Branching patterns of wave trains in the FPU lattice

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Abstract

We study the existence and branching patterns of wave trains in the one-dimensional infinite Fermi-Pasta-Ulam (FPU) lattice. A wave train Ansatz in this Hamiltonian lattice leads to an advance-delay differential equation on a space of periodic functions, which carries a natural Hamiltonian structure. The existence of wave trains is then studied by means of a Lyapunov-Schmidt reduction, leading to a finite-dimensional bifurcation equation with an inherited Hamiltonian structure. While exploring some of the additional symmetries of the FPU lattice, we use invariant theory to find the bifurcation equations describing the branching patterns of wave trains near $p:q$ resonant waves. We show that at such branching points, a generic nonlinearity selects exactly two two-parameter families of mixed-mode wave trains.

1 Introduction

In this paper we investigate the existence of wave trains in the infinite Fermi-Pasta-Ulam (FPU) lattice. The FPU lattice was introduced in [7] as a model for a nonlinear string, formed by identical point masses that interact with their nearest neighbors. It consists of an infinite set of ordinary differential equations for the particle positions $q_j$ and their momenta $p_j$:

$$\frac{dq_j}{dt} = p_j, \quad \frac{dp_j}{dt} = W'(q_{j+1} - q_j) - W'(q_j - q_{j-1}), \quad j \in \mathbb{Z}.$$  \hfill (1.1)

Note that equations (1.1) are Hamiltonian with respect to the formal Hamiltonian function

$$H = \sum_{j \in \mathbb{Z}} \frac{1}{2} p_j^2 + W(q_{j+1} - q_j).$$

Finally, we assume that the interaction potential has a Taylor expansion of the form

$$W(z) = \frac{1}{2} z^2 + \frac{\alpha}{3!} z^3 + \frac{\beta}{4!} z^4 + \ldots.$$
In this paper, we shall be interested in wave train solutions of equation (1.1). Let us make precise here what we mean by a wave train. First of all, we note that the FPU lattice is $\mathbb{Z}$-equivariant with respect to the group of simultaneous particle-shifts: $\{(q_j(t), p_j(t))\}_{j \in \mathbb{Z}}$ is a solution of the equations of motion if and only if $\{(\tilde{q}_j(t), \tilde{p}_j(t))\}_{j \in \mathbb{Z}}$ defined by $\tilde{q}_j(t) := q_{j+1}(t)$, $\tilde{p}_j(t) := p_{j+1}(t)$ is a solution of the equations of motion. We now say that a solution to (1.1) is a wave train if it is a time-periodic solution that is relative periodic with respect to the maximal particle-shift symmetry. In other words, it satisfies

- $\exists \ T > 0$, such that $q_j(t) = q_j(t + T)$ and $p_j(t) = p_j(t + T)$.
- $\exists \ \tau > 0$, such that $q_{j+1}(t) = q_j(t + \tau)$ and $p_{j+1}(t) = p_j(t + \tau)$.

Such solutions have the form

$$q_j(t) = u(\omega t - kj) , \ p_j(t) = v(\omega t - kj) ,$$

(1.2)

where $\omega = 1/T > 0$, $k = \omega \tau$, and $u$ and $v$ are one-periodic functions. One sees that the Ansatz (2.1) produces solutions of equations (1.1) precisely when $u$ and $v$ satisfy the advance-delay differential equations

$$\omega \frac{du(s)}{ds} = v(s) , \ \omega \frac{dv(s)}{ds} = W'(u(s - k) - u(s)) - W'(u(s) - u(s + k)) .$$

(1.3)

It is easy to see that wave trains exist in the linear FPU lattice, i.e. the lattice for which $\alpha = \beta = \ldots = 0$. Indeed, for every $\varepsilon > 0$ and $\phi_0 \in \mathbb{R}/\mathbb{Z}$, the functions

$$q_j(t) = \varepsilon \cos(2\pi \omega t - 2\pi kj + \phi_0) , \ p_j(t) = -2\pi \varepsilon \sin(2\pi \omega t - 2\pi kj + \phi_0)$$

(1.4)

are solutions of the linear FPU equations of motion, exactly if $\omega$ and $k$ are related by the dispersion relation

$$\omega = \pm \omega(k) := \pm \frac{1}{\pi} \sin(k\pi) .$$

The above wave trains are monochromatic and it follows from a Fourier transformation that all motions of the linear lattice are a superposition of such monochromatic wave trains. Some of these superpositions are actually wave trains themselves, for instance if $k$ is “resonant” in the sense that

$$\frac{\sin(qk\pi)}{\sin(pk\pi)} = \frac{q}{p}$$

for some integers $p, q \in \mathbb{Z}$. Writing $\omega := \frac{\sin(qk\pi)}{q} = \frac{\sin(pk\pi)}{p}$, then

$$q_j(t) = \varepsilon_1 \cos(2\pi p\omega t - 2\pi pkj + \phi_0) + \varepsilon_2 \cos(2\pi q\omega t - 2\pi qkj + \phi_1) ,$$

$$p_j(t) = -2\pi p\varepsilon_1 \sin(2\pi p\omega t - 2\pi pkj + \phi_0) - 2\pi q\varepsilon_2 \sin(2\pi q\omega t - 2\pi qkj + \phi_1) ,$$

are wave train solutions of the linear lattice with temporal period $T = q\omega / \omega$ and relative spatial period $\tau = k / \omega$. We call these bichromatic wave trains.

In this paper we address the elementary question whether the monochromatic and bichromatic wave trains of the linear FPU lattice continue to exist in the nonlinear lattice.

The first result is a Lyapunov center theorem for the infinite lattice, corresponding to the nonresonant case.
Theorem 1.1 (Monochromatic wave trains) Let \( n^* \in \mathbb{Z}_{>0} \). When \( \omega^* > 0 \) and \( k^* \) are such that \( \omega^* = \pm \frac{1}{nk^*}\pi \sin(nk^*\pi) \), but \( \omega^* \neq \pm \frac{1}{nk^*}\pi \) for any \( n \in \mathbb{Z}_{\neq 0} \) not equal to \( \pm n^* \), then the nonlinear FPU lattice supports monochromatic wave trains of the form

\[
q_j(t) = \varepsilon \cos(2\pi n^*\omega^*(\varepsilon)t - 2\pi n^*k^*j + \phi_0) + \mathcal{O}(\varepsilon) ,
\]

\[
p_j(t) = -2\pi n^*\omega^*\varepsilon \sin(2\pi n^*\omega^*(\varepsilon)t - 2\pi n^*k^*j + \phi_0) + \mathcal{O}(\varepsilon^2) .
\]

Here, \( \phi_0 \in \mathbb{R}/\mathbb{Z} \) is arbitrary. The function \( \varepsilon \mapsto \omega(\varepsilon) \) satisfies \( \lim_{\varepsilon \downarrow 0} \omega(\varepsilon) = \omega^* \).

This result should not come as a surprise and the proof by Lyapunov-Schmidt reduction that we provide is straightforward. Nevertheless, we did not find this theorem in the literature as formulated and proved here, although a somewhat similar statement can be found in Deift et al. [6], which studies wave trains in a forced semi-infinite lattice. In [6] the wave trains are constructed by means of an explicit iteration that is technically more involved than our approach.

The persistence of bichromatic wave trains is a more delicate matter and our main result concerns precisely these wave trains:

Theorem 1.2 (Bichromatic wave trains) Assume that \( \omega^* > 0 \) and \( k^* \) are such that \( \omega^* = \pm \frac{1}{nk^*}\pi \sin(pk^*\pi) \) and \( \omega^* = \pm \frac{1}{nk^*}\pi \sin(qk^*\pi) \) with \( p, q \in \mathbb{Z}_{>0} \), \( p \neq q \), but \( \omega^* \neq \pm \frac{1}{nk^*}\pi \) sin\((nk^*\pi)\) for any \( n \in \mathbb{Z}_{\neq 0} \) not equal to \( \pm n^* \) and assume that the curves \( \omega = \pm \frac{1}{nk^*}\pi \sin(pk^*\pi) \) and \( \omega = \pm \frac{1}{nk^*}\pi \sin(qk^*\pi) \) in the \((k,\omega)\)-plane intersect transversely at \((k^*,\omega^*)\). Finally, let \( \gcd(p,q) \) be the greatest common divisor of \( p \) and \( q \) and define \( \hat{p} = p/\gcd(p,q) \) and \( \hat{q} = q/\gcd(p,q) \).

Then for generic values of the parameters \( \alpha, \beta, \ldots \), the nonlinear FPU lattice supports bichromatic wave train solutions of the form

\[
q_j(t) = \varepsilon_1 \cos(2\pi p\omega_\pm(\varepsilon)t - 2\pi p k_\pm(\varepsilon)j + \hat{p}\phi_0)
+ \varepsilon_2 \cos(2\pi q\omega_\pm(\varepsilon)t - 2\pi q k_\pm(\varepsilon)j + \hat{q}\phi_0 + \kappa_\pm) + \mathcal{O}(||\varepsilon||^2),
\]

\[
p_j(t) = -2\pi p\omega^*\varepsilon_1 \sin(2\pi p\omega_\pm(\varepsilon)t - 2\pi p k_\pm(\varepsilon)j + \hat{p}\phi_0)
- 2\pi q\omega^*\varepsilon_2 \sin(2\pi q\omega_\pm(\varepsilon)t - 2\pi q k_\pm(\varepsilon)j + \hat{q}\phi_0 + \kappa_\pm) + \mathcal{O}(||\varepsilon||^2) .
\]

Here \( \phi_0 \in \mathbb{R}/\mathbb{Z} \) is arbitrary and \( \kappa_+ = \frac{\pi}{2\hat{p}}, \kappa_- = -\frac{\pi}{2\hat{p}} \) if \( \hat{p} + \hat{q} \) is odd, whereas \( \kappa_+ = 0, \kappa_- = \frac{\pi}{\hat{p}} \) if \( \hat{p} + \hat{q} \) is even. The functions \( \omega_\pm, k_\pm \) satisfy \( \lim_{||\varepsilon|| \downarrow 0} \omega_\pm(\varepsilon) = \omega^* \) and \( \lim_{||\varepsilon|| \downarrow 0} k_\pm(\varepsilon) = k^* \).

It is important to observe that not every bichromatic wave train persists in the nonlinear lattice. In fact, the nonlinearity selects two families of bichromatic wave trains that are exactly in or out of phase. Note that by setting \( \varepsilon_1 = 0 \) or \( \varepsilon_2 = 0 \), we recover the monochromatic waves. The precise meaning of genericity is provided in Theorem 7.3.

We would like to point out the relation of Theorem 1.2 with results obtained by Kriecherbauer in [12], which studies the persistence in the nonlinear FPU lattice of superpositions of monochromatic wave trains with incommensurate spatial frequencies. It should be stressed that such solutions are not wave trains. The problem of continuing these quasi-periodic solutions features small divisors and was solved in [12] by means of a KAM iteration procedure. It is proved that Cantor-families of these solutions persist,

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1We thank one of the referees for pointing out this reference to us.
but phase selection or nontrivial branching patterns are not found.

Other existence results for wave trains can be found in Iooss [10], where, among others, wave trains of small amplitude and long wavelength are found by means of a center manifold reduction. Nonperturbative existence results for wave trains also exist, cf. Filip et al. [8] for the case that the potential energy function $W$ is convex. In the latter paper both a variational proof and a degree theoretic argument are given.

We prove Theorems 1.1 and 1.2 by means of a Lyapunov-Schmidt reduction. This is a way of reducing the advance-delay differential equations (1.3) to a finite-dimensional bifurcation equation. This method is standard in the context of ordinary differential equations and a central aim of this paper is to illustrate its application in Hamiltonian lattice dynamical systems. In particular we want to show how the Hamiltonian structure manifests itself in the reduced bifurcation equation, following the ideas set out in [9].

Although in this paper we focus on the FPU lattice, for its obvious historical and physical importance, we would like to emphasize that our methodology for finding wave train solutions applies equally well to other Hamiltonian lattice dynamical systems, as long as they have the particle-shift $\mathbb{Z}$-equivariance. Additional symmetries of the FPU lattice such as the time-reversal symmetry $(q, p) \mapsto (q, -p)$ are less essential. It is true that the proof of Theorem 1.2 exploits these additional symmetry properties as this facilitates the computations, but the analysis could well be extended to systems with less (or more) symmetry properties.

In the literature, the properties of traveling waves in Hamiltonian lattice dynamical systems and PDEs are often explained from the reversibility of the bifurcation equation that originates from a spatial reflection symmetry in the lattice. But there are interesting examples of Hamiltonian lattices that do not exhibit this reflection symmetry. An example is the tilted Frenkel-Kontorova model [2]. Moreover, although some generic properties of reversible and Hamiltonian systems coincide, the structures are by no means equivalent and may give rise to quite different dynamical phenomena [13]. For a proper insight into generic dynamics and/or bifurcations, it is important to take into account all available structure, and thus in particular also the Hamiltonian structure.

The remainder of this paper is organized as follows. In Section 2 we show in more detail how a wave train Ansatz for a lattice dynamical system leads to an advance-delay equation. In Section 3 we make preparations for the application of Lyapunov-Schmidt reduction, which is introduced in Section 4. The Hamiltonian structure and symmetry properties of the reduced bifurcation equation are examined in Section 5. The final sections are concerned with the proofs of Theorem 1.1 and Theorem 1.2.

**Remark 1.3 (The FPU paradox and periodic solutions)** We would like to briefly discuss how the FPU paradox led to an interest in periodic solutions and wave trains. Recall that a nonlinearity in the interaction between particles ($\alpha$ or $\beta$ or . . . nonzero) may facilitate the transfer of energy between waves of different wavelengths. It was expected by many that this mechanism of nonlinear wave interaction would lead the lattice to, averaged over time, equipartition its energy among waves of different wavelengths, cf. [11]. This process is sometimes called *thermalisation*.

The surprising insight that comes from various numerical investigations, starting with the original experiments by Fermi, Pasta and Ulam [7], is that at low energies therma-
lisation does not occur, see [7] and [11]. The lack of equipartition and the associated
recurrent behavior of the lattice have become known as the FPU problem or FPU para-
dox. Explanations of the FPU recurrence are mostly nonrigorous and have historically
been based on either KAM theory or the integrable KdV approximation, cf. [16, 21] for
an overview.

On the other hand, experiments indicate that equipartition is actually attained when
the initial energy of the lattice is larger then a certain threshold. An “explanation” for
the existence of this energy threshold was suggested in [4]: a well-known mechanism for
instability in Hamiltonian systems occurs when a family of periodic solutions loses its
stability as energy is increased. Above the energy at which this destabilization occurs,
positive Lyapunov exponents emerge, which may in turn enforce chaos, mixing or even
ergodicity at such higher energies. Not surprisingly, this idea has largely stimulated the
search for periodic solutions in the FPU lattice, as well as the investigation of their sta-
bility, cf. [1, 3, 5, 19, 20, 14, 15, 17, 18]. We would like to mention those results that
were not yet discussed above.

Some of these periodic solutions are found from symmetry considerations, cf. [14, 3,
5, 15]. In fact, fixed point sets of symmetries are invariant manifolds. These fixed point
sets were fully classified in [15]. Several of them are two-dimensional and hence fibered
by periodic orbits.

Also, families of periodic solutions are born in bifurcations of the trivial steady state,
such as the Lyapunov families of “q-breathers” discussed in [19, 20], whereas families
of two-mode solutions with Dirichlet boundary conditions were studied in [1]. It should
be pointed out that the results for waves with Dirichlet boundary data require nonreso-
nance conditions and therefore depend heavily on the number theoretic properties of the
eigenvalues of the linear FPU lattice. This problem does not occur for wave trains.

2 Advance-delay equations for wave trains

We have defined wave trains as solutions to the equations of motion (1.1) of the form

\[ q_j(t) = u(\omega t - kj) , \quad p_j(t) = v(\omega t - kj) . \]  

(2.1)

We assume that \( u, v : \mathbb{R}/\mathbb{Z} \to \mathbb{R} \) are Sobolev differentiable functions with period 1 and
the parameters \( \omega \) and \( k \) have the interpretation of the temporal and spatial frequency
of the wave, i.e. the wave speed or phase velocity is \( \omega/k \). The above wave train Ansatz
(2.1) in equations (1.1) leads to the following advance-delay differential equations for \( u \)
and \( v \):

\[ \omega \frac{du(s)}{ds} = v(s) , \quad \omega \frac{dv(s)}{ds} = W'(u(s - k) - u(s)) - W'(u(s) - u(s + k)) . \]

These advance-delay differential equations can be viewed as an operator equation on a
space of periodic functions. An example of such a “loop space” is the Hilbert space of \( s \)
times Sobolev differentiable functions of a 1-periodic variable with average 0. We denote
this space \( H^s_0(\mathbb{R}/\mathbb{Z}) \):

\[ H^s_0 := \{ u : \mathbb{R}/\mathbb{Z} \to \mathbb{R}, u(s) = \sum_{n \in \mathbb{Z}} u_n e^{2\pi ins} \mid ||u||^2_s := \sum_{n \in \mathbb{Z}} (1 + n^2)^s |u_n|^2 < \infty, u_0 = 0 \} . \]
We shall from now on fix an \( s \geq 2 \). We want to find \( u \in H_0^s(\mathbb{R}/\mathbb{Z}) \) and \( v \in H_0^{s-1}(\mathbb{R}/\mathbb{Z}) \) satisfying the above advance-delay equations. We hence search for \( u \) and \( v \) that are zeroes of the analytic map \( F : H_0^s(\mathbb{R}/\mathbb{Z}) \times H_0^{s-1}(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^2 \to H_0^{s-1}(\mathbb{R}/\mathbb{Z}) \times H_0^{s-2}(\mathbb{R}/\mathbb{Z}) \) defined by
\[
F(u, v, \omega, k) := \left( \frac{du}{ds} - v, \frac{dv}{ds} + W'(u - u(\cdot + k)) - W'(u(\cdot - k) - u) \right). \quad (2.2)
\]

Let us remark that the equations of motion (1.1) of the FPU lattice are reversible under the map \((q, p) \mapsto (q, -p)\) and that they are formally Hamiltonian with respect to the Hamiltonian function
\[
H = \sum_{j \in \mathbb{Z}} \frac{1}{2} p_j^2 + W(q_{j+1} - q_j),
\]
and the symplectic form \( \sum_{j \in \mathbb{Z}} dq_j \wedge dp_j \).

In the following proposition, we describe how the operator \( F \) inherits these geometric properties.

**Proposition 2.1 (Geometry of the operator \( F \))**

i) The operator \( F \) is equivariant under the \( \mathbb{R}/\mathbb{Z} \)-action \( \Delta \) defined by
\[
\Delta(u, v, \phi) = (u(\cdot + \phi), v(\cdot + \phi)),
\]
i.e. \( \Delta(F(u, v, \omega, k), \phi) = F(\Delta(u, v, \phi), \omega, k) \).

ii) \( F \) is reversible with respect to the operator
\[
R : (u, v) \mapsto (-u(\cdot), v(\cdot)),
\]
i.e. \( F(R(u, v), \omega, k) = -R(F(u, v, \omega, k)) \).

iii) \( F \) is Hamiltonian with respect to the weak symplectic form
\[
\Omega : (H_0^{s-1}(\mathbb{R}/\mathbb{Z}) \times H_0^{s-2}(\mathbb{R}/\mathbb{Z})) \times (H_0^s(\mathbb{R}/\mathbb{Z}) \times H_0^{s-1}(\mathbb{R}/\mathbb{Z})) \to \mathbb{R}
\]
defined by
\[
\Omega((u_1, v_1), (u_2, v_2)) = \int_{\mathbb{R}/\mathbb{Z}} v_1(s)u_2(s) - v_2(s)u_1(s)ds,
\]
and the Hamiltonian function \( \tilde{H} : H_0^s(\mathbb{R}/\mathbb{Z}) \times H_0^{s-1}(\mathbb{R}/\mathbb{Z}) \to \mathbb{R} \) defined by
\[
\tilde{H}(u, v, \omega, k) = \int_{\mathbb{R}/\mathbb{Z}} \omega u(s) \frac{dv(s)}{ds} + \frac{1}{2} v(s)^2 + W(u(s) - u(s + k))ds,
\]
i.e. \( \Omega(F(u, v, \omega, k), \cdot) = d_{(u,v)} \tilde{H}(u, v, \omega, k) \). \( \tilde{H} \) is invariant under both \( \Delta \) and \( R \).

**Proof:** The \( \Delta \)-equivariance and \( R \)-symmetry are easy to check. It remains to show that \( F \) is a Hamiltonian “vector field” on \( H_0^s(\mathbb{R}/\mathbb{Z}) \times H_0^{s-1}(\mathbb{R}/\mathbb{Z}) \). Firstly, we remark that \( \Omega \) is indeed weakly symplectic, as it is bilinear and anti-symmetric and moreover the map
\[
\Omega^* : H_0^{s-1}(\mathbb{R}/\mathbb{Z}) \times H_0^{s-2}(\mathbb{R}/\mathbb{Z}) \to (H_0^s(\mathbb{R}/\mathbb{Z}) \times H_0^{s-1}(\mathbb{R}/\mathbb{Z}))^*
\]
defined by
\[ \Omega^\#(u_1, v_1)(u_2, v_2) := \Omega((u_1, v_1), (u_2, v_2)) \]
is injective. When \( \Omega^\# \) would have been surjective as well, then \( \Omega \) would have been called a (strong) symplectic form.

If \( \tilde{H} : H^s_0(\mathbb{R}/\mathbb{Z}) \times H^{s-1}_0(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^2 \to \mathbb{C} \) now is a functional for which at every point \((u, v)\) a derivative with respect to the first components \( d_{(u,v)} \tilde{H}(u, v, \omega, k) \in (H^s_0(\mathbb{R}/\mathbb{Z}) \times H^{s-1}_0(\mathbb{R}/\mathbb{Z}))^* \) is defined, then the solution \( F(u, v, \omega, k) \in H^s_0(\mathbb{R}/\mathbb{Z}) \times H^{s-1}_0(\mathbb{R}/\mathbb{Z}) \) to the equation
\[ \Omega(F(u, v, \omega, k), \cdot) = d_{(u,v)} \tilde{H}(u, v, \omega, k) \]
is unique, if it exists. \( F(u, v, \omega, k) \) is then called the Hamiltonian vector field of \( \tilde{H}(\cdot, \omega, k) \) at \((u, v)\) and is denoted \( F(u, v, \omega, k) = X_{\tilde{H}(\cdot, \omega, k)}(u, v) \).

We will now show that the \( F(u, v, \omega, k) \) defined in (2.2) is in fact the Hamiltonian vector field at \((u, v)\) of the functional (2.3). One checks this by computing the Fréchet derivative of \( \tilde{H}(\cdot, \omega, k) \) at \((u, v)\) in the direction of \((u_1, v_1)\). It is given by
\[
\left. \frac{d}{d \varepsilon} \tilde{H}(u + \varepsilon u_1, v + \varepsilon v_1, \omega, k) \right|_{\varepsilon = 0} = \int_{\mathbb{R}/\mathbb{Z}} \omega u(s) \frac{dv_1(s)}{ds} + \omega v_1(s) \frac{dv(s)}{ds} + v(s) v_1(s) + W'(u(s) - u(s + k))(u_1(s) - u_1(s + k)) \, ds.
\]
By a partial integration and a substitution of variables this is equal to
\[
\int_{\mathbb{R}/\mathbb{Z}} (\frac{d\omega v(s)}{ds} + W'(u(s) - u(s + k)) - W'(u(s - k) - u(s))) u_1(s) - (\frac{d\omega u(s)}{ds} - v(s)) v_1(s) \, ds = \Omega(F(u, v, \omega, k), (u_1, v_1)),
\]
which shows that \( F(u, v, \omega, k) = X_{\tilde{H}(\cdot, \omega, k)}(u, v) \). The invariance of the Hamiltonian function \( \tilde{H} \) under \( \Delta \) follows from a substitution of variables, while its invariance under \( R \) is obvious.

3 The derivative of \( F \)

Our purpose is to solve the equation \( F(u, v, \omega, k) = 0 \), for \((u, v) \in H^s_0(\mathbb{R}/\mathbb{Z}) \times H^{s-1}_0(\mathbb{R}/\mathbb{Z}) \) and parameters \((\omega, k) \in \mathbb{R}^2 \).

First of all, we can easily point out a family of trivial solutions: for every fixed parameter-value \((\omega^*, k^*)\), the zero-function \((u, v) = (0, 0)\) solves \( F(u, v, \omega^*, k^*) = (0, 0) \). One would maybe like to prove the uniqueness of these solutions by the implicit function theorem. Thus one would compute the derivative \( D_{(u,v)} F(0, 0, \omega^*, k^*) : H^s_0(\mathbb{R}/\mathbb{Z}) \times H^{s-1}_0(\mathbb{R}/\mathbb{Z}) \to H^s_0(\mathbb{R}/\mathbb{Z}) \times H^{s-1}_0(\mathbb{R}/\mathbb{Z}) \). It is given by
\[
D_{(u,v)} F(0, 0, \omega^*, k^*) \cdot (u_1, v_1) = \left. \frac{d}{d \varepsilon} \right|_{\varepsilon = 0} F(\varepsilon u_1, \varepsilon v_1, \omega^*, k^*) = \\
(\omega \frac{du_1}{ds} - v_1, \omega^* \frac{dv_1}{ds} + 2u_1 - u_1(-k^*) - u_1(+k^*)),
\]
where we have used that $W''(0) = 1$.

The function space $H^0_0(\mathbb{R}/\mathbb{Z}) \times H^{-1}_0(\mathbb{R}/\mathbb{Z})$ is the direct sum over $n \in \mathbb{Z}_{\neq 0}$ of the finite-dimensional subspaces

$$\text{span}_\mathbb{C}\{s \mapsto (e^{2\pi in}s, 0), s \mapsto (0, e^{2\pi in}s)\}_{n \in \mathbb{Z}_{\neq 0}}.$$ These subspaces are invariant for $D_{(u,v)}F(0, 0, \omega^*, k^*)$.  With respect to a basis $\{s \mapsto (e^{2\pi in}s, 0), s \mapsto (0, e^{2\pi in}s)\}$ for these subspaces, the derivative $D_{(u,v)}F(0, 0, \omega^*, k^*)$ has the matrix

$$\begin{pmatrix} 2\pi i \omega^* n & -1 \\ 4\sin^2 \pi nk^* & 2\pi i \omega^* n \end{pmatrix}.$$ (3.1)

The eigenvalues of this matrix are

$$2i(\omega^* n \pi \pm \sin \pi nk^*).$$

From this moment on, we shall assume that $\omega^* \neq 0$. Then the above eigenvalues can only be zero when $\sin k^* \pi n \neq 0$, which implies that the two eigenvalues are different and the kernel of the above matrix can be at most one-dimensional. One checks that if $\omega^* n \pi = \pm \sin \pi nk^*$, then $s \mapsto (e^{2\pi ins}, 2\pi in\omega^* e^{2\pi ins})$ is indeed in its kernel.

Let us summarize this computation as a proposition:

**Proposition 3.1** The kernel of $D_{(u,v)}F(0, 0, \omega^*, k^*)$ is given by

$$K := \ker D_{(u,v)}F(0, 0, \omega^*, k^*)$$

$$= \text{span}_\mathbb{C}\{s \mapsto (e^{2\pi ins}, 2\pi in\omega^* e^{2\pi ins}) \mid n \in \mathbb{Z}_{\neq 0} \text{ and } \pi n \omega^* = \pm \sin(\pi nk^*)\}$$

$$= \{(u(s), v(s)) = \sum_{n : \omega^* n \pi = \pm \sin nk^* \pi} x_n(e^{2\pi ins}, 2\pi in\omega^* e^{2\pi ins}) \mid x_n \in \mathbb{C}\}.$$ The fact that $\omega^* \neq 0$ ensures that $K$ is finite-dimensional and the variables $x_n$, with $\omega^* n \pi = \pm \sin nk^* \pi$, act as coordinates on it. We see that $(u, v) \in K$ is real-valued if and only if $x_n = \bar{x}_{-n}$. The complex coordinates $\{x_n\}_{n \in \mathbb{Z}_{>0} | \pi n \omega^* = \pm \sin nk^* \pi}$ may therefore act as coordinates for the subspace of real-valued elements of $K$.

One checks that $K$ is in fact a symplectic subspace of $H^0_0(\mathbb{R}/\mathbb{Z}) \times H^{-1}_0(\mathbb{R}/\mathbb{Z})$: $\Omega$ restricted to $K \times K$ is strictly nondegenerate. In fact, it can be expressed as

$$\Omega|_{K \times K} = \sum_{n \in \mathbb{Z}_{>0} | \pi n \omega^* = \pm \sin nk^* \pi} 4\pi in\omega^* dx_n \wedge dx_{-n}.$$ $K$ is invariant under the action of $\Delta$ and $R$. These actions are actually given by:

$$\Delta_{\phi}: x_n \mapsto e^{2\pi in\phi} x_n,$$

$$R: x_n \mapsto -x_{-n}.$$ Finally, we note that when $\omega^* \neq 0$, the nonzero eigenvalues of $D_{(u,v)}F(0, 0, \omega^*, k^*)$ are strictly bounded away from zero.
4 Lyapunov-Schmidt reduction

Let us again fix \(\omega^* \neq 0\) and \(k^*\). Then \(F(0, 0, \omega^*, k^*) = 0\) and \(D_{(u,v)}F(0, 0, \omega^*, k^*)\) has a finite dimensional kernel. Let us suppose it is nontrivial. Then in a neighborhood of \((u, v) = (0, 0)\), the solution \((u, v) = (0, 0)\) is not necessarily unique, so that we may find nontrivial solutions to the nonlinear functional equation \(F(u, v, \omega, k) = (0, 0)\) with \((u, v)\) close to \((0, 0)\) in \(H^s_0(\mathbb{R}/\mathbb{Z}) \times H^{s-1}_0(\mathbb{R}/\mathbb{Z})\) and \((\omega, k)\) close to \((\omega^*, k^*)\) in \(\mathbb{R}^2\). We will try to find these by means of a Lyapunov-Schmidt reduction.

We start by defining

\[
\mathcal{M} := \text{im } D_{(u,v)}F(0, 0, \omega^*, k^*) = \text{span}_\mathbb{C}\{ s \mapsto (e^{2\pi i n s}, 0), s \mapsto (0, e^{2\pi i ns}), s \mapsto (e^{2\pi i ns}, -2\pi i n\omega^* e^{2\pi i n s}) | m, n \in \mathbb{Z}_{\neq 0}, \\
\pi m \omega^* \neq \pm \sin \pi n k^*, \pi n \omega^* = \pm \sin \pi n k^* \} \cap (H^{s-1}_0(\mathbb{R}/\mathbb{Z}) \times H^{s-2}_0(\mathbb{R}/\mathbb{Z})) .
\]

If we also define an inner product on \(H^{s-1}_0(\mathbb{R}/\mathbb{Z}) \times H^{s-2}_0(\mathbb{R}/\mathbb{Z})\) by

\[
\langle (u_1, v_1), (u_2, v_2) \rangle := \int_{\mathbb{R}/\mathbb{Z}} u_1(s) \overline{u_2(s)} + v_1(s) \overline{v_2(s)} \, ds ,
\]

then the adjoint operator of \(D_{(u,v)}F(0, 0, \omega^*, k^*)\) with respect to this inner product is given by

\[
(D_{(u,v)}F(0, 0, \omega^*, k^*))^*(u_1, v_1) \mapsto (-\omega^* \frac{du_1}{ds} + 2v_1 - v_1(-k^*) - v_1(\cdot + k^*), -\omega^* \frac{dv_1}{ds} - u_1) .
\]

One can check that indeed

\[
\langle (u_1, v_1), D_{(u,v)}F(0, 0, \omega^*, k^*)(u_2, v_2) \rangle = \langle (D_{(u,v)}F(0, 0, \omega^*, k^*))^*(u_1, v_1), (u_2, v_2) \rangle ,
\]

by integration by parts and a substitution of variables.

With these definitions we can now also define \(\mathcal{K}^* \subset H^{s-1}_0(\mathbb{R}/\mathbb{Z}) \times H^{s-2}_0(\mathbb{R}/\mathbb{Z})\) and \(\mathcal{M}^* \subset H^s_0(\mathbb{R}/\mathbb{Z}) \times H^{s-1}_0(\mathbb{R}/\mathbb{Z})\) as the kernel and formal image of \((D_{(u,v)}F(0, 0, \omega^*, k^*))^*\) respectively:

\[
\mathcal{K}^* = \text{span}_\mathbb{C}\{ s \mapsto (2\pi i n\omega^* e^{2\pi i n s}, -e^{2\pi i n s}) | n \in \mathbb{Z} - \{0\} \text{ and } \pi n \omega^* = \pm \sin \pi n k^* \}
\]

\[
\cap (H^{s-1}_0(\mathbb{R}/\mathbb{Z}) \times H^{s-2}_0(\mathbb{R}/\mathbb{Z})) ,
\]

and

\[
\mathcal{M}^* = \text{span}_\mathbb{C}\{ s \mapsto (e^{2\pi i n s}, 0), s \mapsto (0, e^{2\pi i n s}), s \mapsto (2\pi i n\omega^* e^{2\pi i n s}, e^{2\pi i n s}) | \\
\pi n \omega^* \neq \pm \sin \pi n k^*, \pi n \omega^* = \pm \sin \pi n k^* \} \cap (H^s_0(\mathbb{R}/\mathbb{Z}) \times H^{s-1}_0(\mathbb{R}/\mathbb{Z})) .
\]

By construction, \(\mathcal{K} \perp \mathcal{M}^*\) and \(\mathcal{K}^* \perp \mathcal{M}\) with respect to the inner product and we therefore have the orthogonal direct sum decompositions

\[
H^s_0(\mathbb{R}/\mathbb{Z}) \times H^{s-1}_0(\mathbb{R}/\mathbb{Z}) = \mathcal{K} \oplus \mathcal{M}^* ,
\]

\[
H^{s-1}_0(\mathbb{R}/\mathbb{Z}) \times H^{s-2}_0(\mathbb{R}/\mathbb{Z}) = \mathcal{K}^* \oplus \mathcal{M} .
\]

In fact, one can check that \(\mathcal{K}\) and \(\mathcal{K}^*\) are symplectic spaces, \(\mathcal{M}\) and \(\mathcal{M}^*\) are weak symplectic spaces, and \(\mathcal{K} \perp \mathcal{M}\) and \(\mathcal{K}^* \perp \mathcal{M}^*\) are \(\Omega\)-orthogonal.
Proposition 4.1 The operator

\[ D_{(u,v)}F(0,0,\omega^*,k^*) : H^s_0(\mathbb{R}/\mathbb{Z}) \times H^{-2}_0(\mathbb{R}/\mathbb{Z}) \to H^s_0(\mathbb{R}/\mathbb{Z}) \times H^{-2}_0(\mathbb{R}/\mathbb{Z}) \]

is Fredholm with index zero. \( D_{(u,v)}F(0,0,\omega^*,k^*) \mid _{M^*} : M^* \to M \) is invertible and has bounded inverse.

Proof: Clearly, \( K \) and \( K^* \) are finite-dimensional and their dimensions are equal. Also, \( M \) is closed, being the orthogonal complement of \( K^* \). This means that \( D_{(u,v)}F(0,0,\omega^*,k^*) \) is a Fredholm operator, and its index is zero.

\[ D_{(u,v)}F(0,0,\omega^*,k^*) \mid _{M^*} \text{ is invertible by construction. The boundedness of the inverse follows as the eigenvalues of the matrices (3.1) are bounded from below by } A + Bn \text{ for real constants } A \text{ and } B > 0. \]

We will now perform a Lyapunov-Schmidt reduction with respect to the orthogonal decompositions (4.1) for solving the equation \( F(u,v,\omega,k) = (0,0) \) for \((u,v,\omega,k) \) near \((0,0,\omega^*,k^*) \). This works as follows.

First of all we introduce the orthogonal projections

\[ \pi_{K^*} : H^{-1}_0(\mathbb{R}/\mathbb{Z}) \times H^{-2}_0(\mathbb{R}/\mathbb{Z}) \to K^* \text{ and } \pi_M : H^{-1}_0(\mathbb{R}/\mathbb{Z}) \times H^{-2}_0(\mathbb{R}/\mathbb{Z}) \to M \]

with the help of the inner product. Of course \( F(u,v,\omega,k) = 0 \) if and only if both \( \pi_M F(u,v,\omega,k) = 0 \) and \( \pi_{K^*} F(u,v,\omega,k) = 0 \) and we will solve these equations consecutively.

Therefore let us from now on write \( w = (u,v) \) for elements in \( H^s_0(\mathbb{R}/\mathbb{Z}) \times H^{-1}_0(\mathbb{R}/\mathbb{Z}) \) and denote \( w_K \in K \). Then we have:

Proposition 4.2 (Lyapunov-Schmidt reduction) There is an open neighborhood \( U \) of \((0,\omega^*,k^*) \) \( \subset K \times \mathbb{R}^2 \) and a unique analytic function \( w_{M^*} : U \to M^* \) solving the equation

\[ \pi_M F(w_K + w_{M^*}(w_K,\omega,k),\omega,k) = 0 \]

Proof: The derivative of the map

\[ w_{M^*} \mapsto \pi_M F(w_K + w_{M^*},\omega,k) \mid _{M^*} \text{ to } M \]

at parameter values \( (w_K,\omega,k) = (0,\omega^*,k^*) \) and evaluated in \( w_{M^*} = 0 \) by construction is equal to \( D_{(u,v)}F(0,0,\omega^*,k^*) \mid _{M^*} \), which is invertible and has a bounded inverse. Hence the result of the proposition follows from the implicit function theorem.

It now remains to solve the finite dimensional reduced bifurcation equation \( F(w_K + w_{M^*}(w_K,\omega,k),\omega,k) = 0 \) for \( w_K \in K \) and \((\omega,k) \in \mathbb{R}^2 \), i.e. to find a zero of the reduced analytic mapping

\[ f : (w_K,\omega,k) \mapsto F(w_K + w_{M^*}(w_K,\omega,k),\omega,k) \mid _{K \times \mathbb{R}^2} \to K^* \]

The procedure of reducing the infinite-dimensional equation \( F(w,\omega,k) = 0 \) to a finite-dimensional reduced equation \( f(w_K,\omega,k) = 0 \) is called “Lyapunov-Schmidt reduction”.

10
5 Properties of the reduced equation

In this section we will show that the symmetry, reversibility and Hamiltonian character of $F$ induce symmetry, reversibility and Hamiltonian character of $f$. The results of this section are rather well-known for the Lyapunov-Schmidt reduction for periodic orbits near equilibria of vector fields, see for instance [9], so the reader may want to skip the proofs of the results in this section.

**Proposition 5.1** The reduced map $f : \mathcal{K} \to \mathcal{K}^*$ is equivariant under $\Delta$: $f(\Delta_\phi w, \omega, k) = \Delta_\phi f(w, \omega, k)$ and $f$ is reversible under $R$: $f(Rw, \omega, k) = -RF(w, \omega, k)$.

**Proof:** First of all recall that the actions of $\Delta$ and $R$ leave $\mathcal{K}$ and $\mathcal{K}^*$ invariant and that $F(\Delta_\phi w, \omega, k) = \Delta_\phi F(w, \omega, k)$ and $F(Rw, \omega, k) = -RF(w, \omega, k)$ and note that $\pi_{\mathcal{K}^*}$ and $\pi_{\mathcal{M}}$ commute with $\Delta_\phi$ and $R$, as can easily be verified.

This implies that $\pi_{\mathcal{M}}F(\Delta_\phi w + \Delta_\phi w_{\mathcal{M}^*}(w, \omega, k), \omega, k) = \pi_{\mathcal{M}}F(\Delta_\phi w + w_{\mathcal{M}^*}(w, \omega, k), \omega, k) = \Delta_\phi \pi_{\mathcal{M}}F(w + w_{\mathcal{M}^*}(w, \omega, k), \omega, k) = 0$ so that $w_{\mathcal{M}^*}(\Delta_\phi w, \omega, k) = \Delta_\phi w_{\mathcal{M}^*}(w, \omega, k)$. Similarly we find that $w_{\mathcal{M}^*}(Rw, \omega, k) = Rw_{\mathcal{M}^*}(w, \omega, k)$.

Therefore, for arbitrary $w \in \mathcal{K}$, $f(\Delta_\phi w, \omega, k) = \pi_{\mathcal{K}^*}F(\Delta_\phi w + w_{\mathcal{M}^*}(\Delta_\phi w, \omega, k), \omega, k) = \pi_{\mathcal{K}^*}F(\Delta_\phi w + w_{\mathcal{M}^*}(w, \omega, k), \omega, k) = \pi_{\mathcal{K}^*}F(\Delta_\phi w + w_{\mathcal{M}^*}(w, \omega, k), \omega, k) = \Delta_\phi \pi_{\mathcal{K}^*}F(w + w_{\mathcal{M}^*}(w, \omega, k), \omega, k) = \Delta_\phi f(w, \omega, k)$. A similar argument applies for $R$. □

The following proposition expresses that $f$ is a Hamiltonian vector field on $\mathcal{K}$ with respect to the Hamiltonian function $h(w, \omega, k) = \bar{H}(w + w_{\mathcal{M}^*}(w, \omega, k), \omega, k)$.

**Proposition 5.2** $f(\cdot, \omega, k) : \mathcal{K} \to \mathbb{C}$ is the Hamiltonian vector field of $h(\cdot, \omega, k)$, i.e. $\Omega_{\mathcal{K}^* \times \mathcal{K}}(f(w, \omega, k), \cdot) = dw_h(w, \omega, k)$. Moreover, $h$ is invariant under $\Delta$ and $R$.

**Proof:** Recall that first of all that $F$ is a Hamiltonian vector field on $H^0_\Omega(\mathbb{R}/\mathbb{Z}) \times H^0_{\mathcal{M}^*}(\mathbb{R}/\mathbb{Z})$ with respect to the weak symplectic form $\Omega$ and the Hamiltonian function $\bar{H}$ and that $\mathcal{K}$ and $\mathcal{K}^*$ are symplectic spaces and $\mathcal{M}$ and $\mathcal{M}^*$ are weak symplectic spaces. Because $\mathcal{K}^*$ and $\mathcal{M}^*$ are symplectically orthogonal, $\pi_{\mathcal{M}}F(w + w_{\mathcal{M}^*}(w, \omega, k), \omega, k) = 0$ and $w_{\mathcal{M}^*}(w, \omega, k) \in \mathcal{M}^*$ and hence $dw_{\mathcal{M}^*}(w, \omega, k) \cdot \bar{w}_K \in \mathcal{M}^*$, we see that for every $w$ and $\bar{w}_K$ in $\mathcal{K}$ we have $\Omega(f(w, \omega, k), \bar{w}_K) = \Omega(F(w + w_{\mathcal{M}^*}(w, \omega, k), \omega, k), \bar{w}_K + dw_{\mathcal{M}^*}(w, \omega, k) \cdot \bar{w}_K) = d(w, w_{\mathcal{M}^*}(w, \omega, k) \cdot \bar{w}_K)$.

The invariance of $h$ under $\Delta$ and $R$ follows because $\bar{H}$ is invariant under $\Delta$ and $R$: $h(\Delta_\phi w, \omega, k) = \bar{H}(\Delta_\phi w + w_{\mathcal{M}^*}(\Delta_\phi w, \omega, k), \omega, k) = \bar{H}(\Delta_\phi w + w_{\mathcal{M}^*}(w, \omega, k), \omega, k) = \bar{H}(w + w_{\mathcal{M}^*}(w, \omega, k), \omega, k) = h(w, \omega, k)$. Similarly for $R$. □

This result ensures that, in order to compute the zero’s of $f$, it suffices to compute the stationary points of the parameter-dependent reduced Hamiltonian function $h$. 

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11
6 Families of monochromatic wave trains

In this section we study the existence of nonresonant Lyapunov families of monochromatic wave trains in the FPU lattice.

This is the case that $k^*$ and $\omega^* \neq 0$ solve the equation $\omega = \pm \frac{1}{n\pi} \sin(nk\pi)$ for exactly one pair $n = \pm n^* \in \mathbb{Z}_{\neq 0}$. Then $K$ and $K^*$ are both 2-dimensional, which is the simplest nontrivial case.

**Theorem 6.1 (Monochromatic wave trains)** Let $k^*$ and $\omega^* \neq 0$ be such that $\omega^* = \pm \frac{1}{n^*\pi} \sin(n^*k^*\pi)$, but $\omega^* \neq \pm \frac{1}{n^*\pi} \sin(n^*k^*\pi)$ for all $n \in \mathbb{Z}_{\neq 0}$ with $n \neq \pm n^*$. Then for every $\varepsilon \geq 0$ close enough to 0 and every $\phi_0 \in \mathbb{R}/2\pi\mathbb{Z}$ there is a unique analytic $\omega = \omega(\varepsilon)$ with the property that $\lim_{\varepsilon \to 0} \omega(\varepsilon) = \omega^*$ and such that $x_n^* = \frac{\varepsilon^2}{2} e^{i\phi_0}$ is a real-valued solution of the reduced bifurcation equation $d_b h(x_n^*, x_{-n}^*, \omega(\varepsilon), k^*) = 0$.

**Proof:** The $\Delta$-invariance of $h$ implies that it is a smooth function of $\omega$, $k$ and the invariant $a := x_{n^*} x_{-n^*}$, so that the bifurcation equations $d_b h(x_n^*, x_{-n}^*, \omega, k) = 0$ read $x_n^* \frac{\partial h}{\partial a} = x_{-n^*} \frac{\partial h}{\partial a} = 0$. Except when $x_n^* = x_{-n^*} = 0$ it must therefore be true that $\frac{\partial h}{\partial a} = 0$. Recall also that for real-valued solutions, $x_n^* = x_{-n^*}$, so that both $a = |x_n^*|^2$ and $\frac{\partial h}{\partial a}$ are real.

Now we Taylor expand $h$ near $(x_n^*, x_{-n}^*, \omega^*, k^*) = (0, 0, \omega^*, k^*)$, keeping $k^*$ fixed. For this purpose, we shall for $(u, v) \in H_0^0(\mathbb{R}/\mathbb{Z}) \times H_{-1}^0(\mathbb{R}/\mathbb{Z}) = K \oplus M^*$ write

$$h(u(s), v(s)) = \sum_{n \in \mathbb{Z}_{\neq 0}} x_n(e^{2i\pi ns}, 2i\pi \omega^* e^{2i\pi ns}) + y_n(2i\pi \omega^* e^{2i\pi ns}, e^{2i\pi ns}).$$

(6.1)

Note that the variables $x_{\pm n^*}$ are used to describe the elements of $K$ while the others describe the elements of $M^*$. In terms of the variables $x_n^*, y_n, \omega$ and $k$, the quadratic part of the Hamiltonian function $\tilde{H}$ reads

$$\tilde{H}_2(u, v, \omega, k) := \int_{\mathbb{R}/\mathbb{Z}} \omega u(s) \frac{dv(s)}{ds} + \frac{1}{2} v(s)^2 + \frac{1}{2} (u(s) - u(s + k))^2 ds =$$

$$\sum_{n \in \mathbb{Z}_{>0}} (4 \sin^2 nk\pi + 4n^2 \omega^2(\omega^*)^2 - 8\pi^2 n^2 \omega^* \omega) x_n x_{-n} +$$

$$\sum_{n \in \mathbb{Z}_{>0}} (1 + 8n^2 \omega^* \omega + 16\pi^2 n^2 (\omega^*)^2 \sin^2 nk\pi) y_n y_{-n} +$$

$$\sum_{n \in \mathbb{Z}_{\neq 0}} 2i\pi(4 \sin^2 n^2 (\omega^*)^2 \omega - 4\omega^* \sin^2 nk\pi + \omega^* - \omega) x_n y_{-n}.$$  

(6.2)

For the full Hamiltonian $\hat{H}$ we have that $\hat{H}(u, v, \omega, k) = \tilde{H}_2(u, v, \omega, k) + O(||(u, v)||^3)$, uniform in $\omega$ and $k$.

Now recall that $h$ is obtained from $\tilde{H}$ by viewing in $\tilde{H}$ the dependent variables $x_n(n \neq \pm n^*)$ and $y_n$ as functions of the independent variables $x_{\pm n^*}, \omega, k$ for $K \times \mathbb{R}^2$.

These functions are defined by the relation $\pi_{\mathcal{M}} F((u, v)(x_n^*, x_{-n}^*, \omega, k), \omega, k) = 0$. Differentiation of this identity teaches us that $x_n = O(||(x_n^*, x_{-n}^*, \omega - \omega^*, k - k^*)||^2)$ for $n \neq \pm n^*$ and $y_n = O(||(x_n^*, x_{-n}^*, \omega - \omega^*, k - k^*)||^2)$ for all $n$. With this in mind, one sees from (6.2) that

$$h(x_n^*, x_{-n}^*, \omega, k^*) = -8(n^*)^2 \omega^2(\omega - \omega^*) x_n^* x_{-n}^* + O(||(x_n^*, x_{-n}^*)||^3) + O(||(\omega - \omega^*, x_n^*, x_{-n}^*)||^4).$$
As a result we see that \( \frac{\partial^2 h}{\partial \omega \partial a} \bigg|_{a=0,\omega=\omega^*} \neq 0 \). Therefore, by the implicit function theorem we can for every small positive value of \( a = \frac{\varepsilon}{4} \) find an \( \omega = \omega(\varepsilon) \) such that \( d_x h(\frac{\varepsilon}{2} e^{i\omega a}, \omega(\varepsilon), k^*) = 0 \). \( \square \)

Using that for \((u,v) \in \mathcal{K}\) we wrote \((u(s), v(s)) = x_n^*(e^{2\pi in^*s}, 2\pi in^*\omega^* e^{2\pi in^*s}) + x_{-n^*}(e^{-2\pi in^*s}, -2\pi in^*\omega^* e^{-2\pi in^*s})\), we obtain that our solutions are exactly of the form given in Theorem 1.1 in the introduction. To summarize, we have found for every fixed \( k^* \) a one-parameter family of wave trains, parameterized by their amplitude \( \varepsilon \).

7 Bichromatic wave trains

In this section and the next, we will prove Theorem 1.2 on the existence of bichromatic wave trains. This is the resonant situation, that is we choose parameter values \((k^*, \omega^*)\) with \( \omega^* \neq 0 \) that lie on the intersection of exactly two of the curves \( \{(\omega, k) \mid n\pi \omega = \pm \sin(nk\pi)\}_{n \in \mathbb{Z}^>0} \). We have depicted these dispersion curves in Figure 1, which clearly displays several transversal intersection points. In fact, we will assume that we have

\[
\text{Figure 1: The dispersion curves } \omega(k) = \pm \frac{1}{nk} \sin nk\pi \text{ for } n = 1, 2, 3, 4, 5.
\]

found \( p \neq q \in \mathbb{Z}^>0 \), \( k^* \) and \( \omega^* \neq 0 \) for which \( \omega^* = \pm \frac{1}{pk} \sin(pk^*\pi) \) and \( \omega^* = \pm \frac{1}{qk} \sin(qk^*\pi) \), while \( \omega^* \neq \pm \frac{1}{n\pi} \sin(nk^*\pi) \) for all \( n \in \mathbb{Z}^>0 \) with \( n \neq \pm p, \pm q \). Note that this implies that \( p, q \geq 2 \).

In the case described above, the kernels \( \mathcal{K} \) and \( \mathcal{K}^* \) have dimension 4, whence the reduced bifurcation equation \( d_x h = 0 \) will be considerably more complicated than in the previous section.

Remark 7.1 (2:3 resonance) A nontrivial example can be found by choosing \( p = 2 \) and \( q = 3 \). The intersection of the curves \( 2\pi \omega = \sin 2\pi k \) and \( 3\pi \omega = -\sin 3\pi k \) can be found by equating \( 2 \sin 3k^*\pi = -3 \sin 2k^*\pi \). Expanding left and right hand side in terms of \( \sin k^*\pi \) and \( \cos k^*\pi \) we obtain the equation \( 4 \cos^2 k^*\pi + 3 \cos k^*\pi - 1 = 0 \), whence \( \cos k^*\pi = -1, \frac{1}{4} \). Hence the point \((\omega^*, k^*) = (\frac{1}{16}, \frac{1}{4} \arccos \frac{1}{4}) \approx (0.0771, 0.4196)\) lies on the intersection of the two bifurcation curves \( 2\pi \omega = \sin 2k\pi, 3\pi \omega = -\sin 3k\pi \). Since no other curves pass through this point, the kernels are 4-dimensional, and we can assign to \( \mathcal{K} \) the coordinates \((x_2, x_{-2}, x_3, x_{-3})\). Another nontrivial example is \( p = 3, q = 5 \).
Denote by \( \gcd(p, q) \) the greatest common divisor of \( p \) and \( q \) and define \( \bar{p} = p / \gcd(p, q), \bar{q} = q / \gcd(p, q) \). The invariance of \( h \) under the action of \( \Delta \) implies that \( h \) must be a smooth function of \( \omega, k \) and the invariants

\[
a := x_p x_{-p} , \quad b := x_q x_{-q} , \quad c := i(x_{-p}^\bar{q} x_{\bar{q} p} - x_{\bar{q} p} x_{-p}^\bar{q}) , \quad d = x_{-p}^\bar{q} x_{\bar{q} p} + x_{\bar{q} p} x_{-p}^\bar{q} .
\]

These definitions are such that \( a, b, c \) and \( d \) are real when \( x_p = \bar{x}_{-p} \) and \( x_q = \bar{x}_{-q} \), i.e. when \( (u, v) \) is real-valued. Moreover, the invariants satisfy the relation

\[
c^2 + d^2 = a^\bar{q} b^\bar{q} ,
\]

and \( R \) acts on them as

\[
R : a \mapsto a , \quad b \mapsto b , \quad c \mapsto (-1)^{\bar{p} + \bar{q} + 1} c , \quad d \mapsto (-1)^{\bar{p} + \bar{q}} d .
\]

The invariance of \( h \) under this action of \( R \) implies that \( h \) is in fact a function of \( \omega, k, a, b, c \) and \( d^2 \) if \( \bar{p} + \bar{q} \) is odd, whence relation (7.1) implies that in fact \( h \) is a smooth function of \( \omega, k, a, b, c \) and \( d \) when \( \bar{p} + \bar{q} \) is odd. Similarly, \( h \) is a smooth function of \( \omega, k, a, b, c \) and \( d \) when \( \bar{p} + \bar{q} \) is even.

The following theorem describes the solutions to the reduced bifurcation equation \( d_x h = 0 \) under some mild nondegeneracy conditions.

**Theorem 7.2 (Resonant wave trains)** Assume that \( p, q \in \mathbb{Z}_{>0} \) and \( k^* \) and \( \omega^* > 0 \) satisfy

\[
\omega^* = \pm \frac{1}{p^*} \sin(pk^* \pi) \quad \text{and} \quad \omega^* = \pm \frac{1}{q^*} \sin(qk^* \pi) \quad \text{and} \quad \omega^* \neq \pm \frac{1}{mn} \sin(nk^* \pi) \quad \text{for all} \quad n \in \mathbb{Z}_{\neq 0} \quad \text{not equal to} \quad \pm p, \pm q.
\]

Denote \( C = c \) if \( \bar{p} + \bar{q} \) is odd and \( C = d \) if \( \bar{p} + \bar{q} \) is even and assume that:

i) The matrix

\[
\begin{pmatrix}
\frac{\partial^2 h}{\partial k^2} & \frac{\partial^2 h}{\partial k \partial b} \\
\frac{\partial^2 h}{\partial k \partial a} & \frac{\partial^2 h}{\partial b^2}
\end{pmatrix}
\]

\[
\bigg|_{(a, b, C, \omega, k) = (0, 0, 0, \omega^*, k^*)}
\]

is invertible.

ii) \( \frac{\partial h}{\partial C}(0, 0, 0, \omega^*, k^*) \neq 0 \).

Then there are unique analytic functions \( \omega_{\pm} = \omega_{\pm}(\varepsilon) \) and \( k_{\pm} = k_{\pm}(\varepsilon) \) with the property that \( \lim_{|\varepsilon| \to 0} \omega_{\pm}(\varepsilon) = \omega^* \), \( \lim_{|\varepsilon| \to 0} k_{\pm}(\varepsilon) = k^* \) such that the local solution set to the reduced bifurcation equation \( d_x h(x_p, x_{-p}, x_q, x_{-q}, \omega, k) = 0 \) is given by

\[
x_p = \frac{\varepsilon_1}{2} e^{i\tilde{\phi}_0} , \quad x_q = \frac{\varepsilon_2}{2} e^{i(\tilde{\phi}_0 + \kappa_{\pm})} , \quad \omega = \omega_{\pm}(\varepsilon) , \quad k = k_{\pm}(\varepsilon) ,
\]

for \( \varepsilon_1, \varepsilon_2 \geq 0 \) close enough to 0 and every \( \tilde{\phi}_0 \in \mathbb{R}/2\pi \mathbb{Z} \). Here \( \kappa_+ = \frac{\pi}{2p} \), \( \kappa_- = -\frac{\pi}{2p} \) if \( \bar{p} + \bar{q} \) is odd, whereas \( \kappa_+ = 0 \), \( \kappa_- = \frac{\pi}{p} \) if \( \bar{p} + \bar{q} \) is even.

**Proof:** We shall explicitly treat the case that \( \bar{p} + \bar{q} \) is odd, so that \( C = c \). The analysis is similar in the even case.

First of all, recall that the set of real-valued functions is characterized by the restrictions \( \bar{x}_p = x_{-p}, \bar{x}_q = x_{-q} \). This restriction reduces the four equations \( d_x h(x_p, x_{-p}, x_q, x_{-q}, \omega, k) = 0 \),
\(\omega, k) = 0\) to two independent complex equations in the variables \((x_p, x_q, \omega, k) \in \mathbb{C}^2 \times \mathbb{R}^2\), which, considering \(h\) as a real-valued function of the real variables \(a = x_p \bar{x}_p, b = x_q \bar{x}_q, c = i(\bar{x}_p x_q - x_p \bar{x}_q), \omega\) and \(k\), read:

\[
x_p \frac{\partial h}{\partial a} + \tilde{q} \frac{\partial h}{\partial c} x_p \bar{x}_q^{-1} x^2_p = 0 , \quad x_q \frac{\partial h}{\partial b} - \tilde{p} \frac{\partial h}{\partial c} x_p \bar{x}_q^{-1} = 0 .
\]

The next observation is that the solution set of these equations does not change if we multiply the first equation by \(\bar{x}_p\) and the second by \(\bar{x}_q\), to obtain the equations:

\[
|x_p|^2 \frac{\partial h}{\partial a} + \tilde{q} |x_q|^2 |x_p|^2 \frac{\partial h}{\partial c} x^2_p = 0 , \quad |x_q|^2 \frac{\partial h}{\partial b} - \tilde{p} |x_q|^2 x^2_p \bar{x}_q = 0 .
\]

Obviously, \(x_p = x_q = 0\) constitutes a trivial solution, whereas setting \(x_p = 0, x_q \neq 0\) or \(x_p \neq 0, x_q = 0\) respectively leads to the equations \(\frac{\partial h}{\partial b}(0, b, 0, \omega, k) = 0\) and \(\frac{\partial h}{\partial a}(a, 0, 0, \omega, k) = 0\). In virtue of the nondegenerate dependence of \((\frac{\partial h}{\partial a}, \frac{\partial h}{\partial c})\) on the parameters \(\omega, k\) the first equation can always be solved for a unique \(\omega(b, k)\) or \(k(b, \omega)\) and the second equation can always be solved for \(\omega(a, k)\) or \(\omega(k, a)\).

The case \(x_p, x_q \neq 0\) is of course more interesting: knowing that \(|x_p|^2 \frac{\partial h}{\partial a}, |x_q|^2 \frac{\partial h}{\partial b}\) and \(\frac{\partial h}{\partial c}\) are real and that \(\frac{\partial h}{\partial a} \neq 0\) locally, it must then be true that

\[\tilde{x}_p \tilde{x}_q = \pm i |x_p|^2 |x_q|^\tilde{p} \tilde{q} \in i \mathbb{R}\]

so that \(c = \mp 2 |x_p|^2 |x_q|^\tilde{p}\) and, because \(\tilde{q}, \tilde{p} \geq 2\), by dividing these equations by \(|x_p|^2\) and \(|x_q|^2\) respectively, we obtain the equations

\[
\frac{\partial h}{\partial a} \mp \tilde{q} \frac{\partial h}{\partial c} |x_p|^{-2} |x_q|^{\tilde{p}} = 0 , \quad \frac{\partial h}{\partial b} \pm \tilde{p} \frac{\partial h}{\partial c} |x_q|^{-2} = 0
\]

Due to nondegeneracy, these equations, being smooth in the variables \(|x_p|, |x_q|, \omega\) and \(k\) can be solved simultaneously for unique \(\omega_\pm, k_\pm\) functions of \(|x_p|\) and \(|x_q|\).

It now remains to solve the equation

\[
\text{Re} \left( \tilde{x}_p \tilde{x}_q \right) = 0
\]

which exactly describes the set of points \((x_p, x_q)\) for which there exist \(\omega, k\) such that \(h(\cdot, \omega, k)\) has a stationary point at \((x_p, x_q)\). The above equation is of course not hard to solve and it turns out that its solution set consists exactly of the points

\[
(x_p, x_q) = \left( \frac{\varepsilon_1}{2} e^{i \tilde{q} \phi_0}, \frac{\varepsilon_2}{2} e^{i (\tilde{q} \phi_0 + \kappa_\pm)} \right),
\]

where \(\varepsilon_1 \geq 0\) and \(\varepsilon_2 \geq 0\), \(\phi_0 \in \mathbb{R}/2\pi \mathbb{Z}\) and \(\kappa_+ = \frac{\pi}{2\tilde{p}}, \kappa_- = -\frac{\pi}{2\tilde{p}}\). Each torus \(\{(x_p, x_q) \in \mathbb{C}^2 | |x_p| = \frac{\varepsilon_1}{2} > 0, |x_q| = \frac{\varepsilon_2}{2} > 0\}\) contains exactly two \(\Delta\)-orbits of solutions, which are mapped to each other under \(R\), i.e. they are not invariant under \(R\).

In the case that \(\tilde{p} + \tilde{q}\) is even, the analysis is completely similar, except that it turns out that \((x_p, x_q)\) now are the solutions of \(\text{Im} \left( \tilde{x}_p \tilde{x}_q \right) = 0\). The solutions are then given by a similar formula, with \(\kappa_+ = 0, \kappa_- = \frac{\pi}{\tilde{p}}\), and they are invariant under \(R\). \(\square\)

The following theorem shows that conditions \(i)\) and \(ii)\) of Theorem 7.2 are generically satisfied for the FPU lattice. To formulate it, it is convenient to write the potential energy density function as \(W(z) = \frac{1}{2z^2} + \frac{\alpha}{4} z^3 + \frac{\beta}{4} z^4 + \ldots + \frac{\gamma}{(p+q-1)!} z^{p+q-1} + \frac{\delta}{(p+q)!} z^{p+q} + \ldots\), i.e. \(\gamma = \frac{\beta + q - 1}{d z^{p+q-1}}(0)\) and \(\delta = \frac{\beta + q}{d z^{p+q}}(0)\).
Theorem 7.3 (Generic nondegeneracy conditions) Assume that \( p, q \in \mathbb{Z}_{>0} \) and \( k^* \) and \( \omega^* > 0 \) satisfy \( \omega^* = \pm \frac{1}{pq} \sin(pk\pi) \) and \( \omega^* = \pm \frac{1}{nq} \sin(qk\pi) \) for all \( n \neq \pm p, \pm q, 0 \). Then:

i) The matrix (7.2) is invertible if and only if the curves \( \omega = \pm \frac{1}{pq} \sin(pk\pi) \) and \( \omega = \pm \frac{1}{nq} \sin(qk\pi) \) intersect transversely at \((k^*, \omega^*)\).

ii) For FPU, \( \frac{\partial h}{\partial \omega}(0, 0, 0, \omega^*, k^*) \) is a function of \((\alpha, \beta, \ldots, \gamma, \delta)\). This function is of the form \( \frac{\partial h}{\partial \omega}(0, 0, 0, \omega^*, k^*) = T(\alpha, \beta, \ldots, \gamma) + \tau \delta \) where \( T \) is some smooth function and \( \tau \) is a nonzero constant. In particular, for each value of \((\alpha, \beta, \ldots, \gamma)\) there is at most one \( \delta \) for which \( \frac{\partial h}{\partial \omega}(0, 0, 0, \omega^*, k^*) = 0 \).

The proof of this theorem is based on an explicit computation of Taylor coefficients of \( h \) and is given in the next section.

Theorems 7.2 and 7.3 together prove Theorem 1.2 of the introduction.

8 Proof of generic nondegeneracy conditions

In this section we shall prove Theorem 7.3, by computing a number of derivatives of the reduced Hamiltonian function \( h \) explicitly. The idea is to calculate a Taylor expansion for the reduced Hamiltonian function \( h \) for \((\omega, k)\) close to \((\omega^*, k^*)\) and \( x = (x_p, x_{-p}, x_q, x_{-q}) \) close to zero. It will be convenient to expand \((u, v)\) again in terms of the variables \( \{x_n, y_n\}_{n \in \mathbb{R}^2} \) as in formula (6.1). Now the variables \( \{x_{\pm p}, x_{\pm q}\} \) are used to describe the elements of \( \mathcal{K} \) while the others describe the elements of \( \mathcal{M}^* \).

**Proof (of Theorem 7.3, part i.)** Recall that in terms of the variables \( x_n, y_n, \omega, k \), the quadratic part of the Hamiltonian function is given by formula (6.2) and that \( h \) is obtained from \( \tilde{H} \) by viewing the dependent variables \( x_n(n \neq \pm p, \pm q) \) and \( y_n \) as functions of the six independent coordinates \( x_n(n = \pm p, \pm q), \omega, k \) for \( \mathcal{K} \times \mathbb{R}^2 \) that satisfy \( x_n = \mathcal{O}(||x_p, x_{-p}, x_q, x_{-q}, \omega - \omega^*, k - k^*)||^2) \) for \( n \neq \pm p, \pm q \) and \( y_n = \mathcal{O}(||x_p, x_{-p}, x_q, x_{-q}, \omega - \omega^*, k - k^*)||^2) \) for all \( n \). With this in mind, one sees from (6.2) that

\[
h(x_p, x_{-p}, x_q, x_{-q}, \omega, k) = x_p x_{-p} \left\{ 4p\pi \sin(2pk\pi)(k - k^*) - 8p^2\pi^2 \omega^*(\omega - \omega^*) \right\} \\
+ x_q x_{-q} \left\{ 4q\pi \sin(2qk\pi)(k - k^*) - 8q^2\pi^2 \omega^*(\omega - \omega^*) \right\} \\
+ \mathcal{O}(||x_p, x_{-p}, x_q, x_{-q}||^3) + \mathcal{O}(||x_p, x_{-p}, x_q, x_{-q}, \omega - \omega^*, k - k^*)||^4) 
\]

From this expression one can compute the derivative matrix (7.2) and observe that its determinant is nonzero exactly when the derivatives of \( k \mapsto \pm \frac{1}{pq} \sin(pk\pi) \) and \( k \mapsto \pm \frac{1}{nq} \sin(qk\pi) \) at \( k^* \) are different. This proves part i) of Theorem 7.3. \( \square \)

**Proof (of Theorem 7.3, part ii.)** For the second part of Theorem 7.3 we may set \( \omega = \omega^* \) and \( k = k^* \) to obtain the implicit equations for the dependent variables \( x_n(n \neq \pm p, \pm q) \) and \( y_n \) in terms of the independent variables \( x_{\pm p}, x_{\pm q} \).

It suffices to prove part ii) of Theorem 7.3 under the assumption that \( W(z) = \frac{1}{2}z^2 + \frac{\delta}{(p+q)!}z^{p+q} \). Equating to zero all inner products of \( F(u, v, \omega^*, k^*) \) with basis vectors for \( \mathcal{M} \)
then gives us:

\[ y_n = 0 \ (n \neq \pm p, \pm q) , \]

\[ 4(\pi^2 n^2 (\omega^*)^2 - \sin^2 nk^* \pi)x_n = \delta_C \ (n \neq \pm p, \pm q) , \]

\[ (1 + 4\pi^2 n^2 (\omega^*)^2)^2 y_n = 2\pi i n \omega^* \delta_C \ (n = \pm p, \pm q) , \]

in which for \( n \in \mathbb{Z}_{\neq 0} \),

\[
C_n := \frac{1}{(\bar{p} + \bar{q} - 1)!} \int_{\mathbb{R}/\mathbb{Z}} e^{-2\pi i n s} [(u(s) - u(s + k^*))^\bar{p} + \bar{q} - 1 - (u(s - k^*) - u(s))^\bar{p} + \bar{q} - 1] ds = \\
\frac{1}{(\bar{p} + \bar{q} - 1)!} \sum_{m \in \mathbb{Z}^{\bar{p} + \bar{q} - 1}} \left( \operatorname{Re} \prod_{j=1}^{\bar{p} + \bar{q} - 1} (1 - e^{2\pi i m_j k^*}) \right) \prod_{j=1}^{\bar{p} + \bar{q} - 1} (x_{m_j} + 2\pi i m_j \omega^* y_{m_j}) .
\]

One now observes from these equations that \( \frac{\partial y_n}{\partial x_k} (0, \omega^*, k^*) = 0 \) for all \( n \) and \( \frac{\partial y_n}{\partial x_k} (0, \omega^*, k^*) = \delta^k_n \) (the Kronecker delta), which implies that \( C_n = \mathcal{O}(\| (x_p, x_{-p}, x_q, x_{-q}) \|^{\bar{p} + \bar{q} - 1}) \) and hence

\[
y_n = \mathcal{O}(\| (x_p, x_{-p}, x_q, x_{-q}) \|^{\bar{p} + \bar{q} - 1}) \quad \text{and for } n \neq \pm p, \pm q , \quad x_n = \mathcal{O}(\| (x_p, x_{-p}, x_q, x_{-q}) \|^{\bar{p} + \bar{q} - 1}) .
\]

Now we again compute the reduced Hamiltonian function \( h(\cdot, \omega^*, k^*) \):

\[
h(x_p, x_p, x_q, x_{-q}, \omega^*, k^*) = \sum_{n \in \mathbb{Z}_{>0}} (4\sin^2 nk^* \pi - 4\pi^2 n^2 (\omega^*)^2)x_n x_{-n} + \\
\sum_{n \in \mathbb{Z}_{>0}} (1 + 8\pi^2 n^2 (\omega^*)^2 + 16\pi^2 n^2 (\omega^*)^2 \sin^2 nk^* \pi) y_n y_{-n} + \\
\sum_{n \in \mathbb{Z}_{\neq 0}} 8\pi i n \omega^* (\pi^2 n^2 (\omega^*)^2 - \sin^2 nk^* \pi)x_n y_{-n} + \\
\frac{\delta}{(\bar{p} + \bar{q})!} \sum_{m \in \mathbb{Z}^{\bar{p} + \bar{q}}} \prod_{j=1}^{\bar{p} + \bar{q}} (1 - e^{2\pi i m_j k^*}) (x_{m_j} + 2\pi i m_j y_{m_j}) \\
= \frac{\delta}{(\bar{p} + \bar{q})!} \sum_{m \in \{ \pm p, \pm q \}^{\bar{p} + \bar{q}}} \prod_{j=1}^{\bar{p} + \bar{q}} (1 - e^{2\pi i m_j k^*}) x_{m_j} + \mathcal{O}(\| (x_p, x_{-p}, x_q, x_{-q}) \|^{2(\bar{p} + \bar{q} - 1)}) \\
= g(a, b) \pm \delta \frac{2\bar{p} + \bar{q}}{p|q|!} \sin(\bar{p} \pi k^*) \sin(\bar{q} \pi k^*) C + \mathcal{O}(\| (x_p, x_{-p}, x_q, x_{-q}) \|^{2(\bar{p} + \bar{q} - 1)}) .
\]

**Remark 8.1** The function \( g(a, b) \) appears only when \( \bar{p} + \bar{q} \) is even and is irrelevant for the analysis, as is the plus or minus sign that depends on the exact values of \( \bar{p} \) and \( \bar{q} \).

Finally, we have here used the fact that when \( \sum_{j=1}^{\bar{p} + \bar{q}} m_j = 0 \), then \( \prod_{j=1}^{\bar{p} + \bar{q}} (1 - e^{2\pi i m_j k^*}) = e^{\pi i (\sum_{j=1}^{\bar{p} + \bar{q}} m_j)} k \prod_{j=1}^{\bar{p} + \bar{q}} (e^{-\pi i m_j k} - e^{\pi i m_j k}) = (-2i)^{\bar{p} + \bar{q}} \prod_{j=1}^{\bar{p} + \bar{q}} \sin m_j \pi k .
\]

We have thus proved that \( \tau = \frac{2\bar{p} + \bar{q}}{p|q|!} \sin(\bar{p} \pi k^*) \sin(\bar{q} \pi k^*) \neq 0 \). This concludes the proof of part \( ii \) of Theorem 7.3. \( \square \)
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References


