

THE ERDŐS PROJECT

The Erdős Project is an important focus of the topology group at the VU. In 1940 the eccentric mathematical genius Paul Erdős introduced two subspaces of Hilbert space with unusual properties. These spaces are now called Erdős space and complete Erdős space. Erdős showed that these spaces are totally disconnected yet one-dimensional. Since the Erdős spaces are like Hilbert space topologically equivalent to their own squares the spaces have the counterintuitive property that when squared the dimension does not become two but remains one. This fact makes the Erdős spaces classic and important examples in Dimension Theory. Moreover, since their introduction these spaces have emerged in a number of different contexts. They play a role in Continua Theory and in Complex Dynamics where complete Erdős space has shown up as the end-point set of the Julia set of exponential functions. The primary result of the Erdős Project thus far is the topological characterization of Erdős space and its main application: a complete topological classification of the groups of homeomorphisms of topological manifolds, Hilbert cube manifolds, Menger continua, and Sierpiński carpets that leave a countable dense subset invariant. The group of homeomorphisms of a manifold is the topological symmetry group of the space, which plays a key role in the theory of Group Actions. Menger continua and Sierpiński carpets are fractals that play an important role in Dimension Theory.

In addition, we also found a simple and useful characterization of complete Erdős space solving a problem in the literature. The characterization of complete Erdős space also led to the identification of new representations of that space such as submeasure ideals on the natural numbers. One of the early successes of the project was the proof that complete Erdős space is not topologically equivalent to its countable infinite power, that is, complete Erdős space is unstable. This result was quite unexpected and led to the solution of a series of problems that were posed in the literature. One of the corollaries is that the full homeomorphism groups of the Menger and Sierpiński fractals are not topologically equivalent to complete Erdős space. Our speculation is that the “maximal element” of the class of almost zero-dimensional spaces to which the Erdős spaces belong is not complete Erdős space but its infinite power, stable complete Erdős space. Our main focus for the future is the latter space. It is our expectation that better characterizations of stable complete Erdős space will solve the classification problem of the homeomorphism groups of the Menger and Sierpiński fractals.