\arctan r/R(t)$, with $R(t) \to 0$ as $t \uparrow T$. Recall that the definition of R is $R(t) = \frac{2}{\theta_r(t,0)}$. We introduce a new variable $\xi = r/R(t)$ and obtain for $v(t,\xi) = \theta(t,r)$

$$R^2 v_t - R'R\,\xi v_\xi = v_{\xi\xi} + \frac{1}{\xi} v_\xi - \frac{\sin 2v}{2\xi^2}.$$

Since $R(t)$ becomes small for t close to blowup we formally expand the solution in powers of $R'R$:

$$v(t,\xi) \sim \Phi_0(\xi) + R'(t)R(t)\Phi_1(\xi) + (R'(t)R(t))^2\Phi_2(\xi). \tag{9}$$

A motivation for this expansion is that we anticipate that $R \ll R'$ as $t \to T$. To these orders $R^2 v_t$ does not contribute since the the rescaling is chosen such that the leading order solution is stationary. Of course we need that $R'R \to 0$ as $t \to T$.

One finds that

$$\Phi_0(\xi) = 2\arctan\xi,$$

and $\Phi_1(\xi)$ satisfies

$$\Phi_{1\xi\xi} + \frac{1}{\xi}\Phi_{1\xi} - \frac{\cos 2\Phi_0}{\xi^2}\Phi_1 = -\xi\Phi_{0\xi}.$$

It turns out that Φ_1 is already the interesting term, so that we could have restricted to linearising around Φ_0. The equation for Φ_2 is

$$\Phi_{2\xi\xi} + \frac{1}{\xi}\Phi_{2\xi} - \frac{\cos 4\arctan\xi}{\xi^2}\Phi_2 = \big(1+K\big)\Phi_1 - \xi\Phi_{1\xi} - \frac{\sin(4\arctan\xi)}{\xi^2}\Phi_1^2, \tag{10}$$

where $K = \lim_{t\to T}\frac{R''R}{R'^2}$. Here we see that we need $R''R = O(R'^2)$ as $t \to T$ in order for (9)–(10) to be self-consistent (this includes for example $R \sim (T-t)^a$ and $R \sim t^{-a}$ when $T = \infty$ for any $a > 0$), and this will turn out to be the case with $K = 0$.

The non-uniqueness of Φ_i is resolved by requiring Φ_i to be regular near $\xi = 0$, i.e. $\Phi_i(0) = 0$, and $\Phi_i'(0) = 0$ in view of the definition of $R(t)$. One finds

$$\Phi_1 = \frac{\xi}{1+\xi^2}\int_0^\xi \frac{s(s^4+4s^2\ln s - 1)}{(1+s^2)^2}ds - \frac{\xi^4+4\xi^2\ln\xi - 1}{\xi(1+\xi^2)}\int_0^\xi \frac{s^3}{(1+s^2)^2}ds$$

which, after a quite tedious calculation, can be rewritten as

$$\Phi_1 = \frac{(1-\xi^4)\ln(1+\xi^2) + 2\xi^4 - \xi^2 - 4\xi^2\int_0^\xi \frac{\ln(1+s^2)}{s}ds}{2\xi(1+\xi^2)}.$$

For large ξ the inner approximation thus satisfies

$$v(t,\xi) \sim \pi - 2\xi^{-1} + R'(t)R(t)(-\xi\ln\xi + \xi) \qquad \text{for } t \text{ close to } T \text{ and large } \xi. \tag{11}$$

Here, and in other asymptotic expansions to come, we include all those terms which might be necessary to perform the matching analysis (it is not a priori clear whether all of them will be needed). Let us briefly comment on the inclusion of the term $R'(t)R(t)\xi$ in this expansion. Although it is dominated by $R'(t)R(t)\xi\ln\xi$, it is well known that terms that differ only by orders of $\ln\xi$ can play a role in the matching. We remark that the leading order approximation $v \sim \pi - 2\xi^{-1} - R'(t)R(t)\xi\ln\xi$ is valid if $\xi^4 \gg \frac{1}{|R'R|}$ and $\xi^2 \ll \frac{1}{|R'R|}$, because the next terms are of order $O(\xi^{-3})$ and $O(R'^2R^2\xi^3\ln\xi)$; this last term comes from the large ξ behaviour for the solution of (10).

We remark that we could for most purposes have restricted our attention to the asymptotic equation for Φ_1:

$$\Phi_{1\xi\xi} + \frac{1}{\xi}\Phi_{1\xi} - \frac{1}{\xi^2}\Phi_1 \sim -\frac{2}{\xi},$$

from which we obtain $\Phi_1 \sim -\xi\ln\xi + C\xi$ for large ξ. The value of C can only be determined by solving the full problem for Φ_1 (with boundary conditions at $\xi = 0$) as performed above (i.e. $C = 1$).

2.3 The outer solution

To analyse the outer solution we convert to self-similar coordinates

$$\tau = \ln(T-t)^{-1}, \qquad y = e^{\tau/2}r,$$

where T is the time of blowup. When we set $\zeta(\tau,y) = \theta(t,r)$ then ζ satisfies the equation

$$\zeta_\tau = \zeta_{yy} + (\frac{1}{y} - \frac{y}{2})\zeta_y - \frac{\sin 2\zeta}{2y^2}$$

Linearising around $\zeta = \pi$ one obtains the linear equation

$$\eta_\tau = \mathcal{L}_0 \eta \stackrel{\text{def}}{=} \eta_{yy} + \Big(\frac{1}{y} - \frac{y}{2}\Big)\eta_y - \frac{1}{y^2}\eta \tag{12}$$

We require that $\eta(\tau, y)$ grows less than exponentially for large y, since otherwise it is impossible to match the outer to the remote region. As a boundary condition for small y we take $\eta(\tau, 0) = 0$, at least to leading order. The reason for this choice is not transparent at the moment since it is actually part of the matching process. We will clarify this point in the next section. For now we just stress that this boundary condition is not forced by regularity requirements, but turns out to be required to get consistent matching.

We look for separable solutions of (12) obeying these two 'boundary' conditions (one boundary condition and one growth condition really). The solution of this type which decays most slowly with τ is $\eta = e^{-\tau/2}y$. From a different perspective this means that under the above boundary conditions $\lambda = -\frac{1}{2}$ is the smallest eigenvalue of $\mathcal{L}_0$ (by standard arguments the eigenfunctions are polynomials). Hence one expects η to approach π essentially at rate $e^{-\tau/2}$, so we set

$$\eta \sim \sigma(\tau) e^{-\tau/2}\, y.$$

Since we still have to match to the inner solution (i.e. the true boundary condition is not $\eta(\tau, 0) = 0$) we need to let the coefficient σ depend on τ, but in such a way that σ is not exponential in τ. We thus use 'almost' separable solutions or, in the original variables, 'almost' self-similar solutions. For the solution of (12) we now put forward the Ansatz

$$\eta \sim e^{-\tau/2} \sum_{i=0}^{\infty} \frac{d^i \sigma(\tau)}{d\tau^i} \Omega_i(y), \tag{13}$$

where we anticipate σ to be algebraically decaying. When (with the linear operator $\mathcal{L}_0$ defined in (12))

$$\big(\mathcal{L}_0 + \tfrac{1}{2}\big)\, \Omega_0 = 0 \qquad \text{and} \qquad \big(\mathcal{L}_0 + \tfrac{1}{2}\big)\, \Omega_i = \Omega_{i-1} \qquad i = 1, 2, \ldots,$$

then (13) is formally a solution of (12) for arbitrary $\sigma(\tau)$. In some sense the sequence Ω_i forms a Jordan sequence of the linear operator $\mathcal{L}_0$ at eigenvalue $-\frac{1}{2}$, except that the left boundary condition need no longer be satisfied (in fact, since $-\frac{1}{2}$ is a simple eigenvalue the left boundary condition cannot be satisfied). This shortage of boundary conditions causes non-uniqueness for Ω_i but this is not an issue. The left boundary condition was not a true boundary condition anyway, but merely induced by matching requirements. One obtains

$$\begin{aligned} \Omega_0 &= C_0 y \\ \Omega_1 &= C_0(4y^{-1} - 2y \ln y) + C_1 y. \end{aligned}$$

where the values of the constants are of no significance because they can be absorbed in σ, and without loss of information we set $C_0 = 1$ and $C_1 = 0$. Thus for large τ the outer approximation is

$$\zeta(\tau, y) \sim \pi + e^{-\tau/2}\big[\sigma(\tau) y + \sigma'(\tau)(4y^{-1} - 2y \ln y)\big] \qquad \text{for large } \tau. \tag{14}$$

The function $\sigma(\tau)$ will have to be determined by matching to the inner solution. The outer approximation is valid provided that $y^2 (\ln y)^2 \ll |\frac{\sigma'}{\sigma''}|$ and $y^2 |\ln y| \gg |\frac{\sigma''}{\sigma'}|$ on the side of large and small y respectively, because the next terms are of order $O(\sigma'' y (\ln y)^2)$ for large y and $O(\sigma'' y^{-1})$ for small y. Here we should keep in mind that σ will turn out to be algebraically decaying (in which case $|\frac{\sigma''}{\sigma'}| = O(\tau^{-1})$).

2.4 Matching

To match the outer to the inner solution we rewrite (11) in terms of the self-similar variables:

$$v(t,\xi) = \tilde{v}(\tau, y) \sim \pi - 2e^{\tau/2}\tilde{R}\, y^{-1} + e^{\tau/2}\tilde{R}'\, y(-\ln y + \ln \tilde{R} + \tau/2 + 1),$$

where $\tilde{R}(\tau) = R(t)$ and hence $R'(t) = e^{\tau}\tilde{R}'(\tau)$. Comparing with (14) we obtain

$$O(y^{-1}): \quad 4e^{-\tau/2}\sigma' \sim -2\tilde{R}e^{\tau/2} \tag{15}$$

$$O(y \ln y): \quad -2e^{-\tau/2}\sigma' \sim -\tilde{R}'e^{\tau/2} \tag{16}$$

$$O(y): \quad e^{-\tau/2}\sigma \sim \tilde{R}'e^{\tau/2}(\ln \tilde{R} + \tau/2 + 1). \tag{17}$$

Equations (15) and (16) suggest that $\tilde{R}(\tau) = e^{-\tau}\rho(\tau)$, where $\rho(\tau)$ is algebraic for large τ. Substituting this in (15) and (17) we obtain the following relations for ρ and σ:

$$\rho \sim -2\sigma' \quad \text{and} \quad \tfrac{\tau}{2}\rho \sim \sigma \qquad \text{as } \tau \to \infty,$$

from which we conclude that

$$\sigma(\tau) \sim \frac{\kappa}{2\tau} \quad \text{and} \quad \rho(\tau) \sim \frac{\kappa}{\tau^2} \qquad \text{as } \tau \to \infty,$$

for some $\kappa > 0$. We thus find

$$R(t) \sim \kappa \frac{T-t}{|\ln(T-t)|^2} \qquad \text{as } t \uparrow T. \tag{18}$$

We emphasise that this asymptotic behaviour of $R(t)$ is clearly completely different from the self similar rate $(T-t)^{1/2}$. It is now possible to check that the regions of validity for the inner and outer approximations do indeed overlap: the region of overlap is $(\tau \ln \tau)^{-1/2} \ll y \ll 1$. We note that the matching conditions (15) and (16) convey the same information so that (16) is in some sense redundant, though it provides additional confidence in the matching.

The smallest intersection of $\theta(t,r)$ with π, denoted by $r = S(t)$, can be calculated from (14) using the asymptotic form of σ, which leads to

$$S(t) \sim 2\sqrt{\frac{T-t}{|\ln(T-t)|}} \qquad \text{as } t \uparrow T.$$

Notice that there is no undetermined constant in this leading order formula.

The asymptotic behaviour for small r of the limit profile at $t = T$ is computed by matching the outer approximation to the remote solution

$$\theta(t,r) \sim \Theta(r) + (t-T)\big[\Theta_{rr} + \frac{1}{r}\Theta_r - \frac{\sin 2\Theta}{2r^2}\big] \qquad \text{as } t \to T,$$

with $\Theta(r)$ being the limit profile $\theta(T,r)$. This approximation in the remote region is valid for $r \gg (T-t)^{1/2}$ (at the least). The regions of validity of the outer and remote approximation overlap. Matching with (14) we find

$$\theta(T,r) \sim \pi + \tfrac{1}{4}\kappa \frac{r}{|\ln r|} \qquad \text{for small } r. \tag{19}$$

with the same constant $\kappa > 0$ as in (18). Notice that $\theta(T,r) > \pi$ for small r. A quick way to obtain this behaviour from (14) is by letting τ tend to infinity for fixed y.

We remark that immediately after blowup a different type of inner layer near $r = 0$ appears which describes how analyticity is recovered. In this layer almost self-similar behaviour occurs of the form

$$\theta(t,r) \sim \pi + \frac{\sqrt{t-T}}{|\ln(t-T)|} f_1\big(r/\sqrt{t-T}\big) + \frac{\sqrt{t-T}}{|\ln(t-T)|^2} f_2\big(r/\sqrt{t-T}\big) \qquad \text{as } t \downarrow T \text{ for small } r.$$

Substituting this into the equation for θ one finds that $f_1(z) = Dz$ for some $D \in \mathbb{R}$ to be determined, and $f_2(z)$ is a solution of

$$f_2'' + \left(\frac{1}{z} + \frac{z}{2}\right) f_2' - \left(\frac{1}{z^2} + \frac{1}{2}\right) f_2 = f_1,$$

with the property that $f_2(z) = O(z)$ as $z \to 0$. One may solve for f_2 by reduction of order, but it is sufficient to remark that it follows from the limiting equation for large z that $f_2(z) \sim 2Dz \ln z$ as $z \to \infty$. Matching this with (19) implies that $D = \frac{1}{8}\kappa$, so that the result is

$$\theta(t,r) \sim \pi + \tfrac{1}{8}\kappa \frac{r}{|\ln(t-T)|} + \frac{\sqrt{t-T}}{|\ln(t-T)|^2} f_2\big(r/\sqrt{t-T}\big) \qquad \text{as } t \downarrow T \text{ for small } r.$$

Let us now come back to the point of choosing the boundary condition for the outer solution. In Section 2.3 it was put forward that the left boundary condition for the outer solution is $\eta(\tau, 0) = 0$ but that this is in fact already part of the matching. Here we explain the reasoning behind this. There is a family of separable solutions to (12):

$$\eta = e^{\lambda\tau} f_\lambda(y),$$

where $f = f_\lambda$ obeys

$$\lambda f = \mathcal{L}_0 f \stackrel{\text{def}}{=} f_{yy} + \left(\frac{1}{y} - \frac{y}{2}\right) f_y - \frac{1}{y^2} f. \tag{20}$$

Since we are looking for solutions that do not blow up as $\tau \to \infty$ we restrict to $\lambda \le 0$ (on the other hand this restriction will also be a consequence of what follows).

Now suppose λ is not in the set $\{-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots\}$. We will show that this leads to inconsistent matching conditions. We find two linearly independent solutions $g(y)$ and $h(y)$ of (20) with the following properties. The asymptotic behaviour of $g(y)$ is $g(y) \sim y$ as $y \downarrow 0$ and it grows faster than exponentially as $y \to \infty$; it can therefore be ruled out. The other, linearly independent, solution $h(y)$ is less than exponentially growing as $y \to \infty$ and for small y it behaves as

$$h(y) \sim y^{-1} + \tfrac{2\lambda-1}{4} y \ln y + C_\lambda y$$

for some constant $C_\lambda \in \mathbb{R}$, the value of which is not relevant except for $\lambda = 0$: $C_0 = \frac{1}{8}(4\ln 2 + 1 - \gamma)$ where γ is Euler's constant. We note that $h(y)$ is closely related to a Kummer-U function.

Still assuming that $\lambda \notin \{-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots\}$ we are lead to matching conditions of the form (assuming throughout that $\sigma' \ll \sigma$)

$$\begin{aligned}
O(y^{-1}): &\qquad e^{\lambda\tau}\sigma \sim -2\tilde{R}e^{\tau/2} \\
O(y \ln y): &\quad \tfrac{2\lambda-1}{4} e^{\lambda\tau}\sigma \sim -\tilde{R}' e^{\tau/2} \\
O(y): &\qquad C_\lambda e^{\lambda\tau}\sigma \sim \tilde{R}' e^{\tau/2}(\ln \tilde{R} + \tau/2 + 1).
\end{aligned}$$

Now $\tilde{R} = e^{(\lambda - 1/2)\tau}\rho(\tau)$ where $\rho(\tau)$ is not exponential in τ. Since we require that $R \ll (T-t)^{1/2} = e^{-\tau/2}$ for self-consistency we find that $\lambda \le 0$. For $\rho(\tau)$ one obtains the relations (if $\lambda < 0$)

$$\sigma(\tau) \sim -2\rho(\tau) \qquad \text{and} \qquad C_\lambda \sigma(\tau) \sim \lambda(\lambda - \tfrac{1}{2})\,\tau\rho(\tau),$$

which immediately leads to a contradiction. When $\lambda = 0$ (which would correspond to an almost self-similar blowup rate) we obtain

$$\sigma(\tau) \sim -2\rho(\tau) \qquad \text{and} \qquad C_0\sigma(\tau) \sim -\tfrac{1}{2}\rho(\tau),$$

and since $C_0 \neq \frac{1}{4}$ this leads to a contradiction as well. A more intuitive explanation for the matching failure is that is is impossible to match a leading order term y^{-1} from the outer expansion with a correction term from the inner one.

If $\lambda \in \{-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots\}$ then the solution obeying the growth condition on the right is regular near $y = 0$, i.e. there are no terms of order y^{-1} (or $y \ln y$). The set $\{-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots\}$ thus consists of the eigenvalues of problem (20) with boundary condition $f(0) = 0$ and the growth condition for $y \to \infty$. This explains the choice of boundary conditions in Section 2.3. The case $\lambda = -\frac{1}{2}$ was dealt with in the previous sections. In the next section we consider the remaining possibilities.

2.5 Degenerate (non-generic) cases

There is a whole family of separable solutions of (12) which obey the growth condition (less than exponential) on the right and the boundary condition on the left (regular near $y = 0$). The solution $\eta = e^{-\tau/2} y$ is the first one, i.e. the least rapidly decaying one. In degenerate cases it may however happen that the coefficient σ in front of this term in (14) vanishes. In that case a degenerate situation occurs with co-dimension 1. The outer solution in that case becomes

$$\zeta \sim \pi + e^{-3\tau/2} \sigma(\tau) \, (y - \tfrac{1}{8} y^3),$$

since $\lambda = -\frac{3}{2}$ is the second smallest eigenvalue of $\mathcal{L}_0$, with eigenfunction $y - \frac{1}{8} y^3$. Following the calculation in Section 2.3 the outer approximation now becomes

$$\zeta \sim \pi + e^{-3\tau/2} \big[\sigma(\tau) \, (y - \tfrac{1}{8} y^3) + \sigma'(\tau) \, (2y^{-1} - 2y \ln y + \tfrac{1}{4} y^3 \ln y - \tfrac{3}{2} y) \big]. \tag{21}$$

Notice that the profile is non-monotone for times close to blowup. The matching conditions become

$$\begin{aligned} O(y^{-1}): &\quad 2e^{-3\tau/2} \sigma' \sim -2\tilde{R} e^{\tau/2} \\ O(y \ln y): &\quad -2e^{-3\tau/2} \sigma' \sim -\tilde{R}' e^{\tau/2} \\ O(y): &\quad e^{-3\tau/2} \sigma \sim \tilde{R}' e^{\tau/2} (\ln \tilde{R} + \tau/2 + 1), \end{aligned}$$

so that $\tilde{R}(\tau) = e^{-2\tau} \rho(\tau)$ and $\rho(\tau) \sim C \tau^{-4/3}$, hence

$$R(t) \sim \kappa \frac{(T-t)^2}{|\ln(T-t)|^{4/3}} \qquad \text{as } t \uparrow T,$$

for some $\kappa > 0$.

To calculate the limit profile we have to match into the remote solution $\theta(t, r) \sim \Theta(r) = \theta(T, r)$. To match (21) to the remote region one needs to take into account the highest order terms in τ and y only. For the limit profile one finds

$$\theta(T, r) \sim \pi - \tfrac{3}{8} \kappa \frac{r^3}{|2 \ln r|^{1/3}} \qquad \text{for small } r,$$

with the same constant $\kappa > 0$ as above. Notice that $\theta(T, r) < \pi$ for small r (in contrast to the non-degenerate case). The first two intersections of $\theta(t, r)$ with π, denoted by $S_1(t)$ and $S_2(t)$, behave asymptotically as

$$S_1 \sim \sqrt{\frac{2}{3} \frac{T-t}{|\ln(T-t)|}} \qquad \text{and} \qquad S_2 \sim \sqrt{8(T-t)}.$$

The first intersection comes from the balance between Ω_0 and Ω_1 (i.e. occurs for small y), while the second intersection depends on Ω_0 only (having $y = O(1)$).

In a similar vein, for degenerate cases occurring with co-dimension k ($k = 0, 1, 2, \dots$; the generic case is embedded in this), the $(k+1)$-th eigenvalue of $\mathcal{L}_0$ is $\lambda = -k - \frac{1}{2}$ with as eigenfunction a $(2k+1)$-th order polynomial with only odd terms, say $\Omega_0^k(y)$, which we normalise so that $\Omega_0^k(y) \sim$

y as $y \to 0$. We note that $\Omega_0^k(y)$ can be expressed in terms of a generalised Laguerre polynomial. Then

$$\Omega_0^k(y) = \sum_{i=0}^{k} a_i y^{2i+1} \qquad \text{with} \qquad a_i = (-1)^i \frac{k!}{2^{2i}(k-i)!(i+1)!i!} \quad \text{for } i = 0, 1, \ldots, k.$$

To calculate the next term in the expansion Ω_1^k we have to solve

$$(\mathcal{L}_0 + k + \tfrac{1}{2})\Omega_1^k = \Omega_0^k.$$

After a bit of calculation one finds that

$$\Omega_1^k(y) = \tfrac{4}{k+1} y^{-1} - 2\Omega_0^k(y) \ln y + h_k(y)$$

where h_k is some odd polynomial of degree $2k-1$. The outer approximation becomes

$$\zeta \sim \pi + e^{-(k+1/2)\tau}[\sigma(\tau)\Omega_0^k(y) + \sigma'(\tau)\Omega_1^k(y)],$$

and thus for small y

$$\zeta \sim \pi + e^{-(k+1/2)\tau}\left[\sigma(\tau)\, y + \sigma'(\tau)\, (\tfrac{4}{k+1} y^{-1} - 2y \ln y)\right].$$

The matching condition for the co-dimension k degeneracy become

$$\begin{aligned}
O(y^{-1}): &\quad \tfrac{4}{k+1} e^{-(k+1/2)\tau}\sigma' \sim -2\tilde{R}e^{\tau/2} \\
O(y \ln y): &\quad -2e^{-(k+1/2)\tau}\sigma' \sim -\tilde{R}' e^{\tau/2} \\
O(y): &\quad e^{-(k+1/2)\tau}\sigma \sim \tilde{R}' e^{\tau/2}(\ln \tilde{R} + \tau/2 + 1)
\end{aligned}$$

Hence $\tilde{R} \sim \kappa e^{-(k+1)\tau}\tau^{-(2k+2)/(2k+1)}$, or in the original variables

$$R(t) \sim \kappa \frac{(T-t)^{k+1}}{|\ln(T-t)|^{(2k+2)/(2k+1)}} \qquad \text{as } t \uparrow T,$$

for some $\kappa > 0$. Since $\Omega_0^k \sim a_k y^{2k+1}$ as $y \to \infty$, the limit profile is

$$\theta(T, r) \sim \pi + (-1)^k \kappa \frac{2k+1}{2^{2k+1}k!} \frac{r^{2k+1}}{|2 \ln r|^{1/(2k+1)}} \qquad \text{as } t \uparrow T.$$

This limit profile is again obtained by matching to the outer solution or alternatively by taking the limit $\tau \to \infty$ for fixed y and subsequently $y \to \infty$. One can analyse the short time behaviour just after blowup in a similar way to that in Section 2.4. Finally, the asymptotic behaviour of the first intersection of $\theta(t, r)$ with π, denoted by $S_1(t)$, is

$$S_1(t) \sim 2\sqrt{\frac{1}{(k+1)(2k+1)} \frac{T-t}{|\ln(T-t)|}} \qquad \text{as } t \uparrow T.$$

We have thus found a countable family of non-generic blowup scenarios. The co-dimension 1 situation for example occurs at the borderline between the generic blowup scenario and the case of no blowup (as explained in Section 1); it is characterised by the fact that *two* intersections of $\theta(t, r)$ with π approach the origin simultaneously as $t \uparrow T$ (though at different rates). More generally, in the co-dimension k scenario the profile $\theta(t, r)$ just before blowup has $k+1$ intersections with π which approach the origin as $t \uparrow T$; in other words it corresponds to a non-generic scenario in which the disappearance of sign changes in $\theta - \pi$ coincides with the blowup time (cf. [13] for a detailed discussion of sign change solutions in a different second order parabolic problem). The appearance of degenerate cases indicates that a proof of the formal result that near blowup $R(t) \sim \kappa \frac{T-t}{|\ln(T-t)|^2}$ holds *generically* might be hard to obtain. Restricting to certain classes of monotone initial data excludes the degenerate possibilities, because one can show that the solution then has to remain monotone for all $t > 0$ (see [1]) and all the degenerate blowup scenarios have non-monotone profiles just before blowup. Our analysis strongly suggests, however, that even without such restrictions on the initial data the generic blowup rate is $R(t) \sim \kappa \frac{T-t}{|\ln(T-t)|^2}$.

2.6 Boundary condition $\theta_1 = \pi$

When the boundary condition is $\theta(t,1) = \pi$ there is, beside the finite time blowup scenarios described above, an additional possibility, namely that blowup occurs in infinite time. This infinite time blowup is a co-dimension 0 scenario (we analyse degenerate cases as well). In this case there is no urge to change to self-similar coordinates. Close to blowup the profile is assumed to be near π in the whole of the remote region (and the limit profile is identically equal to π). The linearised equation around π is

$$w_t = \mathcal{L}_1 w \stackrel{\text{def}}{=} w_{rr} + \frac{1}{r} w_r - \frac{1}{r^2} w. \tag{22}$$

We substitute a formal series

$$w \sim \pi + \sum_{i=0}^{\infty} \frac{d^i s(t)}{dt^i} W_i(r), \tag{23}$$

where $s(t)$ is an arbitrary function and

$$\mathcal{L}_1 W_0 = 0 \qquad \text{and} \qquad \mathcal{L}_1 W_i = W_{i-1} \qquad i = 1,2,\ldots.$$

Now (23) is again a formal solution of the linearised differential equation for any $s(t)$, and when we require that $W_i(1) = 0$ then the right boundary condition is satisfied. There is no a priori left boundary condition. We obtain

$$\begin{aligned} W_0 &= C_0(r^{-1} - r) \\ W_1 &= C_0(\tfrac{1}{2} r \ln r + \tfrac{1}{8} r - \tfrac{1}{8} r^3) + C_1(r^{-1} - r), \end{aligned}$$

where we may again set $C_0 = 1$ and $C_1 = 0$ without loss of generality. Hence for the remote region we obtain for small r

$$\theta \sim \pi + s(t)(r^{-1} - r) + s'(t)(\tfrac{1}{2} r \ln r + \tfrac{1}{8} r)$$

Matching in a similar way to Section 2.4 to the inner solution (11) we find that

$$\begin{aligned} O(r^{-1}) : &\qquad s \sim -2R \\ O(r \ln r) : &\qquad \tfrac{1}{2} s' \sim -R' \\ O(r) : &\quad -s + \tfrac{1}{8} s' \sim R'(\ln R + 1). \end{aligned}$$

Hence for large t:

$$R(t) \sim e^{-2\sqrt{t} - 5/4} \qquad \text{as } t \to \infty.$$

We note that this is the one instance where the term $R' R \xi = R' r$ has an influence on the leading order result (it is needed to calculate the multiplicative constant $e^{-5/4}$). The most striking difference with the situation in Section 2.4 is that blowup now occurs in infinite time. We remark that, for suitable initial data, blowup may happen in finite time via the scenario in the previous sections, after which $\theta \to \pi$ uniformly as $t \to \infty$ (generically at rate $e^{-\lambda_1^2 t}$ where λ_1 is the first zero of the Bessel function J_1; see also below). However, this cannot happen for initial profiles $|\theta(0,r)| \le \pi$ for all $r \in [0,1]$, since then $|\theta(t,r)| \le \pi$ for all $t > 0$ (by the comparison principle) and blowup is postponed until $t = \infty$.

As in the previous section there is a hierarchy of degenerate cases. Looking for almost separable solutions one tries, with $\lambda > 0$,

$$w \sim \pi + e^{-\lambda^2 t} \sum_{i=0}^{\infty} \frac{d^i s(t)}{dt^i} W_i(r).$$

Now

$$W_0 = J_1(\lambda r) - \frac{J_1(\lambda)}{Y_1(\lambda)} Y_1(\lambda r),$$

where J_i and Y_i are the Bessel functions of order i (take $W_0 = Y_1(\lambda r)$ if $Y_1(\lambda) = 0$). Analogous to Section 2.4 this does not lead to self-consistent matching unless W_0 is regular at $r = 0$. Therefore we require that $J_1(\lambda) = 0$ and we obtain a nice eigenvalue problem with Dirichlet boundary conditions.

Let $\lambda_k > 0$ be the k-th zero of J_1 and $W_0(r) = J_1(\lambda_k r)$. Then one finds that

$$W_1 = \tfrac{\pi}{8}\left[J_1(\lambda_k r)J_2(\lambda_k r)Y_0(\lambda_k r)r^2 + J_1(\lambda_k r)J_0(\lambda_k r)Y_2(\lambda_k r)r^2 - 2J_0(\lambda_k r)J_2(\lambda_k r)Y_1(\lambda_k r)r^2 + D_k Y_1(\lambda_k r)\right],$$

where

$$D_k \stackrel{\text{def}}{=} 2J_0(\lambda_k)J_2(\lambda_k) < 0.$$

The matching conditions become

$$\begin{aligned}
O(r^{-1}): &\quad -e^{-\lambda_k^2 t}\frac{D_k}{4\lambda_k}s' \sim -2R \\
O(r\ln r): &\quad e^{-\lambda_k^2 t}\frac{D_k\lambda_k}{8}s' \sim -R' \\
O(r): &\quad e^{-\lambda_k^2 t}\tfrac{1}{2}\lambda_k s \sim R'(\ln R + 1).
\end{aligned}$$

We infer that, with co-dimension $k > 0$,

$$R(t) \sim \kappa\, t^{-1+4/D_k\lambda_k^2}\, e^{-\lambda_k^2 t} \qquad \text{as } t \to \infty,$$

for some $\kappa > 0$.

We stress that in the generic case (discussed at the beginning of this subsection) does not obey the boundary condition $W_0(0) = 0$ but nevertheless leads to consistent matching. This is however the only consistent case that has a spatial singularity near the origin in the remote region (the solution is of course not singular in the inner variable). This may be somewhat surprising. Let us clarify the role of the generic and non-generic scenarios.

Notice that for any $\lambda > 0$ the associated eigenfunction $W_0(r)$ has sign changes on $(0, 1)$, leading to intersections of the solution with π close to blowup. Consider initial profiles that lie entirely below π, i.e. $-\pi < \theta(0, r) < \pi$ for $r \in (0, 1)$. A comparison argument shows that for such initial data these blowup profiles are excluded. This indicates that a generic scenario is associated with $\lambda = 0$ (even though the "eigenfunction" (which obeys the boundary condition at $r = 1$) for $\lambda = 0$ is singular).

The degenerate cases act as borderline cases between finite time blowup and infinite time blowup. For example, when the initial data have a sufficiently large bump above π the solution will blowup in finite time, whereas solutions starting from initial data below π blow up in infinite time. The co-dimension 1 scenario found in this subsection acts as the borderline between generic infinite and generic finite time blowup. This is most easily understood for initial condition which have only one crossing with π since for such initial data the finite time co-dimension 1 scenario plays no role (because it has two crossings with π close to blowup). As we have seen in this section, the infinite time blowup phenomenon for $\theta_1 = \pi$ is essentially driven by the linear equation (22) and for such an equation the presence of high co-dimension cases with many sign changes (of $\theta - \pi$) is not unexpected.

2.7 Reverse jumps

As explained in Section 1 weak solutions have the possibility to make a jump in which they *increase* their energy $\mathcal{E}(t)$, see [5, 2]. We consider the situation where $\theta(t, 0)$ jumps from π to 0 ("jumping back") at $t = t_0$. In these jumps the energy $\mathcal{E}$ necessarily increases by 4π at $t = t_0$.

The inner approximation is the same as in Section 2.2 although now $R(t) \to 0$ as $t \downarrow t_0$. Concerning the outer approximation we argue as follows. As in Section 2.3 we turn to self-similar variables ($t > t_0$)

$$\tau = \ln(t - t_0), \qquad y = e^{-\tau/2} r,$$

and obtain for $\zeta(\tau, y) = \theta(t, r)$

$$\zeta_\tau = \zeta_{yy} + (\frac{1}{y} + \frac{y}{2})\zeta_y - \frac{\sin 2\zeta}{2y^2}$$

Since a reverse jump can occur at any moment one generically has $\theta_r(t_0, 0) = C \neq 0$, i.e. the dominant term in the local expansion is $Cr = Ce^{\tau/2} y$. This being a separable solution of the linearised equation one proposes a solution of the form $e^{\tau/2}\eta(y)$ (cf. Section 2.3) where η satisfies

$$\eta_{yy} + (\frac{1}{y} + \frac{y}{2})\eta_y - (\frac{1}{y^2} + \frac{1}{2})\eta = 0.$$

The general solution is

$$\eta = C_0 y + C_1 y \int_y^\infty \frac{e^{-s^2/4}}{s^3} ds,$$

with arbitrary constants C_0, $C_1 \in \mathbb{R}$ and $y_0 > 0$. We see that, as opposed to the situation in Section 2.3, the second independent solution has reasonable growth. In fact it tends to 0 as $y \to \infty$ and behaves as $y^{-1} + \frac{1}{2} y \ln y$ for small y. Therefore the outer approximation becomes

$$\zeta \sim \pi + e^{\tau/2}\big[\alpha(\tau) y + \beta(\tau)(4y^{-1} + 2y \ln y)\big] \qquad \text{for small } y \text{ and } -\tau \gg 1.$$

As matching conditions for $\tau \to -\infty$ we find (writing $\tilde{R}(\tau) = R(t)$)

$$\begin{aligned} O(y^{-1}): &\quad 4\beta e^{\tau/2} \sim -2\tilde{R} e^{-\tau/2} \\ O(y \ln y): &\quad 2\beta e^{\tau/2} \sim -\tilde{R}' e^{-\tau/2} \\ O(y): &\quad \alpha e^{\tau/2} \sim \tilde{R}' e^{-\tau/2}(\ln \tilde{R} - \tau/2 + 1). \end{aligned}$$

We infer that $\tilde{R} \sim e^\tau \rho(\tau)$ with $\rho(\tau) \sim -\frac{2\ell}{\tau}$ for some $\ell = -\alpha(-\infty) > 0$, and $\beta \sim \frac{\ell}{\tau}$, so that $|\beta(\tau)| \ll |\alpha(\tau)|$ as $\tau \to -\infty$. Hence

$$R(t) \sim 2\ell \frac{t - t_0}{|\ln(t - t_0)|} \qquad \text{as } t \downarrow t_0,$$

for some $\ell > 0$. The limit profile is

$$\theta(t_0, r) \sim \pi - \ell r \quad \text{for small } r,$$

which is consistent with the assumption at the start, so that $\ell = -C$. Notice that when the jump is downwards from π to 0 then the profile at $t = t_0$ has to be decreasing for small r.

A whole hierarchy of degenerate cases in which $\theta_r(0, 0) = 0$ can be calculated as well. At some $t_0 > 0$ (i.e. after the solution has started to evolve) it happens with co-dimension k that $\theta(t_0, r) \sim \pi - \tilde{C} r^{2k+1}$ as $r \downarrow 0$ for some $\tilde{C} \neq 0$. Reverse jumps at such non-generic instances can be analysed via the method presented above. The jump (of $\theta(t, 0)$) is downwards when $\tilde{C} > 0$ and upwards when $\tilde{C} < 0$. For example, for the co-dimension 1 scenario one finds

$$R \sim \tfrac{8}{3}\ell \frac{(t - t_0)^2}{|\ln(t - t_0)|} \qquad \text{as } t \downarrow t_0, \tag{24}$$

for some $\ell > 0$ with $\theta(t_0, r) \sim \pi - \ell r^3$ as $r \to 0$. Reverse jumps can also happen at the moment of a forward jump (i.e. $t_0 = T$), see also [18, Section 5]. Then at the self-similar scale a logarithmic correction needs to be applied (cf. the recovery of analyticity in Section 2.4). We note that if $t_0 = T$ and if the forward jump from 0 to π behaves according to the generic scenario then the reverse jump must be from π to 2π (this follows from a comparison of the limiting profiles as $t \uparrow T$ and

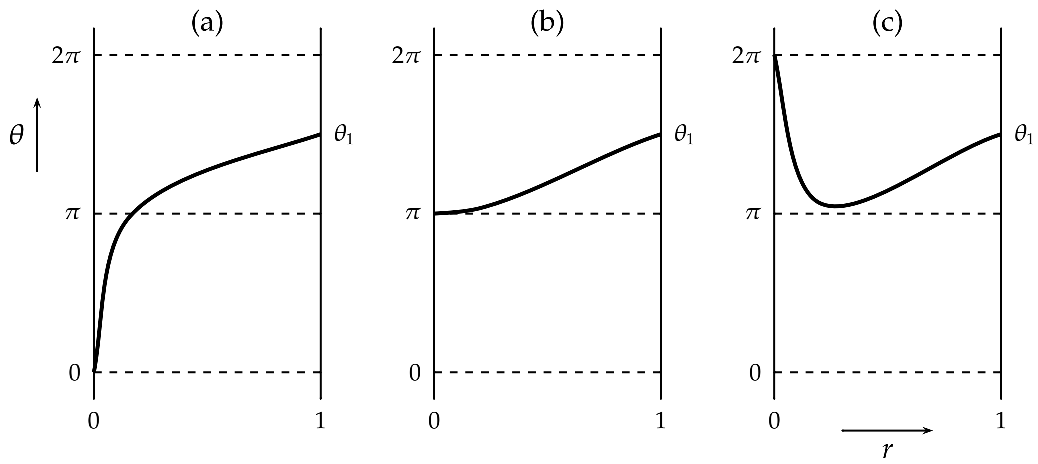

Figure 7: Picture of the situation when $t_0 = T$, i.e. a reverse jump happens at the moment of a forward jump: (a) a time just before $t = T$; (b) $t = T = t_0$; (c) a time just after $t = t_0$.

$t \downarrow t_0$). A schematic picture of the situation is given in Figure 7. We leave the details to the reader and just state the result:

$$R(t) \sim \kappa \frac{t - t_0}{|\ln(t - t_0)|^2} \qquad \text{as } t \downarrow t_0, \tag{25}$$

where $\kappa > 0$ is the same constant as in the forward jump (see Equation (18)). For $t_0 = 0$ one may choose, for example, initial data with $\theta(0, r) \sim \pi - \hat{C}r^a$ as $r \downarrow 0$ for any $a > 0$ and some $\hat{C} \neq 0$, and a reverse jump can then happen instantaneously. The analysis is again along the same lines.

Finally, our analysis suggests that given a jump time t_0 the asymptotic profile of $\theta(t_0, r)$ as $r \downarrow 0$ completely determines the blowup rate $R(t)$. In [2] it is conjectured that from a physical perspective it is most likely that if the solution has first made a forward jump from 0 to π the reverse jump happens at the first instance at which $\theta(t_0, r) < 0$ close to the origin. This corresponds to the system selecting an otherwise degenerate scenario, whereby $\theta(t_0, r) \sim \pi - \ell r^3$ as $r \to 0$ for some $\ell > 0$ and the blowup rate is then given by (24). More degenerate scenarios can of course occur with higher co-dimension. We note that our scenario is subtly different from the one studied in [2], which requires linear behaviour of θ at the origin.

3 The general case

3.1 Preamble

We now analyse the generalisation

$$\theta_t = \theta_{rr} + \frac{1}{r}\theta_r - n^2 \frac{\sin 2\theta}{2r^2}, \qquad 0 \leq r \leq 1. \tag{26}$$

Apart from the fact that the equation yields physically relevant solutions for $n = 1, 2, 3, \ldots$ as explained in Section 1, analysing the dependence of the blowup behaviour on n also enhances the understanding of the special case $n = 1$.

We deal with the inner approximation in Section 3.2 and then have a first attempt at matching without using self-similar coordinates to get the general idea in Section 3.3. In Section 3.4 we deal with $n < 2$, while Section 3.5 deals with $n \geq 2$ and we also remark on the case of an infinite domain. In Section 3.6 we discuss simultaneous blowup at several scales. The special boundary condition $\theta(t, 1) = \pi$ is dealt with in Section 3.7, and Section 3.9 is devoted to reverse jumps.

3.2 The inner approximation

The stationary solutions of (26) are

$$\theta(r) = m\pi + 2\arctan(qr^n), \qquad m \in \mathbb{Z},\ q \in \mathbb{R}.$$

Choosing the scaling $\xi = r/R(t)$ so that

$$R(t)^n\theta(t,r) \sim 2r^n \quad \text{as } r \downarrow 0 \qquad \text{for all } t \text{ up to the blowup time,}$$

the outer limit of the inner approximation $v(t,\xi) = \theta(t,r)$, cf. Section 2.2, becomes for $n > \frac{1}{2}$, $n \neq 1$

$$v \sim \pi - 2\xi^{-n} + R'R(\tfrac{n}{2n-2}\xi^{-n+2} - E_n\xi^n). \tag{27}$$

The coefficient of the term ξ^{-n+2} can be obtained from the asymptotic equation for Φ_1 at large ξ, as explained at the end of Section 2.2. The coefficient E_n of the term ξ^n can only be obtained solving the full problem for Φ_1 (borrowing the notation from Section 2.2):

$$\Phi_{1\xi\xi} + \frac{1}{\xi}\Phi_{1\xi} - n^2\frac{\cos(4\arctan\xi^n)}{\xi^2}\Phi_1 = -\frac{2n\xi^n}{1+\xi^{2n}},$$

with boundary condition $\lim_{\xi\downarrow 0}\frac{\Phi_1(\xi)}{\xi^n} = 0$. We rewrite $\cos(4\arctan\xi^n) = \frac{-6\xi^{2n}+1+\xi^{4n}}{(1+\xi^{2n})^2}$ and note that $\frac{d(2\arctan q\xi^n)}{dq}\big|_{q=1} = \frac{2n\xi^n}{1+\xi^{2n}}$ is a solution of the homogeneous equation. Using variation of constants one finds that the solution we are looking for is

$$\Phi_1 = \frac{(1-\xi^{4n} - 4n\xi^{2n}\ln\xi)\int_0^\xi \frac{s^{2n+1}}{(1+s^{2n})^2}ds + \xi^{2n}\int_0^\xi \frac{s(s^{4n}-1+4ns^{2n}\ln s)}{(1+s^{2n})^2}ds}{\xi^n(1+\xi^{2n})}.$$

This gives for $n > 1$:

$$E_n = \int_0^\infty \frac{s^{2n+1}}{(1+s^{2n})^2}ds = \frac{\pi}{2n^2\sin(\frac{\pi}{n})}.$$

For $n < 1$ the value of E_n is somewhat irrelevant since the term ξ^n in (27) is non-dominant in that case. Nevertheless, for $\frac{1}{2} < n < 1$:

$$E_n = \int_0^1 \frac{s^{2n+1} - 2s^{4n-3} - s^{6n-3}}{(1+s^{2n})^2}ds - 1.$$

For $n = \frac{1}{2}$ the outer limit of the inner approximation is:

$$v \sim \pi - 2\xi^{-1/2} + R'R(-\tfrac{1}{2}\xi^{3/2} + (-\tfrac{9}{2} + 2\ln\xi)\xi^{1/2}),$$

and for $n < \frac{1}{2}$ it becomes

$$v \sim \pi - 2\xi^{-n} + R'R(\tfrac{n}{2n-2}\xi^{2-n} + \tfrac{n(2n^2-n+1)}{4(1-n)^2(\frac{1}{2}-n)}\xi^{2-3n}).$$

3.3 A first try

To get a preliminary idea, let us first attempt to match without going to self-similar coordinates (this turns out to produce the correct generic rate for $n \neq 1$; non-generic cases have to be analysed in self-similar coordinates). Near $\theta = \pi$ the equation can be linearised to

$$w_t = \mathcal{L}_2 w \stackrel{\text{def}}{=} w_{rr} + \frac{1}{r}w_r - \frac{n^2}{r^2}w. \tag{28}$$

The stationary solution is $w = C_0 r^n + C_1 r^{-n}$ with C_0, $C_1 \in \mathbb{R}$ and this forms the inspiration for the formal solution

$$w = \sum_{i=0}^\infty \frac{d^i\alpha}{dt^i}\psi_i(r) + \sum_{i=0}^\infty \frac{d^i\beta}{dt^i}\chi_i(r),$$

for some functions $\alpha(t)$ and $\beta(t)$, and

$$\begin{aligned} \psi_0 &= r^n, & \mathcal{L}_2\psi_i &= \psi_{i-1} \quad i = 1,2,\ldots, \\ \chi_0 &= r^{-n}, & \mathcal{L}_2\chi_i &= \chi_{i-1} \quad i = 1,2,\ldots. \end{aligned}$$

The outer approximation becomes

$$\theta \sim \pi + \alpha(t)r^n + \beta(t)r^{-n} + \beta'(t)\tfrac{1}{4(1-n)}r^{-n+2}.$$

Matching yields (for $n > \frac{1}{2}$)

$$\begin{aligned} O(r^{-n}): &\quad \beta \sim -2R^n \\ O(r^{-n+2}): &\quad \beta'\tfrac{1}{4(1-n)} \sim \tfrac{n}{2n-2}R'R^{n-1} \\ O(r^n): &\quad \alpha \sim -E_nR'R^{1-n}, \end{aligned}$$

from which we conclude that

$$\begin{aligned} &R \sim \left[\tfrac{\beta_0}{2}(T-t)\right]^{1/n} && n < 1 && \text{as } t \uparrow T, \\ &R \sim \left[\tfrac{(2-n)\alpha_0}{E_n}(T-t)\right]^{1/(2-n)} && 1 < n < 2 && \text{as } t \uparrow T, \\ &R \sim \kappa e^{-\frac{\alpha_0}{E_2}t} && n = 2 && \text{as } t \to \infty, \\ &R \sim \left(\tfrac{(n-2)\alpha_0}{E_n}t\right)^{-1/(n-2)} && n > 2 && \text{as } t \to \infty, \end{aligned}$$

for some $\kappa > 0$ and $\alpha_0 = \alpha(T) > 0$ and $\beta_0 = \beta'(T) > 0$ (i.e. $\beta(t) \sim \beta_0(t-T)$). For $n < 1$ we have to consider the three cases ($n < \frac{1}{2}$, $n = \frac{1}{2}$, $n > \frac{1}{2}$) with different asymptotic behaviour for the inner solutions separately; they all lead to results of the same form.

Notice that when we perform the same analysis for $n = 1$ we obtain

$$\begin{aligned} O(r^{-1}): &\quad \beta \sim -2R \\ O(r\ln r): &\quad \frac{1}{2}\beta' \sim -R' \\ O(r): &\quad \alpha \sim R'(\ln R + 1). \end{aligned}$$

One deduces that $R \sim \alpha_0\frac{(T-t)}{|\ln(T-t)|}$, which is the wrong asymptotic behaviour (although it is almost right), see Section 2. The artefact is caused by the fact that one a priori assumes the limit profile to be $\theta(T,r) \sim \pi + Cr$ for some $C \neq 0$, whereas this cannot be fixed a priori but should be determined by matching with the region in which θ is near π; for this one needs to analyse what happens at an intermediate scale, the self-similar one. When one preforms the matching at the self-similar scale (with variable $y = r/(T-t)^{1/2}$), as will be done in Section 3.4, it turns out that in the generic (co-dimension 0) case the solution is y^n ($1 < n < 2$) or y^{-n} ($n < 1$), hence the self-similar scale seems to have no influence. However, this region is crucial since it introduces a selection mechanism (the requirement that the solution does not grow exponentially for large y). Therefore, the analysis of the self-similar region does not appear to influence the result for $n \neq 1$ (in the generic scenario), but for $n = 1$ it corrects the result from the above naive approach.

Finally, it is important to note that our analysis thus far suggests that blowup occurs in infinite time for $n \geq 2$ and in finite time for $n < 2$. This difference causes us to investigate these cases separately.

3.4 The case $n < 2$: finite time blowup

The analysis goes along the same lines as for $n = 1$. The inner approximation has been obtained in Section 3.2. Let us here pay some extra attention to the outer solution. One could formulate this analysis in the same terms as used in Section 2.3 for $n = 1$. The matching for $n \neq 1$ is easier because it turns out that only the dominant term needs to be taken into account. We can therefore use a slightly more straightforward approach.

We look for a self-similar solution to (28) of the form

$$w = (T-t)^{\gamma}\phi\Big(\frac{r}{\sqrt{T-t}}\Big) \tag{29}$$

where γ is not known a priori but has to be determined as part of the process. As the first boundary conditions we require that $\phi(y)$ does not grow exponentially for $y \to \infty$. The second boundary condition is different for $n > 1$ and $n < 1$. For $n > 1$ we require that $\phi(0) = 0$, while for $n < 1$ we require that $\phi = Cy^{-n} + o(y^n)$ for some $C \neq 0$. These boundary conditions are suggested by our preliminary results in Section 3.3: the terms of order y^n and y^{-n} are dominant for $n > 1$ and $n < 1$ respectively. Another, equivalent, point of view is that this boundary condition is in fact a matching condition, as explained in Section 2.4. For both $n > 1$ and $n < 1$ there is a sequence of self-similar solutions of the boundary value problem.

For $n > 1$ the first one is $\gamma = n/2$ and $\phi_0 = C_0 y^n$ with $C_0 \neq 0$; the second one is $\gamma = n/2 + 1$ and $\phi_1 = C_1 y^n(1 - \frac{1}{4n+4}y^2)$. In general, there is a family of solutions $\gamma = n/2 + k$ and $\phi_k = C_k y^n f_{k,n}(y^2)$ for $k = 0, 1, 2, \ldots$, where $f_{k,n}$ is a polynomial of degree k and $f_{k,n}(0) = 1$ (we note that $f_{k,n}$ can be expressed in terms of a generalised Laguerre polynomial). Since the inner approximation (27) thus needs to match into $C_0(T-t)^k r^n$ for some $C_0 \neq 0$, one obtains the matching condition

$$O(r^n): \quad C_0(T-t)^k \sim -E_n R'(t) R(t)^{1-n}.$$

We only use this one term since we have already seen in Section 3.3 that it is the dominant one. Hence with co-dimension $k = 0, 1, 2, \ldots$ (and $1 < n < 2$)

$$R(t) = \kappa(T-t)^{(k+1)/(2-n)} \qquad \text{as } t \uparrow T$$

for some $\kappa > 0$, with limit profile

$$\theta(T, r) \sim \pi + A_{k,n}\kappa^{2-n} r^{n+2k} \qquad \text{as } r \downarrow 0$$

for some constant $A_{k,n}$ which can be calculated from the coefficient of the highest order term in the polynomial $f_{k,n}$ (e.g. $A_{0,n} = \frac{E_n}{4(n+1)(2-n)}$).

For $n < 1$ the first possibly relevant self-similar solution of the form (29) is $(T-t)^{-n/2}y^{-n}$, but this is simply r^{-n} and does not tend to 0 as $t \to T$ and hence it is not suitable (it does not correspond to blowup at $t = T$). The next is $\gamma = -n/2 + 1$ and $\phi_0 = \tilde{C}_{-} y^{-n}(1 - \frac{1}{4(1-n)}y^2)$ with $\tilde{C}_0 \neq 0$. There is again a family of solutions $\gamma = -n/2 + k + 1$ and $\phi_k = \tilde{C}_k y^{-n} \tilde{f}_{k,n}(y^2)$, where $\tilde{f}_{k,n}$ is a polynomial of degree $k+1$ and $\tilde{f}_{k,n}(0) = 1$ (again $f_{k,n}$ can be expressed in terms of a generalised Laguerre polynomial). Because (27) thus needs to match into $\tilde{C}_0(T-t)^{k+1}r^{-n}$ for some $\tilde{C}_0 \neq 0$, one obtains the matching condition

$$O(r^{-n}): \quad \tilde{C}_0(T-t)^{k+1} \sim -2R^n,$$

hence with co-dimension $k = 0, 1, 2, \ldots$ (and $0 < n < 1$)

$$R(t) \sim \kappa(T-t)^{(k+1)/n} \qquad \text{as } t \uparrow T$$

for some $\kappa > 0$, and the limit profile is

$$\theta(T, r) \sim \pi + A_{k,n}\kappa^n r^{2-n+2k} \qquad \text{as } r \downarrow 0,$$

for some constant $A_{k,n}$ which can be calculated from the coefficient of the highest order term in the polynomial $\tilde{f}_{k,n}$ (e.g. $A_{0,n} = \frac{1}{2(1-n)}$).

Immediately after blowup, an inner layer near $r = 0$ appears. For $n < 1$ the leading order behaviour in this layer is simpler than for $n = 1$, being exactly self-similar. In the generic case $(k = 0)$ one finds

$$\theta(t,r) \sim \pi + d_n \kappa^n (t-T)^{(2-n)/2} g_n\big(r/\sqrt{t-T}\big) \qquad \text{as } t \downarrow T \text{ for small } r,$$

for some $d_n > 0$ to be determined. Here

$$g_n(z) = \big(8n(1-n)z^{2-n} + 32n(1-n)^2 z^{-n}\big) \int_0^z \frac{s^{2n-1}e^{-s^2/4}}{(4(1-n)+s^2)^2}\, ds$$

is the solution of the linearised equation in self-similar coordinates with the property that $g_n(z) \sim z^n$ as $z \to 0$. It follows that $g_n(z) \sim 4^{n-1}\Gamma(n+1)z^{2-n}$ as $z \to \infty$. Matching with the limit profile at $t = T$ yields $d_n = \big[2^{2n-1}(1-n)\Gamma(n+1)\big]^{-1}$. Notice that for $1 < n < 2$ special treatment of the short time behaviour after blowup is not necessary (one would just find that $\theta(t,r) \sim \pi + (t-T)^{n/2} g_n\big(r/\sqrt{t-T}\big)$ with $g_n(z) = \frac{E_n}{4(n+1)(2-n)} z^n$).

3.5 The case $n \geq 2$: infinite time blowup

For $n \geq 2$ blowup occurs in infinite time and there are only two scales: an inner and a remote region. In the remote region the limit profile is now known. Namely, for boundary condition $\theta(t,1) = \theta_1 \in (\pi, 2\pi)$ the limit profile is

$$\lim_{t\to\infty} \theta(t,r) \sim \pi + 2\arctan\big(\tan(\tfrac{\theta_1-\pi}{2}) r^n\big). \tag{30}$$

The special case $\theta_1 = \pi$ is dealt with in Section 3.7, and $\theta_1 > 2\pi$ is discussed in Section 3.6. For small r the limit profile (30) behaves as $\pi + 2\tan(\frac{\theta_1-\pi}{2})r^n$. For $\theta_1 \in (\pi, 2\pi)$ we define $\alpha_0 \stackrel{\text{def}}{=} 2\tan(\frac{\theta_1-\pi}{2})$.

Now the inner approximation (27) has to match into $\alpha_0 r$, hence

$$O(r^n): \quad \alpha_0 \sim -E_n R'(t) R(t)^{1-n}.$$

One obtains

$$R \sim \begin{cases} \kappa e^{-\frac{\alpha_0}{E_2}t} & n = 2, \\ \big(\frac{(n-2)\alpha_0}{E_n}t\big)^{-1/(n-2)} & n > 2, \end{cases} \qquad \text{as } t \to \infty \tag{31}$$

for some $\kappa > 0$. Because the limit profile is known a priori (as opposed to $n < 2$), the unknown constant only appears to leading order when $n = 2$, where it is required due to translation invariance in time. Furthermore, we note that there are no degenerate cases; since blowup occurs as $t \to \infty$ it can be determined a priori which of the stationary states will be the final profile. This depends only on the value of θ_1 and $\theta(0,0)$, and since $\theta(0,0) \in \pi\mathbb{Z}$ is not a continuous parameter degenerate (borderline) cases are not needed.

3.6 Multiple blowup

In certain situations it may happen that blowup occurs at several scales simultaneously (a so called "bubble tree", see also [17]), for example when $\theta_1 \geq 2\pi$. For $n < 2$, i.e. finite time blowup, double (or multiple) blowup does not need to happen since the necessary jumps can occur at different instances, and simultaneous blowup is indeed not possible, at least for $n = 1$, see [12] (and our analysis reveals no possible finite time bubble tree). On the other hand, for $n \geq 2$ multi-scale blowup necessarily happens if $\theta_1 \geq 2\pi$ since all blowup occurs as $t \to \infty$.

Let us take $\theta_1 \in (2\pi, 3\pi)$ as an example. The limit profile is now

$$\lim_{t\to\infty} \theta(t,r) = 2\pi + 2\arctan\big(\tan(\tfrac{\theta_1}{2})r^n\big).$$

Define $\alpha_1 = 2\tan(\frac{\theta_1}{2})$ for $\theta_1 \in (2\pi, 3\pi)$. The two blowup rates are $R_1(t)$ and $R_2(t)$ for the jumps from 0 to π and from π to 2π respectively, with $R_1 \ll R_2 \ll 1$. The first blowup rate $R_1(t)$ is defined as before, namely so that $R_1(t)^n\theta(t,r) \sim 2r^n$ as $r \downarrow 0$ for all $t > 0$. The second blowup rate $R_2(t)$ cannot be defined in the same way, and instead we put

$$\theta(t, R_2(t)) = \tfrac{3\pi}{2} \qquad \text{for all } t \text{ close to blowup,}$$

so that in the limit we have for all $\rho > 0$:

$$\lim_{t\to\infty} \theta(t, \rho R_2(t)) = \pi + 2\arctan\rho^n \qquad \text{and} \qquad \lim_{t\to\infty} \theta(t, \rho R_1(t)) = 2\arctan\rho^n.$$

The inner-inner region, $r = O(R_1(t))$, is as analysed in Section 3.2. The analysis of the inner region, $r = O(R_2(t))$, is the same except for the boundary condition on the left. Let $x = r/R_2(t)$, then $v(t,x) = u(t,r)$ behaves for large t as

$$v \sim \pi + 2\arctan x^n + R_2' R_2 \Psi$$

where Ψ obeys

$$\Psi_{xx} + \frac{1}{x}\Psi_x - n^2\frac{\cos(4\arctan x^n)}{x^2}\Psi = -\frac{2nx^n}{1+x^{2n}},$$

with boundary condition $\Psi(1) = 0$. The solution is

$$\Psi = \frac{(1 - x^{4n} - 4nx^{2n}\ln x)(A + \int_1^x \frac{s^{2n+1}}{(1+s^{2n})^2}ds) + x^{2n}\int_1^x \frac{s(s^{4n}-1+4ns^{2n}\ln s)}{(1+s^{2n})^2}ds}{x^n(1+x^{2n})}$$

with arbitrary $A \in \mathbb{R}$. For convenience we define

$$B_n = \int_0^1 \tfrac{s^{2n+1}}{(1+s^{2n})^2}ds.$$

One infers that

$$\Psi \sim (A - B_n)x^{-n} \qquad \text{for small } x,$$

$$\Psi \sim \Big(-\int_1^\infty \tfrac{s^{2n+1}}{(1+s^{2n})^2}ds - A\Big)x^n + \tfrac{n}{2n-2}x^{-n+2} \qquad \text{for large } x.$$

Matching with the remote solution gives (recalling that $E_n = \int_0^\infty \frac{s^{2n+1}}{(1+s^{2n})^2}ds$)

$$O(r^n): \quad \alpha_1 \sim (B_n - E_n - A)R_2' R_2^{1-n}$$

hence

$$R_2 \sim \begin{cases} c_0 e^{-\frac{\alpha_1}{E_2 - B_2 + A}t} & n = 2 \\ \big(\frac{(n-2)\alpha_1}{E_n - B_n + A}t\big)^{-1/(n-2)} & n > 2 \end{cases} \tag{32}$$

for an arbitrary constant $c_0 > 0$.

Matching the inner-inner with the inner gives

$$O(r^{-n}): \qquad -2R_1^n \sim (A - B_n)R_2' R_2^{1+n}, \tag{33}$$

$$O(r^n): \qquad -E_n R_1' R_1^{1-n} \sim 2R_2^{-n}. \tag{34}$$

From (32) and (34) we deduce that

$$R_1 \sim \begin{cases} c_2 e^{-c_1 e^{\frac{2\alpha_1}{E_2 - B_2 + A}t}} & n = 2 \\ \big(\frac{(n-2)(E_n - B_n + A)}{E_n\alpha_1(n-1)}\big)^{-1/(n-2)}\big(\frac{(n-2)\alpha_1}{E_n - B_n + A}t\big)^{-(2n-2)/(n-2)^2} & n > 2 \end{cases}$$

for arbitrary constants c_1, $c_2 > 0$ ($c_1 = \frac{2}{E_2}c_0$). From (33) we then conclude that $A = B_n$, and hence

$$R_1 \sim \begin{cases} c_2 e^{-c_1 e^{\frac{2\alpha_1}{E_2}t}} & n = 2 \\ \big(\frac{n-2}{\alpha_1(n-1)}\big)^{-1/(n-2)}\big(\frac{(n-2)\alpha_1}{E_n}t\big)^{-(2n-2)/(n-2)^2} & n > 2 \end{cases}$$

for arbitrary constants c_1, $c_2 > 0$ (which only appear for $n = 2$), and $\alpha_1 = 2\tan(\frac{\theta_1}{2})$, $E_n = \frac{\pi}{2n^2\sin(\frac{\pi}{n})}$. Notice the doubly-exponential decay for $n = 2$; there are two unknown constants c_1 and c_2 in the leading order asymptotic expression because at both blowup scales there is a scaling invariance. While the first one could be attributed to translation invariance in time, the criticality of the case $n = 2$ is apparent in the appearance of a second unknown constant; for $n > 2$ there are no free constants in the leading order expression. One could generalise this to triple- and higher multi-jumps, but we leave this to the puzzle-minded reader.

When one tries to perform the above analysis for $n < 2$ one readily encounters matching conditions which cannot be fulfilled. We therefore conjecture that there exist no bubble trees for $n < 2$; in particular, there are no finite time bubble trees.

3.7 Boundary condition $\theta_1 = \pi$

For the special boundary condition $\theta(t, 1) = \pi$ we follow the argument of Section 2.6. The remote solution w behaves as

$$w \sim \pi + s(t)(r^n - r^{-n}) + s'(t)\tfrac{1}{4}\left(\tfrac{1}{n-1}r^{2-n} + \tfrac{1}{n+1}r^{2+n} - \tfrac{2n}{n^2-1}r^n\right).$$

Matching with the inner solution we find

$$\begin{aligned} O(r^{-n}): &\qquad -s \sim -2R^n \\ O(r^{-n+2}): &\quad \tfrac{1}{4(n-1)}s' \sim \tfrac{n}{2(n-1)}R'R^{n-1} \\ O(r^n): &\qquad s \sim -E_n R' R^{1-n}. \end{aligned}$$

We obtain

$$R(t) \sim \left(\tfrac{4(n-1)}{E_n}t\right)^{-1/(2n-2)} \qquad \text{as } t \to \infty \qquad \text{for } n > 1.$$

For $n < 1$ matching suggests $R(t) \sim \left(\frac{4(1-n)}{E_n}(T-t)\right)^{1/(2-2n)}$. However, this does not provide consistent matching since it implies that $s' \gg s$. This suggests that we need a left boundary condition on the remote region of the form $w \sim Cr^{-n} + o(r^n)$ as $r \to 0$ for some $C \neq 0$. Hence we need to consider solutions of the form (compare to the degenerate case in Section 2.6)

$$w \sim \pi + C_0 e^{-\nu_n^2 t} r^{-n}$$

for some $C_0 \neq 0$. Here ν_n is the first zero of the n-th order singular Bessel function $\tilde{Y}_n$, which (for the occasion) is defined with the choice that $\tilde{Y}_n(r) \sim \tilde{C}r^{-n} + o(r^n)$, with $\tilde{C} \neq 0$ arbitrary. We remark that $\nu_n \to 0$ as $n \uparrow 1$. Matching now yields

$$O(r^{-n}): \quad C_0 e^{-\nu_n^2 t} \sim -2R^n,$$

hence

$$R(t) \sim \kappa e^{-\frac{\nu_n^2}{n}t} \qquad \text{for } n < 1,$$

for some $\kappa > 0$. To summarise, one finds that for $\theta_1 = \pi$ generically

$$R(t) \sim \begin{cases} \kappa e^{-\frac{\nu_n^2}{n}t} & n < 1 \\ e^{-2\sqrt{t}-5/4} & n = 1 \\ \left(\frac{4(n-1)}{E_n}t\right)^{-1/(2n-2)} & n > 1, \end{cases}$$

provided no finite time blowup occurs (for $n < 2$), for example when one takes initial data which lie entirely between 0 and π.

For $n < 2$ there is again a family of degenerate cases (cf. Section 2.6); the blowup is at an exponential rate determined by the zeros of the Bessel functions $\tilde{Y}_n$ for $n < 1$ and J_n for $1 < n < 2$.

For $n \geq 2$ degenerate scenarios do not exist. We leave the details to the diligent reader. For $n \geq 2$ the cases $\theta_1 = m\pi$, $m = 2, 3, \ldots$ involve multiple blowup and the technique from the previous section may be used.

3.8 The infinite domain

We will now discuss how the results obtained so far have to be adapted when we consider an infinite domain, i.e. $r \in (0, \infty)$, instead of a finite one. In order to have a solution with finite energy $\mathcal{E}(t) = \pi \int_0^\infty \left(r\theta_r^2 + n^2 \frac{\sin^2\theta}{r} \right) dr$ the profile has to approach a multiple of π at a reasonably fast rate as $r \to \infty$; we denote this "boundary" condition by $\lim_{r\to\infty} \theta(t,r) = \tilde{\theta}_1 \in \pi\mathbb{Z}$. We focus on initial data which have compact support in the sense that $\theta(0,r) = m\pi$ for all sufficiently large r or, more generally, data which decay to $m\pi$ exponentially (thus in particular excluding algebraic decay).

Let us first discuss the case $\tilde{\theta}_1 = \pi$. There are several possibilities depending on the initial data. For $n < 2$ a generic possibility is finite time blowup (see in Section 3.4). A priori, another possibility is that there is no blowup and that for large time the solution converges to one of the equilibria $\theta(r) = 2\arctan qr^n$ for some $q > 0$. For $n \geq 2$ this is in fact the only feasible scenario, and no blowup turns out to be a generic scenario for $1 < n < 2$; on the other hand, for $n \leq 1$ blowup always occurs (as is explained below). Regarding non-generic possibilities, consider for example the parameter range $1 < n < 2$ and initial data which have one crossing with π (so that the finite time co-dimension 1 blowup scenario is not possible). We deduce that there should be at least one non-generic infinite time blowup scenario which acts as the borderline between the two generic possibilities.

The large time behaviour away from the origin is described in terms of self-similar variables $\tau = \ln t$ and $y = r/\sqrt{t} = e^{-\tau/2} r$, which leads to the linearised equation

$$\eta_\tau = \eta_{yy} + \left(\frac{1}{y} + \frac{y}{2}\right)\eta_y - \frac{n^2}{y^2}\,\eta.$$

We now analyse the co-dimension 0 and 1 scenarios for various ranges of n (higher co-dimension cases can be analysed in a similar manner).

For $n > 1$ the generic behaviour is described by the solution

$$\eta = C_0 e^{-n\tau/2}\, y^{-n} \int_y^\infty s^{2n-1} e^{-s^2/4}\, ds \tag{35}$$

for some $C_0 \neq 0$; it decays faster than exponentially as $y \to \infty$, and as $y \to 0$ it asymptotically satisfies $\eta \sim C_0 2^{2n-1}\Gamma(n) e^{-n\tau/2} y^{-n} = C_0 2^{2n-1}\Gamma(n) r^{-n}$. Since matching into the inner solution (27) leads to $R(t) \to \kappa$ as $t \to \infty$ for some $\kappa > 0$ with limit profile $\theta_\infty(r) = 2\arctan(r/\kappa)^n$, the generic scenario corresponds to no blowup. To be more precise, for $n > 1$ the matching conditions are

$$\begin{aligned} O(r^{-n}): &\qquad C_0 2^{2n-1}\Gamma(n) \sim -2R^n, \\ O(r^n): &\qquad -\tfrac{1}{2n} C_0 t^{-n} \sim -E_n R' R^{1-n}. \end{aligned}$$

Hence $R(t) \sim \kappa + Q_{q,n} t^{1-n}$ as $t \to \infty$ where $Q_{q,n} = \left[2^{2n-1}\kappa^{1-2n}(n-1)\Gamma(n+1)E_n\right]^{-1} > 0$.

In between the generic possibilities of no blowup and finite time blowup there is a degenerate infinite time blowup scenario. For this co-dimension 1 case we find the outer approximation (writing $\zeta(\tau, y) = \theta(t, r)$)

$$\zeta(\tau, y) \sim \pi + C_1 e^{-(2+n)\tau/2}\, y^n\, e^{-y^2/4} \qquad \text{as } \tau \to \infty,$$

for some $C_1 \neq 0$. Matching with the inner solution we infer that for $1 < n < 2$ the co-dimension 1 blowup rate is

$$R(t) \sim \kappa t^{-n/(2-n)} \qquad \text{as } t \to \infty \qquad \text{for } 1 < n < 2 \tag{36}$$

for some $\kappa > 0$. Away from the origin the decay towards π is at algebraic rate $O(t^{-1-n})$. For $n \geq 2$ matching in the non-generic case is impossible (looking at (36) one could anticipate this), implying that for $\tilde{\theta}_1 = \pi$ there is never blowup when $n \geq 2$.

The case $n = 1$ is again a borderline one; we find that (35) again describes the generic behaviour in the outer region, but we need C_0 to depend on τ in a non-exponential manner. Matching with the inner solution (11) gives matching conditions (with $\tilde{R}(\tau) = R(t)$)

$$\begin{aligned} O(y^{-1}): &\qquad 2C_0(\tau)e^{-\tau/2} \sim -2e^{-\tau/2}\tilde{R}, \\ O(y \ln y): &\qquad C_0'(\tau)e^{-\tau/2} \sim -e^{-\tau/2}\tilde{R}', \\ O(y): &\qquad -\tfrac{1}{2}C_0(\tau)e^{-\tau/2} \sim e^{-\tau/2}\tilde{R}'(\ln \tilde{R} - \tau/2 + 1). \end{aligned}$$

This generic scenario thus describes blowup at rate

$$R(t) \sim \frac{\kappa}{\ln t} \qquad \text{as } t \to \infty \qquad \text{for } n = 1,$$

for some $\kappa > 0$ (since the problem on the infinite domain is scaling invariant a multiplicative constant, whose value depends on the initial data, must again be present). For $r = O(1)$ the solution approaches π at rate $O(1/\ln t)$ as $t \to \infty$. The outer approximation in the co-dimension 1 case is

$$\zeta(\tau, y) \sim \pi + e^{-3\tau/2}e^{-y^2/4}\big[\sigma(\tau)y + \sigma'(\tau)(4y^{-1} - 2y \ln y)\big] \qquad \text{as } \tau \to \infty,$$

for some $\sigma(\tau)$, and matching gives the blowup rate $R \sim \kappa t^{-1}(\ln t)^{-4/3}$ as $t \to \infty$ (i.e. a logarithmically corrected version of (36) with $n = 1$) where $\kappa > 0$ is arbitrary. Away from the origin the rate of decay towards π is of order $O(t^{-1}(\ln t)^{-4/3})$.

For $n < 1$ the outer approximation described by (35) does not lead to consistent matching; we have seen previously that for $n < 1$ the outer solution should behave as $\tilde{C}y^{-n} + o(y^n)$ for small y with $\tilde{C} \neq 0$, which is not satisfied by (35). Therefore, for $n < 1$ the outer approximation in the generic case is

$$\zeta(\tau, y) \sim \pi + C_2 e^{-(2-n)\tau/2}\, y^{-n}\, e^{-y^2/4} \qquad \text{as } \tau \to \infty,$$

for some $C_2 \neq 0$. Matching with the inner solution we find that generically

$$R(t) \sim \kappa t^{-(1-n)/n} \qquad \text{as } t \to \infty \qquad \text{for } n < 1,$$

for some $\kappa > 0$. Away from the origin the decay towards π as $t \to \infty$ is at algebraic rate $O(t^{-1})$. For the co-dimension 1 scenario we find $R \sim \kappa t^{-(2-n)/n}$ as $t \to \infty$ for some $\kappa > 0$.

We note that, although the infinite domain allows scaling invariance, spreading of the form $\theta(t, r) = \Theta(r/\sqrt{t})$ is seen to be impossible by an argument analogous to that given at the end of Section 1. On the other hand, taking initial data with $\theta(t_0, r) \sim \hat{C}r^{-a}$ as $r \to \infty$ for some $\hat{C} \neq 0$ where $0 < a < n$, spreading at a rate slower than the self-similar one will occur as $t \to \infty$. The matching conditions in that case imply that the term of order r^{-n} is dominant for all $n > 0$, which yields $R(t) \sim \kappa t^{(n-a)/2n}$ as $t \to \infty$ for some $\kappa > 0$ and any $0 < a < n$; notice that $\frac{n-a}{2n} < \frac{1}{2}$ so that the self-consistency condition $R'R \to 0$ as $t \to \infty$ holds. For $a = n$ (i.e. $\theta(t_0, r) \sim \hat{C}r^{-n}$ as $r \to \infty$, which is the same rate as a stationary solution) no blowup occurs, while for $a > n$ there is either blowup (for $n \leq 1$) or no blowup (for $n > 1$), but we will not pursue the issue of algebraically decaying initial data any further.

The analysis is similar for $\lim_{r\to\infty} \theta(t, r) = \tilde{\theta}_1 = m\pi$, $m = 2, 3, \ldots$. For $n < 2$ a (finite) number of finite time jumps essentially reduces the situation to the case $\tilde{\theta}_1 = \pi$. For $n \geq 2$ and boundary value $\tilde{\theta}_1 = 2\pi$ blowup will happen as $t \to \infty$ and one of the stationary states $\theta_\infty(r) = \pi + 2\arctan qr^n$ for some $q > 0$ is selected. The analysis is completely analogous to the finite domain case $\theta_1 \in (\pi, 2\pi)$ discussed in Section 3.5 (notice that it thus differs from the finite domain with $\theta_1 = 2\pi$). The result is the same as in Section 3.5 except that the constant $q = \alpha_0$ cannot be determined a priori in the case of an infinite domain (and it should not be since the equation has a scaling invariance). Hence, the asymptotic blowup rate is given by (31), the only alteration being that in the case of an infinite domain $\alpha_0 > 0$ is an arbitrary constant whose value depends on the initial data. For $\tilde{\theta}_1 = m\pi$ with $m = 3, 4, \dots$ (and $n \geq 2$) multi-scale blowup occurs (see Section 3.6) and the analysis of the infinite domain is analogous to that of the finite domain with $\theta_1 \in ((m-1)\pi, m\pi)$.

3.9 Jumping back

Finally, we analyse the possibility of reverse jumps for (26). Analogous to Section 2.7 the profile at any time generically behaves as Cr^n for small r with $C \neq 0$. Hence in the outer region, in self-similar coordinates $\tau = \ln(t - t_0)$ and $y = e^{-\tau/2}r$, we look for a solution of the form $e^{n\tau/2}\psi(y)$. The linear equation for ψ has as solution

$$\psi = C_0 y^n + C_1 y^n \int_y^\infty e^{-s^2/4} s^{-2n-1} ds,$$

with arbitrary constants C_0, $C_1 \in \mathbb{R}$ and $y_0 > 0$. The inner limit of the outer approximation thus becomes

$$\zeta \sim \pi + e^{n\tau/2}\big[\alpha(\tau)y^n + \beta(\tau)(\tfrac{1}{2n}y^{-n} - \tfrac{1}{8(n-1)}y^{2-n})\big] \qquad \text{for small } y \text{ and } -\tau \gg 1.$$

Matching with the inner solution gives (writing $\tilde{R}(\tau) = R(t)$)

$$\begin{aligned} O(y^{-n}): &\qquad \tfrac{1}{2n}\beta e^{n\tau/2} \sim -2\tilde{R}^n e^{-n\tau/2}. \\ O(y^{2-n}): &\quad -\tfrac{1}{8(n-1)}\beta e^{n\tau/2} \sim \tfrac{n}{2(n-1)}\tilde{R}'\tilde{R}^{n-1}e^{-n\tau/2} \\ O(y^n): &\qquad \alpha e^{n\tau/2} \sim -E_n\tilde{R}'\tilde{R}^{1-n}e^{(-1+n/2)\tau} \end{aligned}$$

One concludes that for $n > 1$ the term of order y^n is dominant, hence as $t \downarrow t_0$

$$R \sim \big[\tfrac{\ell(2-n)}{E_n}(t - t_0)\big]^{1/(2-n)} \qquad \text{for } 1 < n < 2,$$

for some $\ell > 0$, the limit profile being $\theta(t_0, r) \sim \pi - \ell r^n$. Considerations about non-generic cases are analogous to those in Section 2.7; for the co-dimension 1 case one finds that

$$R \sim \big[\tfrac{2\ell(n+1)(2-n)}{E_n}\big]^{1/(2-n)}(t - t_0)^{2/(2-n)} \qquad \text{as } t \downarrow t_0,$$

for some $\ell > 0$ with $\theta(t_0, r) \sim \pi - \ell r^{2+n}$ as $r \to 0$. For $n \geq 2$ no consistent matching is found implying the rather strong result that reverse jumps do not seem possible. Notice that these conclusions are in line with what one could expect from Section 3.3.

For $n < 1$ the term of order y^{-n} is dominant, hence the matching conditions give $R \sim \omega_n(t - t_0)$ as $t \downarrow t_0$ for some $\omega_n > 0$ which can be determined using the limit profile $\theta(t_0, r) \sim \pi - \ell r^n$ as $r \to 0$. Since the inner limit of the outer approximation behaves as $(\zeta - \pi)e^{-n\tau/2} \sim -4n\omega_n^n y^{-n} + o(y^n)$ as $y \to 0$, the outer limit of the outer approximation becomes

$$\zeta \sim \pi - 4n\omega_n^n e^{n\tau/2} y^n \Big[-2^{-2n-1}\Gamma(-n) + \int_y^\infty e^{-s^2/4}s^{-2n-1}ds\Big].$$

Hence as $t \downarrow t_0$

$$R \sim \big(\tfrac{\ell 2^{2n-1}}{\Gamma(1-n)}\big)^{1/n}(t - t_0) \qquad \text{for } n < 1,$$

for some ℓ with $\theta(t_0, r) \sim \pi - \ell r^n$ as $r \downarrow 0$. Non-generic cases can also be considered, for example for the co-dimension 1 scenario one finds

$$R \sim \big(\tfrac{\ell 2^{2n+1}}{\Gamma(1-n)}\big)^{1/n}(t - t_0)^{(n+1)/n} \qquad \text{as } t \downarrow t_0,$$

for some ℓ with $\theta(t_0, r) \sim \pi - \ell r^{n+2}$ as $r \downarrow 0$.

Finally, we do not find any self-consistent scenarios for reverse jumps which increase the energy by more than 4π, i.e. no reverse bubble trees for any n (this is analogous to the conclusion in Section 3.6 that there are no normal (energy-decreasing) bubble trees for $n < 2$; for $n \geq 2$ there are normal bubble trees but they are of the infinite time blowup type).

4 Conclusion

We have analysed the blowup rate in the harmonic map heat flow in a family of symmetric settings, leading to a parabolic problem in one space dimension

$$\theta_t = \theta_{rr} + \frac{1}{r}\theta_r - n^2 \frac{\sin 2\theta}{2r^2}, \qquad 0 < r < 1, \tag{37}$$

with boundary conditions $\theta(t, 0) \in \pi\mathbb{Z}$ and $\theta(1, t) = \theta_1$. Here $n > 0$ is a parameter and it corresponds to a well-defined physical situation (e.g. in the context of aligned nematic liquid crystals) when $n = 1, 2, 3, \ldots$. The initial value problem has a unique energy-decreasing solution, the energy being $\mathcal{E}(t) = \pi \int_0^1 (\theta_r^2 + r^{-2} n^2 \sin^2 \theta) r dr$. Equation (37) is the gradient flow associated with this energy.

Without loss of generality we may assume that $\theta(0, 0) = 0$. During the evolution of a solution the value of θ at the origin may jump at some time(s) $t = T \in (0, \infty]$. As the time approaches T the quantity $\lim_{r\downarrow 0} r^{-n}\theta(t, r)$ blows up. In this paper we have determined the asymptotic behaviour of the blowup rate $R(t)$, defined by

$$R(t)^n \theta(t, r) \sim 2r^n \quad \text{as } r \downarrow 0 \qquad \text{for all } t \text{ up to the blowup time,}$$

using formal matched asymptotic expansions. After rescaling with this blowup rate the profile approaches a harmonic map (a stationary state):

$$\lim_{t\uparrow T} \theta(t, \xi R(t)) = 2 \arctan \xi^n \qquad \text{for all fixed } \xi > 0.$$

Since our results suggest important differences between $n < 2$ and $n \geq 2$ let us first summarise the behaviour for $n < 2$. The generic behaviour for $\theta_1 > \pi$ is

$$\begin{aligned}
n < 1: &\qquad R \sim \kappa (T - t)^{1/n} &&\qquad \text{as } t \uparrow T \\
n = 1: &\qquad R \sim \kappa \frac{T - t}{|\ln(T - t)|^2} &&\qquad \text{as } t \uparrow T \\
1 < n < 2: &\qquad R \sim \kappa (T - t)^{1/(2-n)} &&\qquad \text{as } t \uparrow T,
\end{aligned}$$

where $\kappa > 0$ is an arbitrary constant and $T < \infty$ is the time of blowup. There is also a countable family of degenerate cases. We find that with co-dimension k ($k = 0, 1, 2, \ldots$)

$$\begin{aligned}
n < 1: &\qquad R \sim \kappa (T - t)^{(k+1)/n} &&\qquad \text{as } t \uparrow T \\
n = 1: &\qquad R \sim \kappa \frac{(T - t)^{k+1}}{|\ln(T - t)|^{(2k+2)/(2k+1)}} &&\qquad \text{as } t \uparrow T \\
1 < n < 2: &\qquad R \sim \kappa (T - t)^{(k+1)/(2-n)} &&\qquad \text{as } t \uparrow T,
\end{aligned}$$

where $\kappa > 0$ is again an arbitrary constant. In the co-dimension k scenario the profile $\theta(t, r)$ just before blowup has $k + 1$ intersections with π which approach the origin as $t \uparrow T$. Countable families of non-generic blowup rates are encountered in a wide variety of problems, see [10] for an illustrative example.

For boundary data $\theta_1 = \pi$ blowup can occur (in finite time) via the scenario described above, but there is another generic blowup behaviour (in infinite time) of the form

$$\begin{aligned}
n < 1: &\qquad R \sim \kappa e^{-\frac{\nu_n^2}{n} t} && \text{as } t \to \infty \\
n = 1: &\qquad R \sim e^{-2\sqrt{t} - 5/4} && \text{as } t \to \infty \\
1 < n < 2: &\qquad R \sim \big(\tfrac{4(n-1)}{E_n} t\big)^{-1/(2n-2)} && \text{as } t \to \infty,
\end{aligned}$$

with arbitrary constant $\kappa > 0$. Here $E_n = \frac{\pi}{2n^2 \sin(\frac{\pi}{n})}$ and ν_n is the first zero of the Bessel function $\tilde{Y}_n$ (see Section 3.7 for details). There is also a family of non-generic infinite time blowup possibilities.

For $n \geq 2$ all blowup occurs as $t \to \infty$ and there is always a unique blowup scenario. For $\theta_1 \in (0, \pi)$ blowup never occurs (whereas for $n < 2$ this depends on the initial profile). For $\theta_1 \in (\pi, 2\pi)$ one finds

$$\begin{aligned}
n = 2: &\qquad R \sim \kappa e^{-\frac{\alpha_0}{E_2} t} && \text{as } t \to \infty \\
n > 2: &\qquad R \sim \big(\tfrac{(n-2)\alpha_0}{E_n} t\big)^{-1/(n-2)} && \text{as } t \to \infty,
\end{aligned}$$

where $\kappa > 0$ is arbitrary, $E_n = \frac{\pi}{2n^2 \sin(\frac{\pi}{n})}$ and $\alpha_0 = \tan(\frac{\theta_1 - \pi}{2})$. No other (non-generic) scenarios are found. In the case of boundary data $\theta_1 = \pi$ the result is

$$n \geq 2: \qquad R \sim \big(\tfrac{4(n-1)}{E_n} t\big)^{-1/(2n-2)} \qquad \text{as } t \to \infty,$$

which is the same expression as for $1 < n < 2$.

For $\theta_1 \geq 2\pi$ there is blowup over two or more scales (the number being known a priori from the value of θ_1). For $2\pi \leq \theta < 3\pi$ there is blowup over two scales; these are the simplest examples of so called bubble trees. For $\theta_1 = 2\pi$ the blowup rate in the inner most region is

$$\begin{aligned}
n = 2: &\qquad R \sim \kappa e^{-4E_2^{-2} t^2} && \text{as } t \to \infty \\
n > 2: &\qquad R \sim \big(\tfrac{n-2}{3n-2}\big)^{-1/(n-2)} \big(\tfrac{4(n-1)}{E_n} t\big)^{-(3n-2)/(2(n-1)(n-2))} && \text{as } t \to \infty,
\end{aligned}$$

where $\kappa > 0$ is arbitrary. For $\theta_1 \in (2\pi, 3\pi)$ we obtain (for the inner most blowup)

$$\begin{aligned}
n = 2: &\qquad R \sim \kappa_2 e^{-\kappa_1 e^{\frac{2\alpha_1}{E_2} t}} && \text{as } t \to \infty \\
n > 2: &\qquad R \sim \big(\tfrac{n-2}{\alpha_1 (n-1)}\big)^{-1/(n-2)} \big(\tfrac{(n-2)\alpha_1}{E_n} t\big)^{-(2n-2)/(n-2)^2} && \text{as } t \to \infty,
\end{aligned}$$

where $\kappa_1, \kappa_2 > 0$ are arbitrary and $\alpha_1 = 2\tan(\frac{\theta_1}{2})$. Analogous results hold for other values of $\theta_1 \geq 3\pi$.

We now describe the global picture suggested by these formal results. Since the equation is invariant under the discrete symmetries $\theta \mapsto -\theta$ and $\theta \mapsto \theta + \pi$, we may without loss of generality assume that $\theta(0,0) = 0$ and $\theta_1 \geq 0$. For convenience of notation, let the stationary solution $2\arctan(r^n \tan\frac{\theta_1}{2})$ be denoted by $\vartheta_{\theta_1}(r)$. We make a subdivision depending on the value of the boundary data θ_1.

$\underline{0 \leq \theta_1 < \pi}$: No blowup occurs for $n \geq 2$ and the limit profile $\lim_{t\to\infty} \theta(t,r) = \theta_\infty(r) = \vartheta_{\theta_1}(r)$ for all fixed $r > 0$. For $n < 2$ there is, depending on the initial data, either no blowup or blowup at a finite set of finite time moments $\{T_i\}_{i=1}^K$ for some integer K; if $\theta_1 = 0$ then K must be even. At each blowup time T_i the value of θ at the origin jumps by $\pm\pi$ and an amount $\mathcal{E}(T_i) - \lim_{t\uparrow T_i} = 4\pi n$ of energy is lost (a sphere bubbles off). This may either happen via a generic or a non-generic scenario. The blowup instances T_i are all different and there is no blowup as $T \to \infty$. If K is even then the limit profile is $\theta_\infty(r) = \vartheta_{\theta_1}(r)$ while if K is odd it is $\theta_\infty(r) = \pi - \vartheta_{\pi-\theta_1}(r)$. The number of jumps K is a priori bounded from above by $\frac{1}{4\pi n}\max\{\mathcal{E}(0) - \mathcal{E}_{\theta_1}, \mathcal{E}(0) - \mathcal{E}_{\pi-\theta_1}\}$, where $\mathcal{E}(0)$ is the energy of the initial data and $\mathcal{E}_{\theta_1}$ is the energy of the stationary state ϑ_{θ_1}; if $\mathcal{E}(0) < \mathcal{E}_{\pi-\theta_1} + 4\pi n$ then no jump can occur.

$\underline{\theta_1 = \pi}$: For any $n > 0$ the limit profile is $\theta_\infty(r) \equiv \pi$ and blowup has to occur at at least one time $T \in (0, \infty]$. For $n \geq 2$ blowup occurs only as $t \to \infty$ and an energy loss of $4\pi n$ occurs in this limit (i.e. $\lim_{t\to\infty} \mathcal{E}(t) = 4\pi n$ while $\mathcal{E}(\theta_\infty) = 0$). There is only one possible blowup rate. For $n < 2$ there is a finite set of time moments $\{T_i\}_{i=1}^K$ for some integer K with the same properties as for $\theta_1 < \pi$, except that K is odd (and thus $K \geq 1$) and one of the T_i may be equal to ∞, in which case the energy jump at infinity is $4\pi n$ (i.e. the same as for a finite time jump). Both the finite time and the infinite time blowup can happen via a generic or non-generic scenario.

$\underline{\pi < \theta_1 < 2\pi}$: Blowup has to occur at at least one time $T \in (0, \infty]$. For $n \geq 2$ there is blowup only as $t \to \infty$ and the limit profile is $\theta_\infty(r) = \pi + \vartheta_{\theta_1 - \pi}(r)$. The energy loss at infinity is $4\pi n$ and there is only one possible blowup rate. For $n < 2$ the scenario is the same as for $\theta < \pi$ (in particular, there is no infinite time blowup) with the adaptation that $K \geq 1$. If K is odd then the limit profile is $\theta_\infty(r) = \pi + \vartheta_{\theta_1 - \pi}(r)$, while if K is even it is $\theta_\infty(r) = 2\pi - \vartheta_{2\pi - \theta_1}(r)$.

$\underline{\theta_1 = m\pi,\ m = 2, 3, \ldots}$: For any $n > 0$ the limit profile is $\theta_\infty(r) \equiv k\pi$ and blowup has to occur at at least one time moment $T \in (0, \infty]$. For $n \geq 2$ blowup occurs only as $t \to \infty$ and an energy loss of $4\pi nm$ occurs in this limit. There is only one possible blowup rate and blowup occurs over m different scales (a so called bubble tree, the current analysis furnishing simple concrete examples of how such behaviour can occur); there is a unique blowup scenario. For $n < 2$ there is a finite set of time moments $\{T_i\}_{i=1}^K$ for some integer K with the same properties as for $\theta_1 = \pi$ (one of the T_i can be equal to ∞), except that the number of blowup times $K \geq m$ and $K - m$ is always even. There is no bubble tree and the energy loss at each T_i is $4\pi n$.

$\underline{\theta_1 > 2\pi,\ \theta_1 \notin \pi\mathbb{Z}}$: Blowup has to occur at at least one time $T \in (0, \infty]$. For $n \geq 2$ there is blowup only as $t \to \infty$ and the limit profile is $\theta_\infty(r) = M\pi + \vartheta_{\theta_1 - M\pi}(r)$ where M is the largest integer smaller than θ_1. There is blowup over M different scales and the energy loss at infinity is $4\pi nM$ (i.e. a bubble tree). There is just one possible scenario. For $n < 2$ the scenario is the same as for $\pi < \theta < 2\pi$ with the adaptation that $K \geq M$. If $K - M$ is even then the limit profile is $\theta_\infty(r) = M\pi + \vartheta_{\theta_1 - M\pi}(r)$, while if K is even it is $\theta_\infty(r) = (M+1)\pi - \vartheta_{(M+1)\pi - \theta_1}(r)$. Again, at each blowup time one quantum of energy (i.e. $4\pi n$) is lost.

In the case of an infinite domain $r \in (0, \infty)$ and boundary conditions $\lim_{r\to\infty} \theta(t, r) = \tilde{\theta}_1 = m\pi$, $m = 0, 1, 2, \ldots$ (in order for profiles to have finite energy), we restrict our attention to initial data that approach $\tilde{\theta}_1$ sufficiently fast as $r \to \infty$. We again describe the results for $n \geq 2$ and $n < 2$ separately.

For $n \geq 2$ and $\tilde{\theta}_1 = m\pi$, $m = 1, 2, \ldots$ the situation is very similar to that of a finite domain with $\theta_1 \in ((m-1)\pi, m\pi)$ (for $m = 0$ there is no blowup and the solution converges to 0 uniformly as $t \to \infty$). Blowup occurs over $m - 1$ scales and the limit profile is a stationary state $\theta_\infty(r) = (m-1)\pi + 2\arctan qr^n$ for some $q > 0$ (which depends on the initial data). The blowup rate is the same as for the finite domain described previously (with q replacing the constants α_0 and α_1). There are no non-generic scenarios.

For $n < 2$ the solution has $K \geq m$ blowup times (cf. the finite domain case), one of which may be infinity. If $K - m$ is odd then the limit profile is $\theta_\infty(r) = (m \pm 1)\pi \mp 2\arctan qr^n$ for some $q > 0$, while if $K - m$ is even then $\theta_\infty(r) = m\pi$. The finite time blowup rates are the same as for the finite domain (including the possibility of non-generic finite time blowup). Whereas for $1 < n < 2$ the blowup times are all generically finite (and there is thus no generic blowup as $t \to \infty$), for $n \leq 1$

infinite time blowup is generic and the rate is

$$\begin{aligned} n<1: &\qquad R \sim \kappa t^{-(1-n)/n} &&\text{as } t\to\infty\\ n=1: &\qquad R\sim \kappa(\ln t)^{-1} &&\text{as } t\to\infty, \end{aligned}$$

with $\kappa > 0$ arbitrary. For $1 < n < 2$ infinite time blowup can occur with co-dimension 1; the corresponding blowup rate is

$$\begin{aligned} n<1: &\qquad R \sim \kappa t^{-(2-n)/n} &&\text{as } t\to\infty\\ n=1: &\qquad R\sim \kappa t^{-1}(\ln t)^{-4/3} &&\text{as } t\to\infty\\ 1<n<2: &\qquad R\sim \kappa t^{-n/(2-n)} &&\text{as } t\to\infty, \end{aligned}$$

for arbitrary $\kappa > 0$. There is again a countable family of infinite time blowup scenarios with higher co-dimension.

Finally, when one allows for solutions that do not necessarily have decreasing energy (thereby introducing non-uniqueness) then, depending on n, jumps can occur in which the energy increases. The physical interpretation is that the energy stored in the origin at a forward (energy-decreasing) jump is released.

For $n < 2$ reverse jumps can happen at any time. In the nominally generic (co-dimension 0) case we have

$$\begin{aligned} n<1: &\qquad R \sim \left(\tfrac{\ell\, 2^{2n-1}}{\Gamma(1-n)}\right)^{1/n}(t-t_0) &&\text{as } t\downarrow t_0\\ n=1: &\qquad R\sim 2\ell\,\frac{t-t_0}{|\ln(t-t_0)|} &&\text{as } t\downarrow t_0\\ 1<n<2: &\qquad R\sim \left[\tfrac{\ell(2-n)}{E_n}(t-t_0)\right]^{1/(2-n)} &&\text{as } t\downarrow t_0, \end{aligned}$$

where $\ell > 0$ with $\theta(t_0, r) \sim \pi - \ell r^n$. For the co-dimension 1 scenario one finds

$$\begin{aligned} n<1: &\qquad R \sim \left(\tfrac{\ell\, 2^{2n+1}}{\Gamma(1-n)}\right)^{1/n}(t-t_0)^{(n+1)/n} &&\text{as } t\downarrow t_0\\ n=1: &\qquad R\sim \tfrac{8}{3}\ell\,\frac{(t-t_0)^2}{|\ln(t-t_0)|} &&\text{as } t\downarrow t_0\\ 1<n<2: &\qquad R\sim \left(\tfrac{2\ell(n+1)(2-n)}{E_n}\right)^{1/(2-n)}(t-t_0)^{2/(2-n)} &&\text{as } t\downarrow t_0, \end{aligned}$$

where $\ell > 0$ with $\theta(t_0, r) \sim \pi - \ell r^{n+2}$. It is conjectured in [2] that a physical system selects the co-dimension 1 scenario to release the energy stored in the origin. When a forward and a reverse jump occur at the same instant (cf. [18]) the rate is given by (25).

For $n \geq 2$ no reverse jumps are possible; this is not surprising since energy can be stored in the origin only as $t \to \infty$, so none is available for release.

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