FOURTH ORDER ELLIPTIC EQUATIONS WITH CRITICAL GROWTH

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Abstract: *We study a fourth order equation with a limiting Sobolev exponent and the Navier boundary conditions. If one uses the same argument as in the paper of Brezis & Nirenberg for the second order equivalent of this equation, complications arise and therefore some of their techniques have to be adjusted for this case.*

Equations elliptiques de quatrième ordre avec exposants critiques

Résumé: *Nous étudions une équation de quatrième ordre avec exposant de Sobolev critique et conditions de Navier aux limites. En utilisant les arguments de l'article de Brezis & Nirenberg sur l'équivalent de deuxième ordre de cette équation, il y a des complications. C'est pourquoi il a fallu adapter quelques-unes de leurs techniques pour ce cas.*

Version francaise abrégée: Nous etudions l'équation

$$(E) \qquad \Delta^2 u = \lambda u + u|u|^{p^*-1}, \quad u > 0, \quad p^* = \frac{N+4}{N-4},$$

avec les conditions aux limites $u = \Delta u = 0$. Cette équation constitue l'équivalent d'ordre quatre de l'équation $-\Delta u = \lambda u + u|u|^{p-1}$, $p = \frac{N+2}{N-2}$ qui a été etudiée extensivement par Brézis et Nirenberg [BrN]. On obtient les solutions de l'équation (E) en minimisant la fonctionnelle

$$E_\lambda(u) = \int_\Omega |\Delta u|^2 dx - \lambda \int_\Omega u^2 dx,$$

sur la variété $M = \left\{ u \in H^2 \cap H_0^1(\Omega); \quad \int_\Omega |u|^{p^*+1} dx = 1 \right\}$. Comme M n'est pas fermé pour la topologie faible de H^2, la condition supplémentaire

$$\inf_{u \in M} E_\lambda(u) < K_0,$$

est nécessaire. Le nombre K_0 est defini par (1.1) et (2.5).

Soit μ_1 la valeur propre de $(-\Delta)$ avec des conditions de Dirichlet homogènes. Pour $N \geq 8$ nous démontrons l'existence d'une solution quand $\lambda \in (0, \mu_1^2)$, et la non-existence quand $\lambda \geq \mu_1^2$. Si de plus Ω est un domaine etoier, alors il n'y a pas de solutions quand $\lambda \leq 0$. En dimension $5, 6$ et 7, la situation est plus delicate. Nous démontrons que si $\Omega = B_R$ il existe un nombre $\lambda^* > 0$ tel qu'une solution existe pour tout $\lambda \in (\lambda^*, \mu_1^2)$.

0. Introduction. In this note we discuss conditions for existence and non-existence of solutions $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$ of the fourth order elliptic boundary value problem

$$(\mathrm{I}) \begin{cases} \Delta^2 u = \lambda u + u|u|^{p^*-1}, \quad u > 0, \ \text{in} \ \Omega, & (0.1) \\ u = 0, \quad \Delta u = 0, \qquad \text{on} \ \partial\Omega, & (0.2) \end{cases}$$

in which $\Omega \subset \mathbf{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ and $p^* = \frac{N+4}{N-4}$, $N > 4$. In their well-known paper Brezis & Nirenberg [BrN] studied the related second order Dirichlet problem

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$$
\text{(II)} \begin{cases} -\Delta u = \lambda u + u|u|^{p-1}, \quad u > 0 \text{ in } \Omega, & (0.3) \\ \qquad u = 0 \qquad\qquad\qquad\quad \text{on } \partial\Omega, & (0.4) \end{cases}
$$

in which $p = \frac{N+2}{N-2}$. Since then this problem has been studied by many authors who elucidated and refined results of Brezis & Nirenberg (see for instance [ABP], [AP], [BP]).

In our study of Problem (I) we focus on questions which are similar in spirit to those addressed in [BrN] for Problem (II). The question of existence of solutions of equation (0.1) with *Dirichlet* boundary conditions $u = \frac{\partial u}{\partial n} = 0$ has been studied in detail by Edmunds, Fortunato & Janelli [EFJ] (see also [Gr], [NSY]). The case with the Navier boundary conditions (0.2) is more delicate than the Dirichlet case. We shall come back to this in Section 2 (see also [vdV2,3]). Non-existence of solutions of equation (0.1) with Dirichlet boundary conditions was studied by Pucci & Serrin [PS1,PS2].

1. Main results. Whereas for Problem (II) the results were different for $2 < N < 4$ and $N \geq 4$, we find that for Problem (I) they are different for $4 < N < 8$ and $N \geq 8$. We deal with these ranges in succession, beginning with the latter.

We shall denote the principal eigenvalue of the Laplacian $(-\Delta) : H^2 \cap H_0^1(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$ by μ_1 and its associated eigenfunction by ϕ_1.

Theorem 1. *Let $\Omega \subset \mathbf{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$ and $N \geq 8$. Then for every $\lambda \in (0, \mu_1^2)$ Problem (I) has a least one nontrivial positive solution $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$. If $\lambda \geq \mu_1^2$, Problem (I) has no positive solution. Moreover, if Ω is star-shaped, Problem (I) has no nontrivial positive solutions when $\lambda \in (-\infty, 0]$.*

Remark. In the case that $\Omega = B_R$ Theorem 1 provides a complete result concerning positive solutions. Namely, Problem (I) has at least one nontrivial positive solution $u \in C^4(B_R) \cap C^3(\overline{B_R})$ if and only if $\lambda \in (0, \mu_1^2)$.

Next we consider the dimensions $N = 5, 6$ and 7. For arbitrary domains the methods used to prove Theorem 1 are not sufficient in the dimensions $N = 5, 6$ and 7. However it is always possible to obtain existence of a solution for values of λ close to μ_1 by using local bifurcation theory [CR], [Bh]. Using the eigenfunctions as testfunctions we can also derive an existence result for λ close to μ_1^2. Define

$$
K_0 = \pi^2 N(N-4)(N^2-4)\left\{\frac{\Gamma(\frac{N}{2})}{\Gamma(N)}\right\}^{\frac{4}{N}}, \quad \overline{\lambda} = \mu_1^2 - \frac{K_0}{\int_\Omega \tilde{\phi}_1^2 dx}, \tag{1.1}
$$

with ϕ_1 normalized by $\int_\Omega |\tilde{\phi}_1|^{\frac{2N}{N-4}} dx = 1$.

Theorem 2. *Let $\Omega \subset \mathbf{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$ and let $N = 5, 6$ or 7. Then for every $\lambda \in (\overline{\lambda}, \mu_1^2)$ Problem (I) has a least one nontrivial positive solution $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$.*

As in [BrN] and [EFJ] a more detailed study can be performed when $\Omega = B_R$. We define

$$
\lambda^* = \min_{\phi \in X} \frac{\displaystyle\int_0^R |\phi''|^2 \rho^{7-N} d\rho - \alpha_2 \int_0^R \phi''\phi' \rho^{6-N} d\rho + \alpha_3 \int_0^R |\phi'|^2 \rho^{5-N} d\rho}{\displaystyle\int_0^R |\phi(\rho)|^2 \rho^{7-N} d\rho}, \tag{1.2}
$$

with $\alpha_2 = 2(N-7)$ and $\alpha_3 = N^2 - 10N + 33$ and $X = \{\phi \in H^2(0,R);\ \phi(R) = \phi'(0) = 0\}$. It can be shown that $0 < \lambda^* < \mu_1^2$ and λ^* is given by

$$\lambda^* = \left(\frac{\gamma}{R}\right)^4, \tag{1.3}$$

where γ is defined by

$$\gamma = \min\{\rho > 0;\ \zeta_N(\rho) = 0\},$$

with

$$\zeta_N(\rho) = \begin{cases} \left(J_{3/2}(\rho)I_{-3/2}(\rho) - I_{3/2}(\rho)J_{-3/2}(\rho)\right) & \text{if } N = 5, \\ \left(J_2(\rho)K_2(\rho) + \frac{1}{2}I_2(\rho)Y_2(\rho)\right) & \text{if } N = 6, \\ \left(J_{5/2}(\rho)I_{-5/2}(\rho) - I_{5/2}(\rho)J_{-5/2}(\rho)\right) & \text{if } N = 7. \end{cases}$$

Here J_ν and I_ν are the Bessel functions, Y_ν the Hankel functions and K_ν the MacDonald functions. For Problem (I) we have

Theorem 3. *Let $\Omega = B_R$ and let $N = 5, 6$ or 7. Then for every $\lambda \in (\lambda^*, \mu_1^2)$ Problem (I) has a least one nontrivial positive solution $u \in C^4(B_R) \cap C^3(\overline{B_R})$. If either $\lambda \geq \mu_1^2$ or $\lambda \in (-\infty, 0]$, then Problem (I) has no solutions. Moreover, positive solutions of Problem (I) are necessarily radially symmetric.*

For equation (0.1) with Dirichlet boundary conditions Pucci & Serrin [PS2] proved that when $N = 5, 6$ or 7 there exists a number $\lambda^{**} > 0$ so that the equation has no solutions for $\lambda \in (-\infty, \lambda^{**}]$. For Problem (I) we suspect that for $\lambda \in (-\infty, \lambda^*]$ there are no positive solutions. Our proof is restricted to the interval $(-\infty, 0]$.

Remark. In [EFJ] Edmunds, Fortunato & Janelli pose the question as to whether or not (0.1) with the Dirichlet boundary conditions has non-trivial solutions if $\lambda = 0$. It follows from Theorem 1 that when Ω is star-shaped it has no positive solutions for $\lambda = 0$.

2. Some remarks on the proofs of the Theorems 1,2 and 3. The methods used to prove the above theorems are similar to those used in [BrN] and [EFJ]. Thus for $N > 4$ we consider the functional

$$E_\lambda(u) = \int_\Omega |\Delta u|^2 dx - \lambda \int_\Omega u^2 dx, \tag{2.1}$$

which is well-defined on the manifold

$$M = \left\{u \in H^2 \cap H_0^1(\Omega);\ \int_\Omega |u|^{p^*+1} dx = 1\right\}. \tag{2.2}$$

If one can find minimizers $u \in M$ of $E_\lambda(u)$ it is easily seen that a suitably rescaled minimizer satisfies equation (0.1) and boundary conditions (0.2). The required regularity follows from a lemma in [vdV2]. Hence we reformulate Problem (I) as follows: *Find a function $u \in M$ which minimizes the functional $E_\lambda(u)$.* We write

$$K_\lambda = \inf_{u \in M} E_\lambda(u). \tag{2.3}$$

Because the embedding $H^2 \cap H_0^1(\Omega) \hookrightarrow L^{p^*+1}(\Omega)$ is not compact, the manifold M is not weakly closed in the H^2-topology, which complicates the usual techniques for finding a minimizer for $\inf_M E_\lambda$. From [vdV2] we have a precise criterion for the existence of a minimum of K_λ: If

$$\inf_{u \in M} E_\lambda(u) < K_0, \tag{2.4}$$

then the infimum K_λ is achieved in M. The constant K_0 is given by (1.1) and arises as the largest constant in the inequality

$$\|u\|_{L^{p^*+1}} \le K_0^{-1/2}\|\Delta u\|_{L^2}, \quad \forall\, u \in H^2 \cap H_0^1(\Omega). \tag{2.5}$$

The latter was also proved in [vdV2] and both (2.4) and (2.5) are based on the concentration compactness principle of Lions [L1,L2]. It should be noticed that in (2.5) the largest constant remains the same if one restricts to the Sobolev space $H_0^2(\Omega)$. This is proved easily by using the fact that $H_0^2(\Omega)$ can be extended by zero outside Ω (see [EFJ] for a proof). The proof of (2.5) for the Sobolev space $H^2 \cap H_0^1(\Omega)$ is more delicate since the extention by zero outside Ω is not a continuous map from $H^2 \cap H_0^1(\Omega)$ to $H^2(\mathbf{R}^N)$ and therefore different techniques have to be used.

The proofs of the existence parts in the Theorems 1,2 and 3 come down to verifying for which values of λ, $K_\lambda < K_0$. In order to do this we test K_λ with the functions

$$u_\epsilon(x) = C\frac{\phi(x)}{(\epsilon^2 + |x|^2)^{(N-4)/2}}, \quad \epsilon > 0, \tag{2.6}$$

with $\phi \in C_0^\infty(\Omega)$ and $\phi \equiv 1$ in a neighbourhood of $x = 0$. The constant C is an appropriate normalization constant.

The minimization problem K_λ is equivalent to

$$K_\lambda = \inf_{u \in H^2 \cap H_0^1(\Omega)} \frac{\displaystyle\int_\Omega |\Delta u|^2 dx - \lambda \int_\Omega u^2 dx}{\left(\displaystyle\int_\Omega |u|^{p^*+1} dx\right)^{2/(p^*-1)}}. \tag{2.7}$$

If one has a minimizer K_λ one can always find a positive minimizer $v \in H^2 \cap H_0^1(\Omega)$ by solving the equation

$$-\Delta v = |\Delta u|.$$

When $\Omega = B_R$ one can even obtain a positive radially symmetric minimizer by using the Talenti symmetrization comparison principle [T] (see also [L1,L2,vdV2]).

The non-existence parts of the above theorems can be found in [vdV1] and are based on a Pohozaev type integral identity for positive solutions of Problem (I) (see also [Mi], [PS1]).

In the dimensions $N = 5, 6$ and 7 the test-functions $u_\epsilon(x)$ in (2.6), with $\phi \equiv 1$ on $\mathbf{R}^N$, are not in $L^2(\mathbf{R}^N)$. For that reason the verification of (2.4) is more delicate and we restrict ourselves to the case $\Omega = B_R$. The number λ^*, given in (1.2) arises naturally in the verification of (2.4). We believe that this number is the sharp borderline between the existence and non-existence of positive solutions of Problem (I) in the dimensions $N = 5, 6$ and 7.

Detailed proofs of the Theorems presented in this note can be found in [vdV3], where also equation (0.1) with Dirichlet boundary conditions is studied when $\Omega = B_R$ (see also [Gr]). We refer also to a recent paper by Bernis, Garcia-Azorero & Peral [BGP], who studied a variant of Problem (I), and to a paper by Noussair, Swanson & Yang Jiunfu [NSY], who considered the case $\Omega = \mathbf{R}^N$.

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