Derivability and Admissibility of Inference Rules in Abstract Hilbert Systems

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Overview

About the following sections of the report (2003) with the same title:

2 • What is an inference rule?
   • An ‘extensional’ abstract notion of inference rule. Some problems.
   • An ‘intensional’ abstract notion of rules. (Motivated by: abstract reduction systems vs. abstract rewrite systems.)
   • Abstract Hilbert Systems (AHS’s), and
   • Abstract Hilbert Systems with rule/axiom names (n-AHS’s).
   • Three consequence relations on these systems.

3 • Definition of “rule admissibility” in (n-)AHS’s.
   • Definition of three versions of “rule derivability” in (n-)AHS’s.
   • Some basic facts about these notions.
Comparing abstract Hilbert systems w.r.t. consequence relations, rule derivability and admissibility: Introducing relations between abstract Hilbert systems.

“Interrelation Prisms” between these relations.

Three notions of “mimicking derivation”.

Four notions of “rule elimination” in (n)-AHS’s and their relationships with rule derivability and admissibility.

Some notions of “strong rule elimination” in n-AHS’s, and their relationship with rule derivability and admissibility.

(Appendix E) Relationship of (n)-AHS’s with sequent-style “Hilbert systems for consequence” à la Avron.
Rule derivability and admissibility (informal def.’s)

Let $S$ a formal system, $R$ a rule ‘on’ of $S$.

‘Definition’. $R$ is derivable in $S$ if and only if every instance of $R$ can be ‘modelled’, or ‘mimicked’, by an appropriate derivation in $S$.

‘Definition’. Frequently, two versions to define “rule admissibility”: $R$ is admissible in $S$ if and only if . . .

(i) . . . by adding $R$ to $S$ not more theorems become derivable;

[Kleene, 1952; Lorenzen, 1955; Schütte, 1960]

(ii) . . . the theory of $S$ (the collection of theorems of $S$) is closed under applications of $R$ ($R$ is correct for $S$).

Both definitions presuppose the concept of inference rule.
What is an inference rule?

Rules in logic are defined in a variety of ways; here are some examples:

\[
\frac{A \rightarrow B}{\text{MP}} \quad \frac{[A]^u}{\text{I}, u} \\
\frac{D_1}{\exists x A, \Gamma \Rightarrow \Delta} \\
\frac{A[t/x], \Gamma \Rightarrow \Delta}{\text{L}\exists} \\
\frac{p_1 \xrightarrow{a} p_2}{\text{L}+} \\
\frac{p_1 + q \xrightarrow{a} p_2}{\text{UFP}}
\]

Mostly, rules are defined \textit{schematically} (s.a.), using substitution on a meta-language of the formula language.

\textbf{Desirable} for studying general properties of rule derivability and admissibility: an \textit{abstract notion of inference rule} that neglects language-specific details.
pure Hilbert Systems (informally)

- **Formulas, axioms.**

- **Rules** with applications \( \frac{A_1 \ldots A_n}{B} R \) or \( B \quad R \).

- In *derivations* assumptions are allowed to be made.

- Rules are *pure*: An application of a rule \( R \) in a derivations \( D \)

\[
\begin{array}{c}
\mathcal{D}_1 \\
\mathcal{D}_2 \\
\vdots \\
\mathcal{D}_n
\end{array}
\begin{array}{c}
A_1 \\
\ldots \\
A_n
\end{array}
\frac{B}{R}
\]

does not depend on the presence, or absence, of assumptions in the subderivations \( \mathcal{D}_1, \ldots, \mathcal{D}_n \). Example of a *impure* Hilbert-system rule: \( \frac{\phi}{\Box \phi} \text{UG} \)
An ‘extensional’ abstract notion of rule

Definition (“Rule descriptions” in pure Hilbert-systems [Hindley, Seldin]). Let $n \in \omega$, $Fo$ a nonempty set.

A rule description for an $n$-premise rule on $Fo$ is a partial function

$$\Phi : \underbrace{Fo \times \ldots \times Fo}_n \rightarrow Fo ;$$

it describes the rule $R_\Phi$ defined by:

$$\frac{A_1 \ldots A_n}{B} \text{ is application of } R_\Phi \text{ iff } \Phi(A_1, \ldots, A_n) = B .$$

There are, however, some problems connected with rule descriptions.
Problems with rule descriptions (I)

Rules that allow more than one conclusion to be drawn from a given sequence of premises, e.g.:

\[
\begin{align*}
\frac{A}{A \lor B} \lor I_R \\
\frac{\forall x A}{A[t/x]} \forall E
\end{align*}
\]

Definition ("Rule descriptions", generalized version).

A rule description for an \( n \)-premise rule on \( Fo \) is a function

\[
\Phi : (Fo)^n \rightarrow \mathcal{P}(Fo)
\]

it describes the rule \( R_\Phi \) defined by:

\[
\begin{align*}
\frac{A_1 \ldots A_n}{B} \quad \text{is application of } R_\Phi \quad \text{iff} \quad B \in \Phi(A_1, \ldots, A_n).
\end{align*}
\]
Problems with rule descriptions (II)

Rules with ‘behaviourally equivalent’ applications, i.e. applications with the same sequence of premises and the same conclusion:

\[
\frac{A_1 \land A_2}{A_i} \land E \quad (i \in \{1, 2\})
\]

has, for example, the two different applications

\[
\frac{(x = 0) \land (x = 0)}{x = 0} \land E \quad \frac{(x = 0) \land (x = 0)}{x = 0} \land E
\]

Such **syntactic accidents** call for a different abstract framework.
(Problems with) Abstract Reduction Systems

Definition (Klop). An abstract reduction system is a structure \( \langle A, \rightarrow \rangle \) consisting of a set \( A \) with a binary reduction relation.

Example. Consider the TRS \( \mathcal{T} \)

\[
f(x) \rightarrow x.
\]

There are two steps from \( f(f(a)) \),

\[
f(f(a)) \rightarrow f(a) \quad \text{and} \quad f(f(a)) \rightarrow f(a),
\]

both of which give rise to the same step

\[
f(f(a)) \rightarrow_{\mathcal{T}} f(a)
\]

in the extensional description of \( \mathcal{T} \) as abstract reduction system \( (Ter, \rightarrow_T) \); this is called a ‘syntactic accident’ (J.J. Lèvy).
Abstract Rewriting Systems

Definition (van Oostrom, de Vrijer). An *abstract rewriting system* is a quadrupel $\langle A, \Phi, \text{src}, \text{tgt} \rangle$ with

- $A$ a set of *objects*,
- $\Phi$ a set of *steps*,
- and $\text{src}, \text{tgt} : \Phi \rightarrow A$ the *source* and *target* functions.

Visualization of a step as a ‘graph hyperedge’:
An ‘intensional’ abstract notion of rule

premise 1 \rightarrow \text{premise function} \rightarrow \text{rule application} \rightarrow \text{conclusion function} \rightarrow \text{conclusion}

premise 2 \rightarrow \ldots \rightarrow \text{premise } n
An intensional abstract notion of rule

Let, for \( X \) a set, \( Seqs_f(X) \) be the set of \textit{finite sequences} over \( X \).

\textbf{Definition.} Let \( Fo \) be a set.

An \textit{AHS-rule} \( R \) \textit{on} \( Fo \) is a triple \( \langle Apps, \text{prem}, \text{concl} \rangle \) where

- \( Apps \) is the set of \textit{applications} of \( R \),
- \( \text{prem} : Apps \rightarrow Seqs_f(Fo) \) is the \textit{premise} function of \( R \),
- \( \text{concl} : Apps \rightarrow Fo \) is the \textit{conclusion} function of \( R \).

By \( \mathcal{R}(Fo) \) we denote the \textit{class of all AHS-rules} on \( Fo \).

(Later an AHS-rule of \( Fo \) will only be called a \textit{rule on} \( Fo \).)
Visualization of applications of AHS-rules

Visualization as ‘graph hyperedges’ of

– a zero premise application $\alpha$ of an AHS-rule $R_1$, and
– of an application $\alpha'$ of an AHS-rule $R_2$. 
Abstract Hilbert Systems

Definition. An abstract Hilbert system (an AHS) \( \mathcal{H} \) is a triple \( \langle \mathcal{F}_o, \mathcal{A}_x, \mathcal{R} \rangle \) where

- \( \mathcal{F}_o, \mathcal{A}_x \) and \( \mathcal{R} \) the sets of formulas, axioms, and rules of \( \mathcal{H} \),
- \( \mathcal{A}_x \subseteq \mathcal{F}_o \),
- every \( \mathcal{R} \in \mathcal{R} \) is an AHS-rule on \( \mathcal{F}_o \).

We write \( \mathcal{H} \) for the class of all AHS’s.
Derivations in an AHS

For a set $X$, we denote by $\mathcal{M}_f(X)$ the set of finite multisets over $X$.

**Notation.** Let $\mathcal{H}$ be an AHS with formula set $Fo$.

By $\text{Der}(\mathcal{H})$ we denote the set of derivations in $\mathcal{H}$. And for a derivation $D$ in $\mathcal{H}$, we denote by

- $\text{assm}(D) \in \mathcal{M}_f(Fo)$ the multiset of assumptions of $D$, and by
- $\text{concl}(D) \in Fo$ the conclusion of $D$. 
An abstract notion of rule with (rule) names

premise function

application of rule $R$

premise 1

premise $n$

name of $R$

conclusion function

conclusion
Abstract Hilbert systems with names

Definition. An abstract Hilbert system with names (for axioms and rules) (an n-AHS) $\mathcal{H}$ is a quadruple $⟨Fo, Na, nAx, nR⟩$ where

- $Fo$, $Na$, $nAx$ and $nR$ are the formulas, names, named axioms and named rules of $\mathcal{H}$,
- $nAx \subseteq Fo \times Na$,
- $nR \subseteq \mathcal{R}(Fo) \times Na$, (we allow to write $R = \langle R, \text{name}(R) \rangle$, for arbitrary $R \in nR$),
- “axiom names” in $nAx$ are different from “rule names” in $nR$,
  - different rules are differently named in $nR$.

We write $\mathcal{H}_n$ for the class of all n-AHS’s.
Visualization of applications of n-AHS-rules

Visualization as ‘graph hyperedges’ of

– a zero premise application $\alpha$ of a named rule $R_1$, and
– of an application $\alpha'$ of a named rule $R_2$ in an n-AHS $\mathcal{H}$. 
Derivations in an n-AHS

Definition. (Derivations in abstr. Hilbert systems with names).
Let $\mathcal{H} = \langle Fo, Na, nAx, nR \rangle$ be an n-AHS.

A derivation $D$ in $\mathcal{H}$ is the result (a prooftree) of carrying out a finite number of construction steps of the following three kinds:

(i) For every named axiom $\langle A, name \rangle \in nAx$, the prooftree $D$ of the form

$$
\begin{array}{c}
(name) \\
A \\
\end{array}
$$

is a derivation in $\mathcal{H}$ with conclusion $\text{concl}(D) = A$ and without assumptions, i.e. such that $\text{set}(\text{assm}(D)) = \emptyset$ holds.
(ii) For all formulas $A \in F_0$, the prooftree $D$ consisting only of the formula

$A$

is a derivation in $\mathcal{H}$ with assumptions $\text{assm}(D) = \{A\}$ and with conclusion $\text{concl}(D) = A$.

(iii) Let $R = \langle R, \text{name}(R) \rangle \in nR$ a named rule of $\mathcal{H}$, and $\alpha \in \text{Apps}_R$ an appl. of $R$. We distinguish two cases concerning the arity of $\alpha$:

Case 1. $\text{arity}_R(\alpha) = 0$: Given that $\text{concl}_R(\alpha) = A$, the prooftree

$\frac{}{A \ \text{name}(R)}$

is a derivation $D$ in $\mathcal{H}$ that has conclusion $\text{concl}(D) = A$ and no assumptions, i.e. $\text{assm}(D) = \emptyset$ holds.
Case 2. \( \text{arity}_R(\alpha) = n \in \omega \setminus \{0\} \):

Given that \( \text{prem}_R(\alpha) = \langle A_1, \ldots, A_n \rangle \) and that \( \text{concl}_R(\alpha) = A \), and given further that \( D_1, \ldots, D_n \) are derivations in \( \mathcal{H} \) with respective conclusions \( A_1, \ldots, A_n \), the prooftree of the form

\[
\begin{array}{ccc}
D_1 & & D_n \\
A_1 & \ldots & A_n \\
\hline
& & name(R)
\end{array}
\]

is a derivation \( D \) in \( \mathcal{H} \) with conclusion \( \text{concl}(D) = A \) and with assumptions and depth defined by

\[
\text{assm}(D) = \bigcup_{i=1}^{n} \text{assm}(D_i).
\]

We denote by \( \text{Der}(\mathcal{H}) \) the set of all derivations in \( \mathcal{H} \).
Three Consequence Relations on an AHS or n-AHS

**Definition.** For an AHS or n-AHS $\mathcal{H}$ we define:

\[ \Sigma \vdash_{\mathcal{H}} A \iff A \text{ is the conclusion of a derivation in } \mathcal{H} \text{ whose assumptions are contained in the set } \Sigma ; \]

\[ \Sigma \vdash^{(s)}_{\mathcal{H}} A \iff A \text{ is the conclusion of a derivation in } \mathcal{H} \text{ whose assumptions are contained in the set } \Sigma \text{ and that uses every formula in } \Sigma \text{ at least once;} \]

\[ \Gamma \vdash^{(m)}_{\mathcal{H}} A \iff A \text{ is the conclusion of a derivation in } \mathcal{H} \text{ whose assumptions are contained in the multiset } \Gamma \text{ and that uses every formula in } \Gamma \text{ precisely once.} \]
Three Consequence Relations on an AHS or n-AHS

Definition. Let $\mathcal{H}$ be an AHS or n-AHS with formula set $F_0$. We define the consequence relations $\vdash_{\mathcal{H}}$, $\vdash_{\mathcal{H}}^{(s)}$ and $\vdash_{\mathcal{H}}^{(m)}$ by setting for all $A \in F_0$, finite sets $\Sigma$ on $F_0$ and multisets $\Gamma$ on $F_0$:

$$\Sigma \vdash_{\mathcal{H}} A \iff (\exists D \in \text{Der}(\mathcal{H})) \left[ \text{set}(\text{assm}(D)) \subseteq \Sigma \land \text{concl}(D) = A \right] ,$$

$$\Sigma \vdash_{\mathcal{H}}^{(s)} A \iff (\exists D \in \text{Der}(\mathcal{H})) \left[ \text{set}(\text{assm}(D)) = \Sigma \land \text{concl}(D) = A \right] ,$$

$$\Gamma \vdash_{\mathcal{H}}^{(m)} A \iff (\exists D \in \text{Der}(\mathcal{H})) \left[ \text{assm}(D) = \Gamma \land \text{concl}(D) = A \right] ,$$

whereby $\vdash_{\mathcal{H}}$, $\vdash_{\mathcal{H}}^{(s)} \subseteq \mathcal{P}_f(F_0) \times F_0$ and $\vdash_{\mathcal{H}}^{(m)} \subseteq \mathcal{M}_f(F_0) \times F_0$. 

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The neglected consequence relation

Definition. Let $\mathcal{H}$ be an AHS or n-AHS with formula set $F_{o}$.

We define the consequence relation $\vdash_{\mathcal{H}}^{(mw)}$ by letting for all $A \in F_{o}$ and multisets $\Gamma$ on $F_{o}$

$$\Gamma \vdash_{\mathcal{H}}^{(mw)} A \iff \iff (\exists D \in Der(\mathcal{H})) [ \text{assm}(D) \subseteq \Gamma \ & \ \text{concl}(D) = A ],$$

whereby $\vdash_{\mathcal{H}}^{(mw)} \subseteq M_{f}(F_{o}) \times F_{o}$. 
Rule Admissibility

Definition. Let $\mathcal{H}$ be an AHS or n-AHS with formula set $F_0$, and let $R = \langle Apps_R, \text{prem}, \text{concl} \rangle$ be a rule on $F_0$.

The rule $R$ is admissible in $\mathcal{H}$ if and only if it holds that

$$\left( \forall \alpha \in Apps_R \right) \left[ \left( \forall A \in \text{set}(\text{prem}(\alpha)) \right) \left[ \vdash_\mathcal{H} A \right] \Rightarrow \right. \\
\left. \Rightarrow \vdash_\mathcal{H} \text{concl}(\alpha) \right],$$

i.e. iff the theory of $\mathcal{H}$ (the set of theorems of $\mathcal{H}$) is closed under applications of $R$. 
Three Versions of Rule Derivability

Definition. Let $\mathcal{H}$ be an AHS or an n-AHS. We consider a rule $R = \langle \text{Apps}_R, \text{prem}, \text{concl} \rangle$ on $\text{Fo}_\mathcal{H}$.

The rule $R$ is derivable in $\mathcal{H}$ if and only if

$$(\forall \alpha \in \text{Apps}_R) \left[ \text{set} (\text{prem}(\alpha)) \vdash_{\mathcal{H}} \text{concl}(\alpha) \right]$$

holds, that is, for all applications $\alpha$ of $R$, there exists a “mimicking derivation” $\mathcal{D}$ in $\mathcal{H}$, i.e. a derivation $\mathcal{D}$ with conclusion $\text{concl}(\alpha)$ and with its assumptions contained in $\text{set} (\text{assm}(\alpha))$. 
Three Versions of Rule Derivability

**Definition.** Let $\mathcal{H}$ be an AHS or an n-AHS. We consider a rule $R = \langle \text{Apps}_R, \text{prem, concl} \rangle$ on $F_0\mathcal{H}$.

The rule $R$ is *derivable in* $\mathcal{H}$ if and only if

$$(\forall \alpha \in \text{Apps}_R) \left[ \text{set}(\text{prem}(\alpha)) \vdash_H \text{concl}(\alpha) \right]$$

holds.

And we say that $R$ is *s-derivable in* $\mathcal{H}$ or that $R$ is *m-derivable in* $\mathcal{H}$ if and only if, respectively, the assertions (1) and (2) hold:

$$\left( \forall \alpha \in \text{Apps}_R \right) \left[ \text{set}(\text{prem}(\alpha)) \vdash_H^{(s)} \text{concl}(\alpha) \right], \quad (1)$$

$$\left( \forall \alpha \in \text{Apps}_R \right) \left[ \text{mset}(\text{prem}(\alpha)) \vdash_H^{(m)} \text{concl}(\alpha) \right]. \quad (2)$$
Formula Derivability and Admissibility

**Definition.** Let $\mathcal{H}$ be an AHS on an n-AHS with formula set $Fo$.

We call a formula $A \in Fo$ **admissible**, **derivable**, **s-derivable** and **m-derivable** if and only if

$$\vdash_{\mathcal{H}} A$$

holds, i.e. iff $A$ is a theorem of $\mathcal{H}$. 
Admissible and (s-, m-)derivable rules: Examples (I)

Example. Let $\mathcal{H}$ be the AHS without axioms and with the three rules $R_1$, $R_2$ and $R_{AA,B}$ each of which has only one application:

$\frac{C_1}{A} R_1 \frac{C_2}{A} R_2 \frac{A}{B} R_{AA,B}$.

- $C_1 \quad C_2$ is \textit{derivable} in $\mathcal{H}$:
  $\frac{C_1}{A} R_1 \frac{C_2}{A} R_2 \frac{A}{B} R_{AA,B}$.

- $C_1 \quad C_2$ is \textit{s-derivable} in $\mathcal{H}$:
  $\frac{C_1}{A} R_1 \frac{C_2}{A} R_2 \frac{A}{B} R_{AA,B}$.

- $C_1 \quad B$ is \textit{derivable} in $\mathcal{H}$:
  $\frac{C_1}{A} R_1 \frac{C_1}{A} R_1 R_{AA,B}$.

- $C_1 \quad B$ is \textit{s-derivable} in $\mathcal{H}$:
  $\frac{C_1}{A} R_1 \frac{C_1}{A} R_1 R_{AA,B}$.

- $C_1 \quad B$ is \textit{m-derivable} in $\mathcal{H}$:
  $\frac{C_1}{A} R_1 \frac{C_1}{A} R_1 R_{AA,B}$.
Admissible and (s-,m-)derivable rules: Examples (I)

Example. (Continued) Let $\mathcal{H}$ be the AHS \textit{without axioms} and with the three rules $R_1$, $R_2$ and $R_{AA.B}$ each of which has only one application:

\[
\begin{align*}
&\frac{C_1}{A} R_1 \quad \frac{C_2}{A} R_2 \quad \frac{A}{B} R_{AA.B} .
\end{align*}
\]

- $\frac{C_1}{A} C_2$ is (not s-derivable) (not m-derivable) in $\mathcal{H}$: $\frac{C_1}{A} R_1$.

- $\frac{B}{C}$ is (not derivable) (not s-derivable) (not m-derivable) in $\mathcal{H}$: Due to $B \notin Th(\mathcal{H})(= \emptyset)$. 

- $C_1$ $C_2$ is derivable (not m-derivable) in $\mathcal{H}$: $\frac{A}{B}$ $A R_{AA.B}$. 

- $\frac{C_1}{A}$ $R_1$.
Admissible and (s-, m-)derivable rules: Examples (II)

Example. Let $\mathcal{H}$ be the AHS with the single axiom

$$A$$

and with the two rules $R_{A.B}$ and $R_{A.C}$ each of which has only one application:

$$\frac{A}{B}^{R_{A.B}} \quad \frac{A}{C}^{R_{A.C}}$$

- $\frac{D}{C}$ is admissible in $\mathcal{H}$.  $\frac{D}{F}$ is admissible in $\mathcal{H}$.

- $\frac{A}{F}^{D}$ is admissible in $\mathcal{H}$: Since $D \notin Th(\mathcal{H})$. 
Admissible and (s-,m-)derivable rules: Examples (II)

Example. (Continued) Let $\mathcal{H}$ be the AHS with the single axiom $A$

and with the two rules $R_{A.B}$ and $R_{A.C}$ each of which has only one application:

$$
\frac{A}{B} R_{A.B} \quad \frac{A}{C} R_{A.C}
$$

• \( \frac{A}{D} C \) is not admissible in $\mathcal{H}$: Since $A, C \in Th(\mathcal{H})$ and $D \notin Th(\mathcal{H})$.

• \( \frac{A}{B} C \) is (not s-derivable) in $\mathcal{H}$: \( \frac{A}{B} R_{A.B} \) is (not m-derivable)
Rule Derivability and Admissibility: Basic Facts

Lemma. (Hindley, Seldin [except (iv)]).
Let $\mathcal{H}$ be an AHS and let $R$ be a rule on the set of formulas of $\mathcal{H}$.

(i) $R$ is admissible in $\mathcal{H} \iff$ the AHS $\mathcal{H} + R$ does not possess more theorems than $\mathcal{H}$.

(ii) $R$ is derivable in $\mathcal{H} \implies R$ is also admissible in $\mathcal{H}$.
(The inverse implication does not hold in general.)

(iii) $R$ is derivable in $\mathcal{H} \implies R$ is derivable in every extension of $\mathcal{H}$ that is obtained by adding new formulas, axioms and/or rules.

(iv) $R$ is $m$-derivable in $\mathcal{H} \implies R$ is $s$-derivable in $\mathcal{H} \implies R$ is derivable in $\mathcal{H}$. (The inverse implications aren’t true in general).
Rule Derivability and Admissibility: Basic Facts

**Theorem.**  Let $\mathcal{H}$ be an AHS with set $F_0$ of formulas, and $R$ a rule on $F_0$.

Then the following three statements are equivalent:

(i) $R$ is **derivable** in $\mathcal{H}$.

(ii) $R$ is **admissible** in the AHS $\mathcal{H} + \Sigma$, for every set $\Sigma$ on $F_0$.

(iii) $R$ is **admissible** in every extension of $\mathcal{H}$ that is obtained by adding new formulas, axioms and/or rules (in every extension by enlargement of $\mathcal{H}$).
(Mutual) Inclusion Relations between Abstract Hilbert Systems

We will define inclusion relations $\preceq_{P,Q}$ between AHS’s by stipulating, for all $\mathcal{H}_1, \mathcal{H}_2 \in \mathcal{H}$,

$$\mathcal{H}_1 \preceq_{P,Q} \mathcal{H}_2 \iff \begin{cases} \text{Every formula in } \mathcal{H}_1 \text{ is also a formula of } \mathcal{H}_2, \\ \text{and every object in } \mathcal{H}_1 \text{ having property } \hspace{1em} P \hspace{1em} \text{ appears in } \mathcal{H}_2 \text{ as an object with property } \hspace{1em} Q. \end{cases}$$

for properties $P$ and $Q$ of ‘objects’ in AHS’s (objects like theorems, rules, . . . , and properties like “is theorem” or “is derivable rule”).

And, for every inclusion relation $\preceq_{P,Q}$, we will define the induced mutual inclusion relation $\sim_{P,Q}$ by stipulating for all $\mathcal{H}_1, \mathcal{H}_2 \in \mathcal{H}$:

$$\mathcal{H}_1 \sim_{P,Q} \mathcal{H}_2 \iff \mathcal{H}_1 \preceq_{P,Q} \mathcal{H}_2 \hspace{1em} \& \hspace{1em} \mathcal{H}_2 \preceq_{P,Q} \mathcal{H}_1.$$
Relations between Abstract Hilbert Systems (I)

Definition. We define the inclusion relation $\preceq_{th}$ on the class $\mathcal{H}$ by stipulating for all $\mathcal{H}_1, \mathcal{H}_2 \in \mathcal{H}$:

$$\mathcal{H}_1 \preceq_{th} \mathcal{H}_2 \iff \text{Fo}_{\mathcal{H}_1} \subseteq \text{Fo}_{\mathcal{H}_2} \& \text{Th}(\mathcal{H}_1) \subseteq \text{Th}(\mathcal{H}_2).$$

We define the inclusion relations $\preceq_{rth}$, $\preceq_{(s)}$, and $\preceq_{(m)}$ on $\mathcal{H}$ by stipulating for all $\mathcal{H}_1, \mathcal{H}_2 \in \mathcal{H}$:

$$\mathcal{H}_1 \preceq_{rth} \mathcal{H}_2 \iff \text{Fo}_{\mathcal{H}_1} \subseteq \text{Fo}_{\mathcal{H}_2} \& \vdash_{\mathcal{H}_1} \subseteq \vdash_{\mathcal{H}_2},$$

$$\mathcal{H}_1 \preceq_{(s)} \mathcal{H}_2 \iff \text{Fo}_{\mathcal{H}_1} \subseteq \text{Fo}_{\mathcal{H}_2} \& \vdash_{\mathcal{H}_1}^{(s)} \subseteq \vdash_{\mathcal{H}_2}^{(s)},$$

$$\mathcal{H}_1 \preceq_{(m)} \mathcal{H}_2 \iff \text{Fo}_{\mathcal{H}_1} \subseteq \text{Fo}_{\mathcal{H}_2} \& \vdash_{\mathcal{H}_1}^{(m)} \subseteq \vdash_{\mathcal{H}_2}^{(m)}.$$
**Relations between Abstract Hilbert Systems (I)**

**Definition.** We define the inclusion relation $\preceq_{th}$ on the class $\mathcal{H}$ by stipulating for all $\mathcal{H}_1, \mathcal{H}_2 \in \mathcal{H}$:

$$\mathcal{H}_1 \preceq_{th} \mathcal{H}_2 \iff Fo_{\mathcal{H}_1} \subseteq Fo_{\mathcal{H}_2} \land (\forall A \in Fo_{\mathcal{H}_1})[ (\vdash_{\mathcal{H}_1} A) \Rightarrow (\vdash_{\mathcal{H}_2} A) ].$$

We define the inclusion relations $\preceq_{rth}$ and $\preceq_{rth}^{(m)}$ on $\mathcal{H}$ by stipulating for all $\mathcal{H}_1, \mathcal{H}_2 \in \mathcal{H}$

$$\mathcal{H}_1 \preceq_{rth} \mathcal{H}_2 \iff Fo_{\mathcal{H}_1} \subseteq Fo_{\mathcal{H}_2} \land (\forall \Sigma \in \mathcal{P}(Fo_{\mathcal{H}_1})) (\forall A \in Fo_{\mathcal{H}_1})[ (\Sigma \vdash_{\mathcal{H}_1} A) \Rightarrow (\Sigma \vdash_{\mathcal{H}_2} A) ],$$

$$\mathcal{H}_1 \preceq_{rth}^{(m)} \mathcal{H}_2 \iff Fo_{\mathcal{H}_1} \subseteq Fo_{\mathcal{H}_2} \land (\forall \Gamma \in \mathcal{M}_f(Fo_{\mathcal{H}_1})) (\forall A \in Fo_{\mathcal{H}_1})[ (\Gamma \vdash^{(m)}_{\mathcal{H}_1} A) \Rightarrow (\Gamma \vdash^{(m)}_{\mathcal{H}_2} A) ].$$
These four inclusion relations \emph{induce} respective \emph{mutual inclusion relations}: For all $\mathcal{H}_1, \mathcal{H}_2 \in \mathcal{H}$, we let

$$\mathcal{H}_1 \sim_{th} \mathcal{H}_2 \iff \mathcal{H}_1 \preceq_{th} \mathcal{H}_2 \land \mathcal{H}_2 \preceq_{th} \mathcal{H}_1$$

(if $\mathcal{H}_1 \sim_{th} \mathcal{H}_2$ holds, we say that $\mathcal{H}_1$ and $\mathcal{H}_2$ are \emph{(theorem) equivalent}; and we use analogous stipulations for the \emph{mutual inclusion relations}

$$\sim_{rth}, \sim_{rth}^{(s)} \text{ and } \sim_{rth}^{(m)}$$

(if $\mathcal{H}_1 \sim_{rth} \mathcal{H}_2$ holds, we say that $\mathcal{H}_1$ and $\mathcal{H}_2$ are \emph{equivalent with respect to relative theoremhood}).
Relations between Abstract Hilbert Systems (II)

**Definition.** We define the inclusion relation $\preceq_{adm}$ on the class $\mathcal{H}$ by stipulating for all $\mathcal{H}_1, \mathcal{H}_2 \in \mathcal{H}$:

$$\mathcal{H}_1 \preceq_{adm} \mathcal{H}_2 \iff \text{Fo}_{\mathcal{H}_1} \subseteq \text{Fo}_{\mathcal{H}_2} \land$$
$$\land (\forall A \in \text{Fo}_{\mathcal{H}_1}) \left[ A \text{ is adm. in } \mathcal{H}_1 \Rightarrow A \text{ is adm. in } \mathcal{H}_2 \right] \land$$
$$\land (\forall R \text{ rule on } \text{Fo}_{\mathcal{H}_1})$$

$$\left[ R \text{ is admissible in } \mathcal{H}_1 \Rightarrow R \text{ is admissible in } \mathcal{H}_2 \right].$$

The inclusion relations $\preceq_{der}$, $\preceq_{(s)der}$ and $\preceq_{(m)der}$ are defined analogously by using ‘derivable’, ‘s-derivable’ and ‘m-derivable’ instead of ‘admissible’.

The *induced* mutual incl. relations: $\sim_{adm}$, $\sim_{der}$, $\sim_{(s)der}$ and $\sim_{(m)der}$. 
Relations between Abstract Hilbert Systems (III)

**Definition.** We define the inclusion relation \( \leadsto_{r/adm} \) on the class \( \mathcal{H} \) by stipulating for all \( \mathcal{H}_1, \mathcal{H}_2 \in \mathcal{H} \):

\[
\mathcal{H}_1 \leadsto_{r/adm} \mathcal{H}_2 \iff \text{Fo}_{\mathcal{H}_1} \subseteq \text{Fo}_{\mathcal{H}_2} \land \\
\land (\forall A \in Ax_{\mathcal{H}_1})[A \text{ is admissible in } \mathcal{H}_2] \land \\
\land (\forall R \in R_{\mathcal{H}_1})[R \text{ is admissible in } \mathcal{H}_2].
\]

The inclusion relations \( \leq_{r/der} \), \( \preceq_{r/der}^{(s)} \) and \( \preceq_{r/der}^{(m)} \) are defined analogously by using ‘derivable’, ‘s-derivable’ and ‘m-derivable’ instead of ‘admissible’.

These four relations on \( \mathcal{H} \) *induce* the four mutual inclusion relations \( \sim_{r/adm} \), \( \sim_{r/der} \), \( \sim_{r/der}^{(s)} \) and \( \sim_{r/der}^{(m)} \) on \( \mathcal{H} \), respectively.
Relationships between (mutual) inclusion relations

\[ \mathcal{H}_1 \preceq_{\text{der}} \mathcal{H}_2 \]
\[ \mathcal{H}_1 \preceq_{\text{adm}} \mathcal{H}_2 \]
\[ \mathcal{H}_1 \preceq_{\text{th}} \mathcal{H}_2 \]

\[ \mathcal{H}_1 \preceq_{\text{r/der}} \mathcal{H}_2 \]
\[ \mathcal{H}_1 \preceq_{\text{r/adm}} \mathcal{H}_2 \]
Relationships between (mutual) inclusion relations

Theorem. (Interrelation Prisms)

(i) The implications and equivalences shown in the interrelations prisms hold, for all AHS’s \( H_1 \) and \( H_2 \), between statements \( H_1 \preceq H_2 \) (where \( \preceq \) is an introduced inclusion relation), and respectively, between statements of the form \( H_1 \sim H_2 \) (where \( \sim \) is an introduced inclusion relation).

(ii) Not inverted arrows indicate that the implication in the opposite direction does not hold in general.

(iii) In the case of the int. rel. prism for the incl. relations, in general no implication holds in either direction between \( H_1 \preceq_{r/adm} H_2 \) and any of \( H_1 \preceq_{r/der} H_2 \), \( H_1 \preceq_{(s)}^{r/der} H_2 \) or \( H_1 \preceq_{(m)}^{r/der} H_2 \).
A Consequence of the Interrelation Prisms (I)

Corollary. (Characterizations of rule admissibility, derivability and m-derivability)

Let $\mathcal{H}$ be an AHS and let $R$ be a rule on the set of formulas of $\mathcal{H}$. Then the following hold:

- $R$ is admissible in $\mathcal{H} \iff \mathcal{H} + R \sim_{th} \mathcal{H}$,
- $R$ is derivable in $\mathcal{H} \iff \mathcal{H} + R \sim_{rth} \mathcal{H}$,
- $R$ is $s$-derivable in $\mathcal{H} \iff \mathcal{H} + R \sim_{rth}^{(s)} \mathcal{H}$,
- $R$ is $m$-derivable in $\mathcal{H} \iff \mathcal{H} + R \sim_{rth}^{(m)} \mathcal{H}$.
A Consequence of the Interrelation Prisms (II)

Theorem. (Reformulation of a theorem by Schütte).
For all abstract Hilbert systems $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ it holds:

$$\mathcal{H}_1 \preceq_{r/\text{der}} \mathcal{H}_2 \quad \& \quad \mathcal{H}_2 \preceq_{r/\text{adm}} \mathcal{H}_3 \quad \Rightarrow \quad \mathcal{H}_1 \preceq_{r/\text{adm}} \mathcal{H}_3.$$

Wrong Proof. For all AHS’s $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ it holds:

$$\mathcal{H}_1 \preceq_{r/\text{der}} \mathcal{H}_2 \quad \& \quad \mathcal{H}_2 \preceq_{r/\text{adm}} \mathcal{H}_3 \quad \Rightarrow \quad$$

$$\Rightarrow \quad \mathcal{H}_1 \preceq_{r/\text{adm}} \mathcal{H}_2 \quad \& \quad \mathcal{H}_2 \preceq_{r/\text{adm}} \mathcal{H}_3 \quad \text{(int.rels. prisma)}$$

$$\Rightarrow \quad \mathcal{H}_1 \preceq_{r/\text{adm}} \mathcal{H}_2 \quad \text{(if } \preceq_{r/\text{adm}} \text{ were transitive)}.$$

However, the relation $\preceq_{r/\text{adm}}$ is not transitive.
A Consequence of the Interrelation Prisms (II)

Proof. For all AHS's $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ it holds:

$$\mathcal{H}_1 \preceq_{r/\text{der}} \mathcal{H}_2 \quad \& \quad \mathcal{H}_2 \preceq_{r/\text{adm}} \mathcal{H}_3 \quad \Rightarrow$$

$$\Rightarrow \quad \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3 \sim_{r/\text{der}} \mathcal{H}_2 + \mathcal{H}_3 \quad \& \quad \mathcal{H}_2 + \mathcal{H}_3 \sim_{r/\text{adm}} \mathcal{H}_3$$

(due to defs.)

$$\Rightarrow \quad \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3 \sim_{r/\text{adm}} \mathcal{H}_2 + \mathcal{H}_3 \quad \& \quad \mathcal{H}_2 + \mathcal{H}_3 \sim_{r/\text{adm}} \mathcal{H}_3$$

(due int.rels. prisma)

$$\Rightarrow \quad \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3 \sim_{r/\text{adm}} \mathcal{H}_3$$

($\sim_{r/\text{adm}}$ is transitive)

$$\Rightarrow \quad \mathcal{H}_1 \preceq_{r/\text{adm}} \mathcal{H}_2$$

(def. of $\preceq_{r/\text{adm}}$).
Three notions of “mimicking derivation”

Let $H_1$ and $H_2$ be AHS’s or n-AHS’s, and let $D_1 \in Der(H_1)$ and $D_2 \in Der(H_2)$ be derivations.

We say that $D_1$ mimics $D_2$ (denoted by $D_1 \preceq D_2$) if and only if

\[ \text{set}(\text{assm}(D_1)) \subseteq \text{set}(\text{assm}(D_2)) \quad \& \quad \text{concl}(D_1) = \text{concl}(D_2), \]

i.e. $D_1$ and $D_2$ have the same conclusion and all assumptions of $D_1$ are contained in the set of assumptions of $D_2$. 

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Three notions of “mimicking derivation”

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be AHS’s or n-AHS’s, and let $D_1 \in \text{Der}(\mathcal{H}_1)$ and $D_2 \in \text{Der}(\mathcal{H}_2)$ be derivations.

We say that $D_1$ mimics $D_2$ (denoted by $D_1 \preccurlyeq D_2$) if and only if

$$\text{set}\left(\text{assm}(D_1)\right) \subseteq \text{set}\left(\text{assm}(D_2)\right) \quad \& \quad \text{concl}(D_1) = \text{concl}(D_2) ,$$

Furthermore, we stipulate that $D_1$ s-mimics $D_2$ (symb. $D_1 \simeq^{(s)} D_2$), and that $D_1$ m-mimics $D_2$ (symb. $D_1 \simeq^{(m)} D_2$) if and only if respectively (3) and (4) hold:

$$\text{set}\left(\text{assm}(D_1)\right) = \text{set}\left(\text{assm}(D_2)\right) \quad \& \quad \text{concl}(D_1) = \text{concl}(D_2) , \quad (3)$$

$$\text{assm}(D_1) = \text{assm}(D_2) \quad \& \quad \text{concl}(D_1) = \text{concl}(D_2) . \quad (4)$$
Examples. (The notions \(\preceq\), \(\preceq^{(s)}\) and \(\preceq^{(m)}\) of mimicking deriv.).

(a) \[
\begin{array}{c}
\frac{C_1}{A} R_1 \\
\frac{A}{B} R_{A,B}
\end{array}
\begin{array}{c}
\preceq^{(s)} \\
\preceq^{(m)}
\end{array}
\begin{array}{c}
\frac{C_1}{A} R_1 \\
\frac{A}{B} R_{A,A,B}
\end{array}
\]

(b) \[
\begin{array}{c}
\frac{C_1}{A} R_1 \\
\frac{A}{B} R_{A,B}
\end{array}
\begin{array}{c}
\preceq^{(s)} \\
\preceq^{(m)}
\end{array}
\begin{array}{c}
\frac{C_1}{A} R_1 \\
\frac{A}{B} R_{A,A,B}
\end{array}
\]

(c) \[
\begin{array}{c}
\frac{C_1}{A} R_1 \\
\frac{C_2}{A} R_{A,A,B}
\end{array}
\begin{array}{c}
\preceq^{(s)} \\
\preceq^{(m)}
\end{array}
\begin{array}{c}
\frac{C_2}{A} R_2 \\
\frac{C_1}{A} R_{A,A,B}
\end{array}
\]

ZIC, TU Eindhoven, 22nd June, 2004
Proposition. (The notions $\preceq$, $\sim^{(s)}$ and $\sim^{(m)}$ of mimicking deriv.).

(i) $\preceq$ is reflexive and transitive.

(ii) $\sim^{(s)}$ and $\sim^{(m)}$ are equivalence relations.

(iii) For all derivations $D_1$ and $D_2$

$$D_1 \sim^{(s)} D_2 \iff D_1 \preceq D_2 \land D_2 \preceq D_1.$$  

holds, i.e. $\sim^{(s)} = \preceq \cap \preceq$, where $\preceq = (\preceq)^{-1}$.

(iv) $\sim^{(m)} \not\subseteq \sim^{(s)} \not\subseteq \preceq$. 

Four notions of “rule elimination”

**Definition.** Let $\mathcal{H}$ be an AHS or n-AHS, and let $R$ be a (named) rule of $\mathcal{H}$.

(i) We say that $R$-elimination holds in $\mathcal{H}$ if and only

\[(\forall D \in \text{Der}(\mathcal{H})) \left[ \text{set}(\text{assm}(D)) = \emptyset \implies \exists D' \in \text{Der}(\mathcal{H} - R) \right. \left. \left[ D' \prec D \right] \right],\]

i.e. iff every derivation $D$ in $\mathcal{H}$ without assumptions can be mimicked by a derivation $D'$ in $\mathcal{H} - R$. 
(ii) We say that \textit{R-elimination holds in} $\text{Der}(\mathcal{H})$ \textit{with respect to} $\sim$ if and only if

$$(\forall \mathcal{D} \in \text{Der}(\mathcal{H})) \ (\exists \mathcal{D}' \in \text{Der}(\mathcal{H} - R)) \ [\mathcal{D}' \sim \mathcal{D}],$$

i.e. iff every derivation $\mathcal{D}$ of $\mathcal{H}$ can be mimicked by a derivation $\mathcal{D}'$ of $\mathcal{H} - R$.

We say that \textit{R-elimination holds in} $\text{Der}(\mathcal{H})$ \textit{with respect to} $\sim^{(s)}$, and that \textit{R-elimination holds in} $\text{Der}(\mathcal{H})$ \textit{with respect to} $\sim^{(m)}$ if and only if respectively (5) and (6) are the case:

$$(\forall \mathcal{D} \in \text{Der}(\mathcal{H})) \ (\exists \mathcal{D}' \in \text{Der}(\mathcal{H} - R)) \ [\mathcal{D}' \sim^{(s)} \mathcal{D}], \quad (5)$$

$$(\forall \mathcal{D} \in \text{Der}(\mathcal{H})) \ (\exists \mathcal{D}' \in \text{Der}(\mathcal{H} - R)) \ [\mathcal{D}' \sim^{(m)} \mathcal{D}] \quad (6)$$
How do these notions of rule elimination relate to rule derivability and admissibility?

**Theorem.** Let $\mathcal{H}$ be an AHS or an $n$-AHS, and let $R$ be a (named) rule of $\mathcal{H}$. Then the following statements hold:

- $R$-elimination holds in $\mathcal{H} \iff R$ is admissible in $\mathcal{H} - R$,

- $R$-elimination holds in $\text{Der}(\mathcal{H}) \ w.r.t. \sim \iff R$ is derivable in $\mathcal{H} - R$,

- $R$-elimination holds in $\text{Der}(\mathcal{H}) \ w.r.t. \sim^{(s)} \implies R$ is $s$-derivable in $\mathcal{H} - R$,

- $R$-elimination holds in $\text{Der}(\mathcal{H}) \ w.r.t. \sim^{(m)} \iff R$ is $m$-derivable in $\mathcal{H} - R$. 

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Effective rule elim. by “mimicking steps” in n-AHS’s

Let $\mathcal{H}$ be an n-AHS, and let $R$ be a named rule of $\mathcal{H}$.

A mimicking step for $R$-elimination in $\mathcal{H}$ is a transition of the form

$$
\phi : \frac{D_1}{A_1} \ldots \frac{D_n}{A_n} \xrightarrow{\text{name}(R)} \frac{(R)}{\text{mim}} \frac{D_{i_1}}{(A_{i_1})} \ldots \frac{D_{i_k}}{(A_{i_k})}
$$

where the derivation $D_\alpha \in Der(\mathcal{H} - R)$ mimics the application $\alpha$ of $R$ displayed in the left derivation.

Observation: If $R$ is derivable in $\mathcal{H} - R$, then each $R$-application in an $\mathcal{H}$-derivation can be eliminated by a mimicking step.
ARS’s of rule elimination by mimicking steps

Let again $\mathcal{H}$ be an n-AHS and $R$ a named rule of $\mathcal{H}$.

The described kind of steps give rise to the $ARS \rightarrow^{(R)}_{\text{mim}}(\mathcal{H})$ of $R$-elimination on $Der(\mathcal{H})$ by mimicking steps

$$\rightarrow^{(R)}_{\text{mim}}(\mathcal{H}) = \langle Der(\mathcal{H}), \Phi^{(R)}_{\text{mim}}(\mathcal{H}), \text{src}, \text{tgt} \rangle,$$

where $\Phi^{(R)}_{\text{mim}}(\mathcal{H})$ the set of mimicking steps for $R$-elimination on $Der(\mathcal{H})$, and src and tgt the source and target functions on $\Phi^{(R)}_{\text{mim}}(\mathcal{H})$. 
Effective rule elim. by s- and m-mimicking steps

Let $\mathcal{H}$ be an n-AHS and $R$ a named rule of $\mathcal{H}$. We define similarly:

- **s-mimicking steps** for $R$-elimination in $\mathcal{H}$ replace $R$-applications in $\mathcal{H}$-derivations by s-mimicking derivations.

- **m-mimicking steps** for $R$-elimination in $\mathcal{H}$ replace $R$-applications in $\mathcal{H}$-derivations by m-mimicking derivations.

Analogously as before, these notions give rise to

$$\rightarrow^{(R)}_{\text{s-mim}}(\mathcal{H}) \quad \text{and} \quad \rightarrow^{(R)}_{\text{m-mim}}(\mathcal{H}),$$

the **ARS of $R$-elimination on $\text{Der}(\mathcal{H})$ by s-mimicking steps**, and the **ARS of $R$-elimination on $\text{Der}(\mathcal{H})$ by m-mimicking steps**.
Weak normalization of rule elimination by mimicking steps

For an ARS $\rightarrow$ we denote by $\mathcal{NF}(\rightarrow)$ the set of its normal forms.

**Lemma.** Let $\mathcal{H}$ be an $n$-AHS. Let $R$ be a named rule of $\mathcal{H}$ that is derivable in $\mathcal{H} - R$.

(i) $\mathcal{NF}(\rightarrow^{(R)}_{\text{mim}}(\mathcal{H})) = \text{Der}(\mathcal{H} - R)$,

i.e. a derivation of $\mathcal{H}$ is a normal form of $\rightarrow^{(R)}_{\text{mim}}(\mathcal{H})$ if and only if it does not contain applications of $R$.

(ii) $\rightarrow^{(R)}_{\text{mim}}(\mathcal{H})$ is weakly normalizing.

Analogous statements hold for $\rightarrow^{(R)}_{\text{s-mim}}(\mathcal{H})$ and $\rightarrow^{(R)}_{\text{m-mim}}(\mathcal{H})$. 
Correctness of rule elim. by (s-,m-)mimicking steps

**Theorem.** Let $\mathcal{H}$ be an $n$-AHS and $R$ be a named rule of $\mathcal{H}$. Then it holds:

(i) $R$-elim. by mimicking steps in $\text{Der}(\mathcal{H})$ is correct w.r.t. $\preceq$:

$$(\forall \mathcal{D}, \mathcal{D}' \in \text{Der}(\mathcal{H}))$$

$$(\exists \phi) \left[ \phi : \mathcal{D} \xrightarrow{\text{mim}}^*(R) \mathcal{D}' \text{ & } \mathcal{D}' \in \text{Der}(\mathcal{H} - R) \right] \implies \mathcal{D}' \preceq \mathcal{D} .$$

(ii) $R$-elimination in $\text{Der}(\mathcal{H})$ by s-mim. steps is correct w.r.t. $\preceq$; but it is not in general also correct w.r.t. $\simeq^{(s)}$.

(iii) $R$-elimination in $\text{Der}(\mathcal{H})$ by m-mim. steps is correct w.r.t. $\simeq^{(m)}$. 

ZIC, TU Eindhoven, 22nd June, 2004
Termination of rule elimination by mimicking steps

**Lemma.** Let $\mathcal{H}$ be an $n$-AHS, and let $R$ be a named rule of $\mathcal{H}$.

(i) If $R$ is derivable in $\mathcal{H} - R$, then the ARS $\to^{(R)\text{ mim}} (\mathcal{H})$ is strongly normalizing.

(ii) If $R$ is $s$-derivable in $\mathcal{H} - R$, then the ARS $\to^{(R)\text{ s-mim}} (\mathcal{H})$ is strongly normalizing.

(iii) If $R$ is $m$-derivable in $\mathcal{H} - R$, then the ARS $\to^{(R)\text{ m-mim}} (\mathcal{H})$ is strongly normalizing.

**Proof:** Reducing the termination problem of these ARS’s to a multiset-ordening. ($\sim$: Colonies of amoebae have a finite life-span).
Strong rule elimination by (s-, m-) mimicking steps

Definition. Let $\mathcal{H}$ be an $n$-AHS and let $R$ be a named rule of $\mathcal{H}$.

Strong $R$-elimination by mimicking steps holds in $\text{Der}(\mathcal{H})$ iff

$$\text{SN}(\rightarrow^{(R)}_{\text{mim}}(\mathcal{H})),$$

i.e. $\rightarrow^{(R)}_{\text{mim}}(\mathcal{H})$ is strongly normalizing,

and $\text{NF}(\rightarrow^{(R)}_{\text{mim}}(\mathcal{H})) = \text{Der}(\mathcal{H} - R)$.

And similarly, we say that strong $R$-elimination by s-mimicking steps holds in $\text{Der}(\mathcal{H})$, and that strong $R$-elimination by m-mimicking steps holds in $\text{Der}(\mathcal{H})$ iff respectively (7) and (8) holds:

$$\text{SN}(\rightarrow^{(R)}_{\text{s-mim}}(\mathcal{H})),$$

and $\text{NF}(\rightarrow^{(R)}_{\text{s-mim}}(\mathcal{H})) = \text{Der}(\mathcal{H} - R)$, \hspace{1cm} (7)

$$\text{SN}(\rightarrow^{(R)}_{\text{m-mim}}(\mathcal{H})),$$

and $\text{NF}(\rightarrow^{(R)}_{\text{m-mim}}(\mathcal{H})) = \text{Der}(\mathcal{H} - R)$. \hspace{1cm} (8)
How do these notions of strong rule elimination relate to rule derivability and admissibility?

**Theorem.** Let $\mathcal{H}$ be an $n$-AHS and let $R$ be a named rule of $\mathcal{H}$.

Then the following three logical equivalences hold:

- Strong $R$-elimination by mimicking steps holds in $\text{Der}(\mathcal{H})$ if and only if $R$ is derivable in $\mathcal{H} - R$,
- strong $R$-elimination by s-mimicking steps holds in $\text{Der}(\mathcal{H})$ if and only if $R$ is s-derivable in $\mathcal{H} - R$,
- strong $R$-elimination by m-mimicking steps holds in $\text{Der}(\mathcal{H})$ if and only if $R$ is m-derivable in $\mathcal{H} - R$. 
How do the notions of strong rule elimination relate to the notions of rule elimination?

Corollary. Let $\mathcal{H}$ be an $n$-AHS and let $R$ be a named rule of $\mathcal{H}$. Then the following three statements hold:

- Strong $R$-elimination by mimicking steps holds in $\text{Der}(\mathcal{H})$ if and only if $R$-elimination holds in $\text{Der}(\mathcal{H})$ w.r.t. $\preceq$, strong $R$-elimination by s-mimicking steps holds in $\text{Der}(\mathcal{H})$ if and only if $R$-elimination holds in $\text{Der}(\mathcal{H})$ w.r.t. $\simeq$,
- strong $R$-elimination by m-mimicking steps holds in $\text{Der}(\mathcal{H})$ if and only if $R$-elimination holds in $\text{Der}(\mathcal{H})$ w.r.t. $\simeq^{(m)}$. 

ZIC, TU Eindhoven, 22nd June, 2004
Sequent-style Hilbert systems à la Avron

Definition. A Hilbert system for consequence (a HSC) $\mathcal{HC}$ in the language $L$ is an axiomatic system such that:

1. The formulas of $\mathcal{HC}$ are sequents in $L$, i.e. expressions $\Gamma \Rightarrow \Delta$ with $\Gamma, \Delta$ multisets of wff in $L$.

2. – The axioms of $\mathcal{HC}$ include $A \Rightarrow A$ for all $A$.
   – All other axioms of $\mathcal{HC}$ are of the form $\Rightarrow \Rightarrow A$.

3. Every rule $R$ of $\mathcal{HC}$ is an $n$-premise rule for some $n \in \omega$.

4. With the possible exception of the structural rules weakening and contraction and of the cut rule, all rules of $\mathcal{HC}$ fulfill the left-hand side property.
Left-hand side property of HSC-rules

The set of formulas that appear on the left-hand side of the conclusion of a rule is the union of the sets of formulas that appear on the left-hand side of the premises.

– An \( n \)-premise rule (where \( n \in \omega \setminus \{0\} \)) in a HSC has the left-hand side property if and only if for all its applications of the form

\[
\frac{\Gamma_1 \Rightarrow \Delta_1 \ldots \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}
\]

holds: \( \text{set}(\Gamma) = \bigcup_{i=1}^{n} \text{set}(\Gamma_i) \).  

– A zero-premise rule of \( \mathcal{HC} \) fulfills the left-hand side property if and only if all of its applications are of the form

\[
\Rightarrow \Delta
\].  

Pure Rules in HSC’s

**Definition.** Let $\mathcal{HC}$ be a HSC with language $L$, and $R$ a rule of $\mathcal{HC}$.

The rule $R$ is called **pure** if and only if the following holds: Whenever, for some $n \in \omega \setminus \{0\}$,

$$
\frac{\Gamma_1 \Rightarrow \Delta_1 \ldots \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}
$$

is an application of $R$, then

$$
\Gamma = \Gamma_1 \ldots \Gamma_n
$$

holds, and for all multisets $\Gamma'_1, \ldots, \Gamma'_n$ of formulas in $L$, also

$$
\frac{\Gamma'_1 \Rightarrow \Delta_1 \ldots \Gamma'_n \Rightarrow \Delta_n}{\Gamma'_1 \ldots \Gamma'_n \Rightarrow \Delta}
$$

is an application of $R$ (hence zero-premise rules are pure trivially).
Structural Rules and Cut for HSC’s

Weakening and contraction rules:

\[
\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \text{ Weak}_l
\]

\[
\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \text{ Contr}_l
\]

\[
\left(\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\right) \text{ Weak}_r
\]

\[
\left(\frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\right) \text{ Contr}_r
\]

Cut rule:

\[
\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A, \Gamma' \Rightarrow \Delta'} \frac{A, \Gamma' \Rightarrow \Delta'}{\Gamma \Gamma' \Rightarrow \Delta \Delta'} \text{ Cut}
\]
Cut-elimination in pure, single-conclusioned HSC’s

Proposition. Cut-elimination holds in every pure, single-conclusioned Hilbert system for consequence $\mathcal{HC}$, that is, for all sequents $\Gamma \Rightarrow A$ in $\mathcal{HC}$ it holds:

$$\vdash_{\mathcal{HC}} \Gamma \Rightarrow A \iff \vdash_{\mathcal{HC}-\text{Cut}} \Gamma \Rightarrow A.$$

Moreover: Every derivation $D$ in $\mathcal{HC}$ can effectively be transformed into a cut-free derivation $D'$ in $\mathcal{HC}$ with the same conclusion.
Correspondence between AHS’s and HSC’s

**Theorem.** For every AHS \( \mathcal{H} \) there exists a pure, single-conclusioned HSC \( \mathcal{HC}(\mathcal{H}) \) without structural rules such that for all \( A \in \text{Fo}_\mathcal{H} \) and \( \Gamma \in \mathcal{M}_f(\text{Fo}_\mathcal{H}) \) and \( \Sigma \in \mathcal{P}_f(\text{Fo}_\mathcal{H}) \) the following assertions hold:

\[
\begin{align*}
\Gamma \vdash^{(\text{m})}_\mathcal{H} A & \iff \vdash_{\mathcal{HC}(\mathcal{H})} \Gamma \Rightarrow A , \\
\Gamma \vdash^{(\text{mw})}_\mathcal{H} A & \iff \vdash_{\mathcal{HC}(\mathcal{H}) + \text{Weak}} \Gamma \Rightarrow A , \\
\Sigma \vdash^{(\text{s})}_\mathcal{H} A & \iff \vdash_{\mathcal{HC}(\mathcal{H}) + \text{Contr \ mset}(\Sigma)} \Rightarrow A , \\
\Sigma \vdash \mathcal{H} A & \iff \vdash_{\mathcal{HC}(\mathcal{H}) + \text{Weak} + \text{Contr \ mset}(\Sigma)} \Rightarrow A .
\end{align*}
\]

\(^1\)\text{Fo}_\mathcal{H} \text{ is the set of formulas of } \mathcal{H}.
Summary

We have introduced / we have found:

2. • Abstract Hilbert Systems (AHS’s), and
   • Abstract Hilbert Systems with rule/axiom names (n-AHS’s).
   • Three consequence relations on these systems.

3. • Definition of rule admissibility in (n-)AHS’s.
   • Definition of three versions of rule derivability in (n-)AHS’s (derivability, s- and m-derivability).
   • Some basic facts about these notions. A theorem that characterizes derivability of a rule $R$ in an AHS $\mathcal{H}$ by admissibility of $R$ in extensions of $\mathcal{H}$.
4 • (Mutual) inclusion relations \([2 \times 12\) relations].
• Two Interrelation Prisms between these relations.
• As a corollary: alternative characterizations of rule admissibility and rule \((m-)\)derivability.

5 • Three notions of mimicking derivation between derivations in an AHS or n-AHS.
• Four notions of rule elimination in AHS’s and n-AHS’s. Correspondences with rule admissibility and \((s-,m-)\)derivability.
• Three notions of strong rule elimination in n-AHS’s, and their correspondences with the three notions of rule derivability.

E (Appendix E) A close relationship of \((n)\)-AHS’s with sequent-style Hilbert systems for consequence à la Avron.
References


