HOMEOMORPHISM GROUPS OF SIERPIŃSKI CARPETS AND ERDŐS SPACE

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ABSTRACT. Erdős space $E$ is the ‘rational’ Hilbert space, that is the set of vectors in $\ell^2$ the coordinates of which are all rational. Erdős proved that $E$ is one-dimensional and homeomorphic to its own square $E \times E$, which makes it an important example in dimension theory. Dijkstra and van Mill found topological characterizations of $E$. Let $M_n$, $n \in \mathbb{N}$, be the $n$-dimensional Menger continuum in $\mathbb{R}^{n+1}$, also known as the $n$-dimensional Sierpiński carpet, and let $D$ be a countable dense subset of $M_n$. We consider the topological group $\mathcal{H}(M_n, D)$ which consists of all autohomeomorphisms of $M_n$ that map $D$ onto itself equipped with the compact-open topology. We show that under some appropriate conditions on $D$ we have that $\mathcal{H}(M_n, D)$ is homeomorphic to $E$ for $n \in \mathbb{N} \setminus \{3\}$.

1. Introduction

All spaces in this paper are assumed to be separable and metrizable. If $X$ is locally compact then we equip the group of homeomorphisms $\mathcal{H}(X)$ of $X$ with the compact-open topology. If $A$ is a subset of $X$ then $\mathcal{H}(X, A)$ stands for the subgroup $\{h \in \mathcal{H}(X) : h(A) = A\}$ of $\mathcal{H}(X)$.

Let $D$ be a countable dense subset of a locally compact space $X$. In [5] Dijkstra and van Mill show that if $X$ contains a nonempty open subset homeomorphic to $\mathbb{R}^n$ for $n \geq 2$, an open subset of the Hilbert cube, or an open subset of some universal Menger continuum $\mu^n$ for $n \in \mathbb{N}$, then $\mathcal{H}(X, D)$ is homeomorphic to $E$. In line with these results we consider in this paper the topological group $\mathcal{H}(M_n, D)$ for $n \in \mathbb{N}$. Here $M_n$ is the $n$-dimensional Menger continuum in $\mathbb{R}^{n+1}$ (see Engelking [6, §1.11]), also known as the $n$-dimensional Sierpiński carpet, and $D$ is a countable dense subset of $M_n$. In our main result, Theorem 3.1, we show that under some appropriate conditions on $D$ we have that $\mathcal{H}(M_n, D)$ is homeomorphic to $E$ for $n \in \mathbb{N} \setminus \{3\}$. The proof of this result is based on the proof of [5, Theorem 10.4] where Dijkstra and van Mill use their characterization of $E$.
to deal with the \( \mu^n \) case. We also heavily rely on Dijkstra [4, §5] where it is shown that there are closed imbeddings of Erdős-type subspaces of \( \ell^1 \) (see Theorem 2.14) in \( \mathcal{H}(M_n^{n+1}) \), if \( n \in \mathbb{N} \setminus \{3\} \). The main complication is that \( M_n^{n+1} \) is, in contrast to the \( n \)-dimensional universal Menger continuum considered in [5, Theorem 10.4], not homogeneous.

2. Preliminaries

Let \( \mathbb{R}^+ = [0, \infty) \). We shall use a number of compactifications of \( \mathbb{R}^m \). Let \( S^m \) denote the one-point compactification of \( \mathbb{R}^m \). We let \( \hat{\mathbb{R}} \) denote the compactification \( [-\infty, \infty] \) of \( \mathbb{R} \). We shall use the convention that \( \pm \infty + t = \pm \infty \) when \( t \in \mathbb{R} \). This extends the addition operation on \( \mathbb{R}^m \times \mathbb{R}^m \) to a continuous function from \( \hat{\mathbb{R}}^m \times \hat{\mathbb{R}}^m \) to \( \hat{\mathbb{R}}^m \). An \( m \)-cell is any space that is homeomorphic to \( I^m \), where \( I = [0, 1] \). For a set \( A \) in a topological space we let \( \partial A \) denote the boundary of \( A \) and \( \text{Int}(A) \) the interior of \( A \).

Recall that for a compact space \( X \) the compact-open topology on \( \mathcal{H}(X) \) coincides with the topology of uniform convergence. We denote the identity element of \( \mathcal{H}(X) \) by \( e_X \). If \( O \) is an open subset of \( X \) then we say that \( h \in \mathcal{H}(X) \) is supported on \( O \) if \( h \) is equal to the identity on \( X \setminus O \), i.e. if \( h|_{(X \setminus O)} = e_{X \setminus O} \). We write \( \mathcal{H}_O(X) \) for the subgroup of \( \mathcal{H}(X) \) consisting of all homeomorphisms of \( X \) that are supported on \( O \), so \( \mathcal{H}_O(X) = \{ h \in \mathcal{H}(X) : h|_{(X \setminus O)} = e_{X \setminus O} \} \). Furthermore, we let \( \mathcal{H}_O(X, A) \) stand for the subgroup \( \mathcal{H}_O(X) \cap \mathcal{H}(X, A) \) of \( \mathcal{H}(X) \).

We need the following elementary result; see [5, Lemma 10.3].

**Lemma 2.1.** Let \( f : X \to Y \) and \( g : Y \to Z \) be continuous. If \( g \circ f \) is a closed imbedding then so is \( f \).

We give the definition of an \( n \)-dimensional Sierpiński carpet.

**Definition 2.2.** Let \( n \in \mathbb{N} \). A nowhere dense subset \( X \) of \( S^{n+1} \) is called an \( n \)-dimensional Sierpiński carpet if the collection of components \( \{ U_i : i \in \mathbb{N} \} \) of \( S^{n+1} \setminus X \) forms a null sequence such that the closures of the \( U_i \)'s are a pairwise disjoint collection and every \( S^{n+1} \setminus U_i \) is an \( (n+1) \)-cell.

The Menger continuum \( M_n^{n+1} \), constructed according to the ‘middle third’ method, see Engelking [6, §1.11], is a standard example of an \( n \)-dimensional Sierpiński carpet. The following characterization theorem is due to Whyburn [11] (for \( n = 1 \)) and Cannon [2] (for \( n \geq 2 \)).

**Theorem 2.3.** Let \( X \) and \( Y \) be two \( n \)-dimensional Sierpiński carpets for \( n \in \mathbb{N} \setminus \{3\} \) and let \( U \) and \( V \) be components of \( S^{n+1} \setminus X \), respectively \( S^{n+1} \setminus Y \).
If \( h \) is a homeomorphism from the boundary of \( U \) to the boundary of \( V \), then \( h \) can be extended to a homeomorphism from \( X \) to \( Y \).

**Remark 2.4.** In Theorem 2.3, let \( S \) and \( T \) be components of \( S^{n+1} \setminus X \), respectively \( S^{n+1} \setminus Y \), such that \( S \neq U \) and \( T \neq V \). The proofs of Lemma 1 and Theorem 1 in [2] together with the Annulus Theorem ([2]), which enables one to control where the boundary of a component of \( S^{n+1} \setminus X \) is mapped to, yield that we can extend \( h \) to a homeomorphism \( \overline{h} : X \to Y \) in such a way that \( \overline{h}(\partial S) = \partial T \).

**Definition 2.5.** A point \( x \) of an \( n \)-dimensional Sierpiński carpet \( X \) is called a **boundary point** of \( X \) if it lies on a non-separating copy \( S \) of \( S^n \) in \( X \), that is, \( X \setminus S \) is connected. If \( x \) is not a boundary point we call it an **interior point** of \( X \).

Using the notation of Definition 2.2, it follows easily from Brouwer Invariance of Domain [8, Theorem 3.6.8] and the generalized Jordan Curve Theorem [9, Theorem 36.3] that \( x \) is a boundary point of \( X \) if and only if \( x \in \bigcup_{i=1}^{\infty} \partial U_i \) and that every \( \partial U_i \) is homeomorphic to \( S^n \). Note that these definitions of boundary point and interior point of \( X \) do not coincide with the usual meaning of these notions since \( \operatorname{Int}(X) = \emptyset \). We have that boundary points and interior points are two topologically different types of points in \( X \), both of which are represented in \( X \). This means that \( X \) is not homogeneous. It is well known that these points are topologically the only two different types of points in \( X \) if \( \dim X \neq 3 \), cf. Theorem 2.3 and Lemma 2.7.

**Lemma 2.6.** Let \( n \in \mathbb{N} \setminus \{3\} \) and suppose that \( x \in \partial U \), where \( U \) is a component of \( S^{n+1} \setminus M_{n+1}^n \). Then there is a local basis \( B_x \) at \( x \) such that for every \( B \in B_x \) and every \( y \in B \cap \partial U \) there is a homeomorphism \( h \) of \( M_{n+1}^n \) with \( h(x) = y \) that is supported on \( B \).

**Proof.** Take \( n \neq 3 \) and note that it follows from Theorem 2.3 and the homogeneity of \( S^n \) that all boundary points of \( M_{n+1}^n \) are topologically equivalent. Therefore, it is enough to consider the boundary point \( x = (0,0,\ldots,0) \in \partial(I^{n+1}) \), where \( \partial(I^{n+1}) \) is the boundary of the unbounded component of \( \mathbb{R}^{n+1} \setminus M_{n+1}^n \). For \( B_x \) we take the collection \( \{ B_i : i \in \omega \} \), where \( B_i = M_{n+1}^n \cap [0,3^{-i}]^{n+1} \). Now take \( i \in \omega \) and a point \( y \in B_i \cap \partial I^{n+1} \). If \( y = x \) then the identity map on \( M_{n+1}^n \) is obviously a homeomorphism that satisfies the requirements of the lemma, so we suppose that \( y \neq x \). The closure of \( B_i \) in \( M_{n+1}^n \) is the space \( C_i = M_{n+1}^n \cap [0,3^{-i}]^{n+1} \), so \( C_i = 3^{-i} M_{n+1}^n \),
which means that $C_i$ is again an $n$-dimensional Sierpiński carpet. Note that $D_i = \partial([0, 3^{-i}]^{n+1})$ is the boundary of the unbounded component of $\mathbb{R}^{n+1} \setminus C_i$. Since $B_i \cap D_i$ is open and connected in $D_i$ and $D_i$ is homeomorphic to $S^n$, we have that $B_i \cap D_i$ is path connected and we can use the strong local homogeneity of $S^n$ to see that there is a homeomorphism $g_i: D_i \to D_i$ with $g_i(x) = y$ and that is supported on $B_i \cap D_i$. By Theorem 2.3 we can extend $g_i$ to a homeomorphism $\overline{g_i}$ of $C_i$. If we now define $h_i: M^{n+1}_i \to M^{n+1}_i$ by

$$h_i(x) = \begin{cases} \overline{g_i}(x), & \text{if } x \in C_i, \\ x, & \text{otherwise}, \end{cases}$$

then $h_i$ is as required. \hfill \Box

We want to derive a similar result for the interior points of $M^{n+1}_i$ with $n \in \mathbb{N} \setminus \{3\}$. For this we use the following lemma. Note that $\partial(I^{n+1})$ is the boundary of the unbounded component of $\mathbb{R}^{n+1} \setminus M^{n+1}_i$.

**Lemma 2.7.** Let $n \in \mathbb{N} \setminus \{3\}$ and let $x$ and $y$ be interior points of $M^{n+1}_i$. Then there is a homeomorphism $h: M^{n+1}_i \to M^{n+1}_i$ with $h(x) = y$ and $h|\partial(I^{n+1}) = e_{\partial(I^{n+1})}$.

**Proof.** If $x = y$ we can take $h = e_{M^{n+1}_i}$, so suppose that $x \neq y$. Clearly, we can find quotient mappings $q_x, q_y: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with $q_x^{-1}(\{x\}) = I^{n+1}$, respectively $q_y^{-1}(\{y\}) = I^{n+1}$ and such that $q_x: \mathbb{R}^{n+1} \setminus I^{n+1} \to \mathbb{R}^{n+1} \setminus \{x\}$, respectively $q_y: \mathbb{R}^{n+1} \setminus I^{n+1} \to \mathbb{R}^{n+1} \setminus \{y\}$, are homeomorphisms. Then $q_x^{-1}(M^{n+1}_i) \setminus \text{Int } I^{n+1}$ and $q_y^{-1}(M^{n+1}_i) \setminus \text{Int } I^{n+1}$ are Sierpiński carpets and we denote them by $S_x$ respectively $S_y$.

Let $B_x$, respectively $B_y$, be the boundary of the unbounded component of $\mathbb{R}^{n+1} \setminus S_x$, respectively $\mathbb{R}^{n+1} \setminus S_y$. So $B_x = q_x^{-1}(\partial I^{n+1})$ and $B_y = q_y^{-1}(\partial I^{n+1})$. Note that $g = (q_y^{-1} \circ q_x)|B_x$ is a homeomorphism from $B_x$ to $B_y$ such that $q_y \circ g = q_x|B_x$. It follows from Remark 2.4 that we can extend $g$ to a homeomorphism $\overline{g}: S_x \to S_y$ such that $g(\partial I^{n+1}) = \partial I^{n+1}$.

Now define the function $h: M^{n+1}_i \to M^{n+1}_i$ by

$$h(z) = \begin{cases} y, & \text{if } z = x, \\ (q_y \circ \overline{g} \circ q_x^{-1})(z), & \text{if } z \neq x. \end{cases}$$

It is easy to see that $h$ is a bijection such that $h \circ q_x = q_y \circ \overline{g}$. Since $q_x$ is a quotient mapping and $q_y \circ \overline{g}$ is continuous we have that $h$ is continuous. By compactness of $M^{n+1}_i$ we see that $h$ is a homeomorphism.

Take $z \in \partial(I^{n+1})$. Then $q_x^{-1}(z) \in B_x$ and since $\overline{g}$ is an extension of $g$ we see that

$$h(z) = (q_y \circ \overline{g})(q_x^{-1}(z)) = (q_y \circ g)(q_x^{-1}(z)) = q_x(q_x^{-1}(z)) = z.$$
This shows that $h|\partial(I^{n+1}) = e_{\partial(I^{n+1})}$, so $h$ is as required.

\begin{lemma}
Let $n \in \mathbb{N} \setminus \{3\}$ and suppose that $x$ is an interior point of $M_n^{n+1}$. Then there is a local basis $\mathcal{B}_x$ at $x$ such that for every $B \in \mathcal{B}_x$ and every interior point $y$ of $M_n^{n+1}$ in $B$ there is a homeomorphism $h$ of $M_n^{n+1}$ with $h(x) = y$ that is supported on $B$.
\end{lemma}

\begin{proof}
Let $x$ be an interior point of $M_n^{n+1}$. It follows from the construction of $M_n^{n+1}$ that $x$ has arbitrarily small open neighbourhoods $B$ in $M_n^{n+1}$ such that $\overline{B}$, the closure of $B$ in $M_n^{n+1}$ (or in $\mathbb{R}^{n+1}$), is homeomorphic to $M_n^{n+1}$ and the boundary $\partial B$ of $B$ in $M_n^{n+1}$ is the boundary of the unbounded component of $\mathbb{R}^{n+1} \setminus \overline{B}$. Let $\mathcal{B}_x$ be the collection of these neighbourhoods $B$ of $x$. Clearly, $\mathcal{B}_x$ is a local base at $x$. If $y$ is an interior point of $M_n^{n+1}$ such that $y$ is an element of a $B \in \mathcal{B}_x$, then $y$ is an interior point of $\overline{B}$. It follows from Lemma 2.7 that we can find a homeomorphism of $\overline{B}$ that maps $x$ onto $y$ and that is equal to the identity on the boundary of $B$ in $M_n^{n+1}$. This homeomorphism can be extended to $M_n^{n+1}$ by taking the identity on $M_n^{n+1} \setminus \overline{B}$. We showed that the local basis $\mathcal{B}_x$ at $x$ is as required.
\end{proof}

\begin{lemma}
Let $O$ be an open subset of $M_n^{n+1}$ for $n \in \mathbb{N} \setminus \{3\}$ and let $D_1$ and $D_2$ be countable subsets of $O$. Suppose that for $j \in \{1, 2\}$ the interior points of $M_n^{n+1}$ contained in $D_j$ are dense in $O$ and $D_j \cap \partial U_i$ is dense in $\partial U_i \cap O$ for all $i$. Then there is a homeomorphism $h$ of $M_n^{n+1}$ that is supported on $O$ and that satisfies $h(D_1) = D_2$.
\end{lemma}

\begin{proof}
This proof uses a well known back-and-forth construction; see for instance [1] or [8, Theorem 1.6.9]. Write $D_1 = D_1^i \cup D_1^b$, where $D_1^i$ is the set of all points of $D_1$ that are interior points of $M_n^{n+1}$ and $D_1^b$ is the set of all points of $D_1$ that are boundary points of $M_n^{n+1}$. Similarly, write $D_2 = D_2^i \cup D_2^b$. Let $\{a_1, a_2, \ldots\}$ and $\{\bar{a}_1, \bar{a}_2, \ldots\}$ be enumerations of $D_1^i$, respectively $D_1^b$, and let $\{b_1, b_2, \ldots\}$ and $\{\bar{b}_1, \bar{b}_2, \ldots\}$ be enumerations of $D_2^i$, respectively $D_2^b$. Using the Inductive Convergence Criterion [8, 1.6.2] we construct a sequence $(h_m)_{m \in \mathbb{N}}$ of homeomorphisms of $M_n^{n+1}$ such that $h = \lim_{m \to \infty} h_m \circ \cdots \circ h_1$ exists and is a homeomorphism and such that the following conditions are satisfied:

1. $h_m$ is supported on $O$ for all $m \in \mathbb{N}$;
2. $h_m \circ \cdots \circ h_1(a_j) = h_{4j-2} \circ \cdots \circ h_1(a_j) \in D_2^i$ for all $j$ and $m \geq 4j - 2$;
3. $(h_m \circ \cdots \circ h_1)^{-1}(b_j) = (h_{4j-1} \circ \cdots \circ h_1)^{-1}(b_j) \in D_1^i$ for all $j$ and $m \geq 4j - 1$;
4. $h_m \circ \cdots \circ h_1(\bar{a}_j) = h_{4j} \circ \cdots \circ h_1(\bar{a}_j) \in D_2^b$ for all $j$ and $m \geq 4j$;

\end{proof}
These conditions ensure that \( h \in \mathcal{H}_O(M_n^{m+1}) \) and that \( h(D_1^j) = D_2^j \) and \( h(D_1^{j+1}) = D_2^{j+1} \). Put \( h_1 = e_{M_n^{m+1}} \) and assume that \( h_1, \ldots, h_{4j-3} \) are defined for certain \( j \in \mathbb{N} \).

If \( h_{4j-3} \circ \cdots \circ h_1(a_j) \in D_2^1 \), take \( h_{4j-2} = e_{M_n^{m+1}} \). Otherwise, we use Lemma 2.8 to find a small neighbourhood \( V_{4j-2} \subset O \) of \( h_{4j-3} \circ \cdots \circ h_1(a_j) \) which is disjoint from the finite set

\[ \{b_1, \ldots, b_{j-1}, \tilde{b}_1, \ldots, \tilde{b}_{j-1}\} \cup h_{4j-3} \circ \cdots \circ h_1(\{a_1, \ldots, a_{j-1}, \tilde{a}_1, \ldots, \tilde{a}_{j-1}\}) \]

and moreover has the property that we can map \( h_{4j-3} \circ \cdots \circ h_1(a_j) \) on every other interior point of \( M_n^{m+1} \) in \( V_{4j-2} \) with a homeomorphism that is supported on \( V_{4j-2} \). Since \( D_2^1 \) is dense in \( O \) we have that \( D_2^1 \cap V_{4j-2} \neq \emptyset \). This means that we can find a homeomorphism \( f_{4j-2} \) of \( M_n^{m+1} \) supported on \( V_{4j-2} \) such that

\[ f_{4j-2} \circ h_{4j-3} \circ \cdots \circ h_1(a_j) \in D_2^1. \]

We put \( h_{4j-2} = f_{4j-2} \).

If \( (h_{4j-2} \circ \cdots \circ h_1)^{-1}(b_j) \in D_1^j \), we take \( h_{4j-1} = e_{M_n^{m+1}} \). Otherwise, we use Lemma 2.8 again to find a small neighbourhood \( V_{4j-1} \subset O \) of \( b_j \) that is disjoint from the finite set

\[ \{b_1, \ldots, b_{j-1}, \tilde{b}_1, \ldots, \tilde{b}_{j-1}\} \cup h_{4j-2} \circ \cdots \circ h_1(\{a_1, \ldots, a_{j-1}, \tilde{a}_1, \ldots, \tilde{a}_{j-1}\}) \]

and has the property that we can map \( b_j \) on every other interior point of \( M_n^{m+1} \) in \( V_{4j-1} \) with a homeomorphism that is supported on \( V_{4j-1} \). Since \( (h_{4j-2} \circ \cdots \circ h_1)^{-1}(D_1^j) \) is dense in \( O \) by property (1) we know that \( (h_{4j-2} \circ \cdots \circ h_1)(D_1^j) \cap V_{4j-1} \neq \emptyset \). This means that there is a homeomorphism \( f_{4j-1} \) of \( M_n^{m+1} \) supported on \( V_{4j-1} \) such that

\[ f_{4j-1}^{-1}(b_j) \in (h_{4j-2} \circ \cdots \circ h_1)(D_1^j). \]

We put \( h_{4j-1} = f_{4j-1} \).

Using the same argumentation as above, now using Lemma 2.6 instead of Lemma 2.8, we find appropriate neighbourhoods \( V_{4j}, V_{4j+1} \subset O \) of \( h_{4j-1} \circ \cdots \circ h_1(\tilde{a}_j) \), respectively \( \tilde{b}_j \), and homeomorphisms \( h_{4j}, h_{4j+1} \) in \( \mathcal{H}_{V_{4j}}(M_n^{m+1}) \), respectively \( \mathcal{H}_{V_{4j+1}}(M_n^{m+1}) \), such that

\[ h_{4j} \circ h_{4j-1} \circ \cdots \circ h_1(\tilde{a}_j) \in D_2^b, \]

and

\[ h_{4j+1}^{-1}(\tilde{b}_j) \in h_{4j} \circ \cdots \circ h_1(D_1^b). \]
If the neighbourhoods $V_{4j-2}, V_{4j-1}, V_{4j}, V_{4j+1}$ are chosen small enough, then the conditions of the Inductive Convergence Criterion are satisfied.

Remark 2.10. It follows immediately from this lemma that if $D_1 \cap \partial U_i$ and $D_2 \cap \partial U_i$ are dense in $\partial U_i \cap O$ for every $i$ with $\partial U_i \cap O \neq \emptyset$ and $D_1$ and $D_2$ do not contain any interior points of $M_n^{n+1}$, there is a homeomorphism $h$ of $M_n^{n+1}$ that is supported on $O$ that maps $D_1$ onto $D_2$. Similarly, if $D_1$ and $D_2$ both entirely consist of interior points of $M_n^{n+1}$ there is a homeomorphism that maps $D_1$ onto $D_2$ which is supported on $O$.

Now let $p \geq 1$ and consider the Banach space $\ell^p$. This space consists of all sequences $z = (z_0, z_1, \ldots) \in \mathbb{R}^\omega$ such that $\sum_{n=0}^{\infty} |z_n|^p < \infty$. The topology on $\ell^p$ is generated by the $p$-norm $\|z\|_p = \left(\sum_{n=0}^{\infty} |z_n|^p\right)^{1/p}$. It is well known that $\| \cdot \|_p$ is a Kadec norm with respect to the coordinate projections, that is, the norm topology is the weakest topology that makes all the coordinate projections $z \mapsto z_n$ and the norm function continuous. This fact can also be formulated as follows: the norm topology on $\ell^p$ is generated by the product topology (that is inherited from $\mathbb{R}^\omega$) together with the sets $\{z \in \ell^p : \|z\|_p < t\}$ for $t > 0$. We extend the $p$-norm over $\mathbb{R}^\omega$ by putting $\|z\|_p = \infty$ for each $z \in \mathbb{R}^\omega \setminus \ell^p$.

Definition 2.11. Let $X$ be a space. A function $f : X \to \widehat{\mathbb{R}}$ is called lower semi-continuous (abbreviated LSC) if $f^{-1}\left((t, \infty]\right)$ is open in $X$ for every $t \in \mathbb{R}$.

Note that the norm as a function from $\widehat{\mathbb{R}}^\omega$ to $[0, \infty]$ is LSC but not continuous because the norm topology on $\ell^p$ is much stronger than the topology that this space inherits from $\mathbb{R}^\omega$. It is easily checked that $f : X \to \widehat{\mathbb{R}}$ is LSC if and only if for every convergent sequence $(x_n)_{n \in \omega}$ in $X$ we have that $f\left(\lim_{n \to \infty} x_n\right) \leq \liminf_{n \to \infty} f(x_n)$.

We define Erdős space

$\mathcal{E} = \{x \in \ell^2 : x_n \in \mathbb{Q} \text{ for all } n \in \omega\}$.

Let $\mathcal{I}$ be the zero-dimensional topology that $\mathcal{E}$ inherits from $\mathbb{Q}^\omega$. We noted that $\mathcal{I}$ is weaker than the norm topology so we have that clopen sets separate points, that is, $\mathcal{E}$ is totally disconnected. By the remark above we see that the graph of the norm function, when seen as a function from $(\mathcal{E}, \mathcal{I})$ to $\mathbb{R}^+$, is homeomorphic to $\mathcal{E}$. This means that we can informally think of $\mathcal{E}$ as a ‘zero-dimensional space with some LSC function declared continuous’.

We point out the following connection between the two topologies on $\mathcal{E}$. Because the norm is LSC on $\mathbb{R}^\omega$ every closed $\varepsilon$-ball in $\mathcal{E}$ is also closed in the
zero-dimensional space $\mathbb{Q}^\omega$. This means that every point in $\mathcal{E}$ has arbitrarily small neighbourhoods which are intersections of clopen sets.

**Definition 2.12.** A subset $A$ of a space $X$ is called a $C$-set in $X$ if $A$ can be written as an intersection of clopen subsets of $X$. A space is called almost zero-dimensional if every point of the space has a neighbourhood basis consisting of $C$-sets of the space. If $Z$ is a set that contains $X$ then we say that a (separable metric) topology $\mathcal{T}$ on $Z$ witnesses the almost zero-dimensionality of $X$ if $\dim(Z, \mathcal{T}) \leq 0$, $O \cap X$ is open in $X$ for each $O \in \mathcal{T}$, and every point of $X$ has a neighbourhood basis in $X$ consisting of sets that are closed in $(Z, \mathcal{T})$. We will also say that the space $(Z, \mathcal{T})$ is a witness to the almost zero-dimensionality of $X$.

Thus $\mathcal{E}$ is almost zero-dimensional. The space $\mathbb{Q}^\omega$ is a witness to the almost zero-dimensionality of Erdős space. More generally, if $\varphi: Z \to \mathbb{R}$ is an LSC function with a zero-dimensional domain then it follows easily that $Z$ is a witness to the almost zero-dimensionality of the graph of $\varphi$. Clearly, a space $X$ is almost zero-dimensional if and only if there is a topology on $X$ witnessing this fact. Oversteegen and Tymchatyn [10] proved that every almost zero-dimensional space is at most one-dimensional.

**Definition 2.13.** Let $X$ be a space and let $\mathcal{A}$ be a collection of subsets of $X$. The space $X$ is called $\mathcal{A}$-cohesive if every point of the space has a neighbourhood that does not contain nonempty clopen subsets of any element of $\mathcal{A}$. If a space $X$ is $\{X\}$-cohesive then we simply call $X$ cohesive.

Again, let $p \geq 1$. As a generalization of the construction of $\mathcal{E}$, consider a fixed sequence $E_0, E_1, E_2, \ldots$ of subsets of $\mathbb{R}$ and let

$$\mathcal{E} = \{z \in \ell^p : z_n \in E_n \text{ for every } n \in \omega\}.$$ 

The following two results were proved in Dijkstra [3].

**Theorem 2.14.** Assume that $\mathcal{E}$ is not empty and that every $E_n$ is zero-dimensional. The following statements are equivalent:

1. there exists an $x \in \prod_{n=0}^\infty E_n$ with $\|x\|_p = \infty$ and $\lim_{n \to \infty} x_n = 0$,
2. every nonempty clopen subset of $\mathcal{E}$ is unbounded,
3. $\mathcal{E}$ is cohesive, and
4. $\dim \mathcal{E} > 0$.

Recall that if $A_0, A_1, \ldots$ is a sequence of subsets of a space $X$ then

$$\limsup_{n \to \infty} A_n = \bigcap_{n=0}^\infty \bigcup_{k=n}^\infty A_k.$$
Corollary 2.15. If 0 is a cluster point of $\limsup_{n \to \infty} E_n$ then every nonempty clopen subset of $\mathcal{E}$ is unbounded (and hence $\dim \mathcal{E} \neq 0$).

We need some new notions. The following definitions are taken from Dijkstra and van Mill [5].

Definition 2.16. If $A$ is a nonempty set then $A^{<\omega}$ denotes the set of all finite strings of elements of $A$, including the null string $\lambda$. If $s = a_0a_1 \ldots a_{k-1} \in A^{<\omega}$ for some $k \in \omega$, then $|s|$ denotes its length $k$. In this context the set $A$ is called an alphabet. Let $A^\omega$ denote the set of all infinite strings $a_0a_1 \ldots$ of elements of $A$. If $s \in A^{<\omega}$ and $\sigma \in A^{<\omega} \cup A^\omega$ then we put $s \prec \sigma$ if $s$ is an initial substring of $\sigma$, that is, there is a $\tau \in A^{<\omega} \cup A^\omega$ with $s \sigma = \tau$, where $\sigma$ denotes concatenation of strings. If $\sigma = a_0a_1 \ldots \in A^{<\omega} \cup A^\omega$ and $k \in \omega$ with $k \leq |\sigma|$, then $\sigma|k = a_0a_1 \ldots a_{k-1}$.

Definition 2.17. A tree $T$ on an alphabet $A$ is a subset of $A^{<\omega}$ that is closed under initial segments, that is, if $s \in T$ and $t \prec s$ then $t \in T$. An infinite branch of $T$ is an element $\sigma$ of $A^\omega$ such that $\sigma|k \in T$ for every $k \in \omega$. The body of $T$, written as $[T]$ is the set of all infinite branches of $T$. If $s, t \in T$ are such that $s \prec t$ and $|t| = |s| + 1$ then we say that $t$ is an immediate successor of $s$ and $\text{succ}(s)$ denotes the set of immediate successors of $s$ in $T$.

Now we introduce the concept of an anchor.

Definition 2.18. Let $T$ be a tree and let $(X_s)_{s \in T}$ be a system of subsets of a space $X$ such that $X_s \subseteq X_t$ whenever $s \prec t$. A subset $A$ of $X$ is called an anchor for $(X_s)_{s \in T}$ in $X$ if for every $\sigma \in [T]$ we have $X_{\sigma|k} \cap A = \emptyset$ for some $k \in \omega$ or the sequence $X_{\sigma|0}, X_{\sigma|1}, \ldots$ converges to a point in $X$.

Example 2.19. As noted before $Q^\omega$ is a witness to the almost zero-dimensionality of $\mathcal{E}$. Let $\mathcal{T}$ be the topology that $\mathcal{E}$ inherits from $Q^\omega$. Put $T = Q^{<\omega}$ and let for $s = q_0 \ldots q_{k-1} \in T$, with $k \in \omega$, the closed subset $Q^\omega_s$ of $Q^\omega$ be given by

$$Q^\omega_s = \{x \in Q^\omega : x_i = q_i \text{ for } 0 \leq i \leq k - 1\}.$$

Put $\mathcal{E}_s = Q^\omega_s \cap \mathcal{E}$ for $s \in T$ and let $B$ be a bounded subset of $\mathcal{E}$. We show that $B$ is an anchor for $(\mathcal{E}_s)_{s \in T}$ in $(\mathcal{E}, \mathcal{T})$. Let $\sigma = q_0q_1 \ldots \in [T]$ be such that $\mathcal{E}_{\sigma|k} \cap B \neq \emptyset$ for all $k \in \omega$. It is clear that $\mathcal{E}_{\sigma|k}$ converges to the point $\sigma \in Q^\omega$ in the product topology of $Q^\omega$, where we identify the string $q_0q_1 \ldots$ with the sequence $(q_0, q_1, \ldots)$. It suffices to show that $\sigma \in \mathcal{E}$. Since $B$ is bounded there is an $M \in \mathbb{N}$ such that $B \subseteq \{x \in Q^\omega : \|x\| \leq M\}$ and because
\( \mathcal{E}_{\sigma|k} \cap B \neq \emptyset \) for all \( k \in \omega \) this means that \( \| (q_0, q_1, \ldots, q_{k-1}, 0, 0, \ldots) \| \leq M \) for all \( k \geq 0 \). We have that

\[
\| \sigma \| = \lim_{k \to \infty} \| (q_0, q_1, \ldots, q_{k-1}, 0, 0, \ldots) \| \leq M,
\]

so \( \sigma \in \mathcal{E} \).

Dijkstra and van Mill [5, §8] introduced the following class of spaces \( \mathcal{E}' \).

**Definition 2.20.** \( \mathcal{E}' \) is the class of all nonempty spaces \( E \) such that there exists an \( F_{\sigma\delta} \)-topology \( \mathcal{T} \) on \( E \) that witnesses the almost zero-dimensionality of \( E \) and there exist a nonempty tree \( T \) over a countable set and subspaces \( E_s \) of \( E \) that are closed with respect to \( \mathcal{T} \) for each \( s \in T \setminus \{ \lambda \} \) such that

1. \( E_\lambda \) is dense in \( E \) and \( E_s = \bigcup \{ E_t : t \in \text{succ}(s) \} \) whenever \( s \in T \),
2. each \( x \in E \) has a neighbourhood \( U \) that is an anchor for \( (E_s)_{s \in T} \) in \( (E, \mathcal{T}) \),
3. for each \( s \in T \setminus \lambda \) and \( t \in \text{succ}(s) \) we have that \( E_t \) is nowhere dense in \( E_s \), and
4. \( E \) is \( \{ E_s : s \in T \} \)-cohesive.
5. \( E \) can be written as a countable union of nowhere dense subsets that are closed with respect to \( \mathcal{T} \).

In [5, Theorem 8.13] Dijkstra and van Mill prove

**Theorem 2.21.** A space \( E \) is homeomorphic to \( \mathcal{E} \) if and only if \( E \in \mathcal{E}' \).

As an illustration we show that \( \mathcal{E} \) satisfies the conditions of Definition 2.20. Let \( \mathcal{T} \) be the product topology that \( \mathcal{E} \) inherits from \( \mathbb{Q}^\omega \), put \( T = \mathbb{Q}^{<\omega} \) and let \( \mathcal{E}_s \) for \( s \in T \) be as defined in Example 2.19. Since \( \mathbb{Q} \) is a \( \sigma \)-compact space, it is easy to see that \( \mathbb{Q}^\omega \) is an absolute \( F_{\sigma\delta} \)-space. Furthermore, \( \mathcal{E} \) is an \( F_{\sigma} \) subset of \( \mathbb{Q}^\omega \), which means that \( \mathcal{T} \) is indeed an \( F_{\sigma\delta} \)-topology on \( \mathcal{E} \) that witnesses the almost zero-dimensionality of \( \mathcal{E} \). It is clear that \( \mathcal{E}_s \) is closed in \( (\mathcal{E}, \mathcal{T}) \) for all \( s \in T \) and conditions (1'), (3') and (5') are easily seen to be satisfied. For (2') and (4') note that it follows from Example 2.19 and Corollary 2.15 that every bounded neighbourhood of a point \( x \) in \( \mathcal{E} \) is an anchor for \( (\mathcal{E}_s)_{s \in T} \) in \( (\mathcal{E}, \mathcal{T}) \) that contains no nonempty clopen subsets of any \( \mathcal{E}_s \).

3. **Homeomorphism groups of a Sierpiński carpet**

We prove the following theorem for \( n \)-dimensional Sierpiński carpets as an extension of the results in [5, Chapter 10].
Theorem 3.1. Let \( n \in \mathbb{N} \setminus \{3\} \), let \( \{U_i : i \in \mathbb{N}\} \) be the collection of components of \( S_n^{n+1} \setminus M_n^{n+1} \), and let \( D \) be a countable dense subset of \( M_n^{n+1} \). If \( O \) is a nonempty open subset of \( M_n^{n+1} \) such that either \( D \cap \partial U_i = \emptyset \) for every \( i \) with \( \partial U_i \subset O \) or \( D \cap \partial U_i \) is dense in \( \partial U_i \) for every \( i \) with \( \partial U_i \subset O \), then \( \mathcal{H}_U(M_n^{n+1}, D) \) is homeomorphic to Erdős space for every open \( U \) that contains \( O \).

As noted before, \( M_n^{n+1} \) is not homogeneous, which is why we need the conditions on \( D \) here. If we choose for instance a set \( D \subset M_n^{n+1} \) such that \( |D \cap \partial U_i| = i \) for every \( i \) then \( \mathcal{H}(M_n^{n+1}, D) \) contains only the identity map.

Note that if \( D \cap \partial U_i = \emptyset \) for all \( \partial U_i \subset O \), there can still be \( j \in \mathbb{N} \) with \( D \cap \partial U_j \cap O \neq \emptyset \). Similarly, if \( D \cap \partial U_i \) is dense in \( \partial U_i \) for all \( \partial U_i \subset O \), there can still be \( j \in \mathbb{N} \) such that \( D \cap \partial U_j \cap O \) is not dense in \( \partial U_j \cap O \). The following claim shows that for the proof of Theorem 3.1 we can avoid these situations. Furthermore, it shows that if \( D \cap \partial U_i \) is dense in \( \partial U_i \) for all \( \partial U_i \subset O \), we may assume that the set of interior points of \( M_n^{n+1} \) contained in \( D \cap O \) is either empty or dense in \( O \). This observation will also be useful in the proof of Theorem 3.1.

Claim 3.2. It suffices to prove Theorem 3.1 for the following three cases:

(i) \( D \cap O \) consists entirely of interior points of \( M_n^{n+1} \);
(ii) \( D \cap \partial U_i \cap O \) is dense in \( \partial U_i \cap O \) for every \( i \in \mathbb{N} \) and the interior points of \( M_n^{n+1} \) contained in \( D \cap O \) are dense in \( O \);
(iii) \( D \cap \partial U_i \cap O \) is dense in \( \partial U_i \cap O \) for every \( i \in \mathbb{N} \) and \( D \cap O \) contains no interior points of \( M_n^{n+1} \).

Proof. Suppose that we are in the situation of Theorem 3.1. Let \( D_i \) be the set of all points of \( D \) that are interior points of \( M_n^{n+1} \). We define \( O' \subset O \) by

\[
O' = \begin{cases} 
O \setminus \overline{D_i}, & \text{if } O \setminus \overline{D_i} \neq \emptyset, \\
O, & \text{otherwise}.
\end{cases}
\]

Clearly, \( O' \) is a nonempty open subset of \( M_n^{n+1} \) such that either \( D_i \cap O' = \emptyset \) or \( D_i \cap O' \) is dense in \( O' \). Next we define \( O'' \subset O' \) by

\[
O'' = O' \setminus \bigcup \{ \partial U_i : \partial U_i \setminus O' \neq \emptyset \}.
\]

Since the interior points of \( M_n^{n+1} \) are dense in \( M_n^{n+1} \) and the collection \( \{U_i : i \in \mathbb{N}\} \) forms a null sequence, we have that \( O'' \) is a nonempty open subset of \( M_n^{n+1} \). Furthermore, if \( \partial U_i \cap O'' \neq \emptyset \) then \( \partial U_i \subset O'' \subset O \). It is clear that \( O'' \) satisfies one of the conditions (i), (ii) or (iii) and if we prove the theorem for \( O'' \) then we have also proved it for \( O \). \( \square \)
We introduce some notation.

**Definition 3.3.** We define subspaces $E_2$ and $E_4$ of $\ell^1$ as follows:

$$E_2 = \{ z \in \ell^1 : 2^i z_i \in \omega \text{ for all } i \in \omega \}, \text{ and}$$

$$E_4 = \{ z \in \ell^1 : 4^i z_i \in \omega \text{ for all } i \in \omega \}.$$  

We write $Z_2$ for the space consisting of the set $E_2$ equipped with the zero-dimensional topology this set inherits from the product space $\mathbb{R}^\omega$, that is, the topology generated by the coordinate projections. Similarly, we write $Z_4$ for the set $E_4$ equipped with the zero-dimensional topology it inherits from $\mathbb{R}^\omega$. For $i \in \omega$ we let $\xi_i : E_4 \to E_4$ denote the projection that is given by $\xi_i(z) = (z_0, z_1, \ldots, z_i, 0, 0, \ldots)$.

We will use the following proposition in the proof of Theorem 3.1.

**Proposition 3.4.** Let $n \in \mathbb{N} \setminus \{3\}$ and let $O \subset M_n^{n+1}$ be open and nonempty. Then there exists a closed imbedding $G : E_4 \ni z \to G_z \in \mathcal{H}_O(M_n^{n+1})$, a copy $\hat{\mathbb{R}}_c$ of $\hat{\mathbb{R}}$ in $O$ and a sequence $p_1, p_2, \ldots \in O \setminus \hat{\mathbb{R}}_c$ such that

(a) $\lim_{i \to \infty} p_i = 0_c \in \mathbb{R}_c$, where $\mathbb{R}_c = \hat{\mathbb{R}}_c \setminus \{ \pm \infty \}$,

(b) for each $r \in \hat{\mathbb{R}}_c$ and $z \in E_4$ we have $G_z(r) = r + \| z \| \in \hat{\mathbb{R}}_c$,

(c) for each $x \in M_n^{n+1} \setminus \mathbb{R}_c$ there is an $i \in \omega$ such that $G_z(x) = G_{\xi_i(z)}(x)$ for every $z \in E_4$, and

(d) $\beta \circ G : Z_4 \to \beta(\mathcal{H}(M_n^{n+1}))$ is a closed imbedding, where $A = \{ \infty_c, p_1, p_2, \ldots \}$ and $\beta : \mathcal{H}(M_n^{n+1}) \to (M_n^{n+1})^A$ is given by $\beta(h) = h | A$. ($h | A$ denotes the restriction of the map $h$ to $A$ and is an element of the infinite product space $(M_n^{n+1})^A$.)

The sets $\mathbb{R}_c$ and $A$ can be chosen such that either both consist of interior points of $M_n^{n+1}$ or both consist of boundary points of $M_n^{n+1}$. Moreover, for $n = 1$ the sets $\mathbb{R}_c$ and $A$ can be chosen such that $\mathbb{R}_c$ consists of interior points of $M_1^2$ and $A$ consists of boundary points of $M_1^2$.

**Proof.** Take $n \in \mathbb{N} \setminus \{3\}$. Dijkstra [4, Remark 3] showed that there exists a closed imbedding $\overline{H}$ of $E_2$ in $\mathcal{H}(\overline{B})$, where $\overline{B}$ is a topological copy of $M_n^{n+1}$ that contains a copy $\hat{\mathbb{R}}_c$ of $\hat{\mathbb{R}}$ and a sequence $p_1, p_2, \ldots \in \overline{B} \setminus \hat{\mathbb{R}}_c$ such that properties (a)–(d) are satisfied. Note that we can imbed $E_4$ in $E_2$ by the map $g : E_4 \to E_2$ given by

$$g(z_0, z_1, \ldots) = (z_0, 0, z_1, 0, z_2, 0, \ldots).$$

Now $g$ is even an isometry such that $g(E_4)$ is closed with respect to the weak and (therefore also) the strong topology on $E_2$. This means that we may
assume that $\overline{H}$ is a closed imbedding of $E_4$ in $\mathcal{H}(\overline{B})$ satisfying properties (a)–(d). We prove the proposition for $n = 1$ and $n \geq 2$ separately.

**Case I :** $n \in \mathbb{N} \setminus \{1, 3\}$. This is the easy case because in the construction of Dijkstra [4, §5] the points of $\hat{\mathbb{R}}_c \cup \{p_1, p_2, \ldots\}$ all lie in the same boundary $\partial U$ of some component $U$ of the complement of $\overline{B}$. With [4, Remark 4] it follows immediately that there is an imbedding $G$ as described in the proposition and such that $\hat{\mathbb{R}}_c$ and $A$ both consist of boundary points of $M_n^{n+1}$.

To show that there is also a suitable imbedding $G$ such that both $\mathbb{R}_c$ and $A$ consist of interior points of $M_n^{n+1}$ we consider two disjoint copies $B_1, B_2$ of $\overline{B}$ in $S^{n+1}$. Let $\partial U_1$, respectively $\partial U_2$, be the boundary of the component of $S^{n+1} \setminus B_1$, respectively $S^{n+1} \setminus B_2$, that contains the set $\hat{\mathbb{R}}_c \cup \{p_1, p_2, \ldots\}$ in $B_1$, respectively $B_2$. Then, using Theorem 2.3, we make a new Sierpiński carpet $B$ from $B_1$ and $B_2$ by identifying the points of $\partial U_1$ with the corresponding points on $\partial U_2$. This means that the set $\hat{\mathbb{R}}_c \cup \{p_1, p_2, \ldots\} \subset B_1$ now only contains interior points of $B$. The imbeddings of $E_4$ in $\mathcal{H}(B_1)$ and $\mathcal{H}(B_2)$ by Dijkstra naturally give rise to an imbedding $G$ of $E_4$ in $\mathcal{H}(B)$ that satisfies the requirements of the proposition and is such that $\mathbb{R}_c$ and $A$ both consist of interior points of $B$. Applying [4, Remark 4] we see that there exists an imbedding $G$ as in the proposition with the property that $\mathbb{R}_c$ and $A$ both consist of interior points of $M_n^{n+1}$.

**Case II :** $n = 1$. In this case the set $\hat{\mathbb{R}}_c$ consists of boundary points of $\overline{B}$ and the sequence $p_1, p_2, \ldots$ consists, with the exception of one point, of interior points of $\overline{B}$; see [4, §5]. We note that all points of $\hat{\mathbb{R}}_c \cup \{p_1, p_2, \ldots\}$ that are boundary points of $\overline{B}$ lie in the boundary of the same component of $S^{2} \setminus \overline{B}$. This means that we can use the same argumentation as in the case $n \in \mathbb{N} \setminus \{1, 3\}$ to show that we can find an imbedding $G$ as required and such that $\mathbb{R}_c$ and $A$ both consist of interior points of $M_2^{2}$.

Now we observe that it follows from the definition of $g$ and the construction of Dijkstra that all the points $p_i$ might as well be chosen as boundary points of $\overline{B}$. With [4, Remark 4] it can then be shown that we can find an imbedding $G$ that is as desired and with the property that $\mathbb{R}_c$ and $A$ both consist of boundary points of $M_2^{2}$.

Consider now two disjoint copies $B_1, B_2$ of $\overline{B}$ in $S^{2}$ and assume that all the points $p_i$ in $B_1$ consist of boundary points. Let $\partial U_1$, respectively $\partial U_2$, be the boundary of the component of $S^{2} \setminus B_1$, respectively $S^{2} \setminus B_2$, that contains the set $\hat{\mathbb{R}}_c$. We have that $\hat{\mathbb{R}}_c$ in $B_1$ is an arc in the simple closed curve $\partial U_1$ and similarly, the set $\hat{\mathbb{R}}_c$ in $B_2$ is an arc in $\partial U_2$. This means that,
using Theorem 2.3, we can form a new Sierpiński carpet $B$ from $B_1$ and $B_2$ by simply identifying the points of the set $\mathbb{R}_c$ in $B_1$ with the corresponding points of the set $\mathbb{R}_c$ in $B_2$. Then we have that $\mathbb{R}_c \subseteq B$ now consists of interior points of $B$ and the points $\pm \infty$ are boundary points of $B$. The imbeddings of $E_4$ in $\mathcal{H}(B_1)$ and $\mathcal{H}(B_2)$ by Dijkstra naturally extend to an imbedding $G$ of $E_4$ in $\mathcal{H}(B)$ that satisfies the properties (a)–(d) and is such that $\mathbb{R}_c$ consists of interior points of $B$ and $A$ consists of boundary points of $B$. Applying [4, Remark 4] we see that there exists an imbedding $G$ as in the proposition with the property that $\mathbb{R}_c$ consists of interior points of $M_i^2$ and $A$ consists of boundary points of $M_i^2$. □

We can now give the proof of Theorem 3.1.

Proof of Theorem 3.1. Take an open subset $U$ of $M_n^{n+1}$ that contains $O$. Let $\rho$ be a metric on $M_n^{n+1}$ and let $\hat{\rho}$ be the induced metric on $\mathcal{H}(M_n^{n+1})$: $\hat{\rho}(f, g) = \max_{x \in M_n^{n+1}} \rho(f(x), g(x))$ for $f, g \in \mathcal{H}(M_n^{n+1})$. Note that $\hat{\rho}$ is right-invariant: $\hat{\rho}(f \circ h, g \circ h) = \hat{\rho}(f, g)$ for any $h \in \mathcal{H}(M_n^{n+1})$. We prove the theorem by showing that $\mathcal{H}_U(M_n^{n+1}, D)$ satisfies the conditions of Definition 2.20. The result then follows from Theorem 2.21. Without loss of generality we may assume that $D \cap (M_n^{n+1} \setminus U)$ is dense in $M_n^{n+1} \setminus U$. Let $\mathcal{T}$ be the topology that $\mathcal{H}_U(M_n^{n+1}, D)$ inherits from the zero-dimensional product space $D^D$ via the injection $h \mapsto h|D$. It follows from [5, Theorem 10.1] that $\mathcal{T}$ is an $F_{\sigma\delta}$-topology that witnesses the almost zero-dimensionality of $\mathcal{H}_U(M_n^{n+1}, D)$.

Consider the spaces $E_4$ and $Z_4$ and the projection map $\xi_i: E_4 \to E_4$ for $i \in \omega$ as given in Definition 3.3. We let $P$ be the countable dense subset $\bigcup_{i=0}^{\infty} \xi_i(E_4)$ of $E_4$. Consider now the Cantor set

$$C' = \{ z \in E_4 : z_i \in \{0, 4^{-i}\} \text{ for } i \in \omega \},$$

and note that since $\sum_{i=0}^{\infty} 4^{-i} < \infty$ the norm topology and the product topology coincide on $C'$. Let $\delta: C' \to \mathbb{R}^+$ be the imbedding that is given by the rule $\delta(z) = \|z\|$. We define $C = \delta(C')$, $\gamma = \delta^{-1}|C$, and $Q = \delta(C' \cap P)$. Thus $C$ is a Cantor set with $Q$ as a countable dense subset and $\|\gamma(r)\| = r$ for each $r \in C$. We define subspaces $\mathcal{E}_c$ and $\mathcal{E}$ of $\ell^1$ as follows

$$\mathcal{E}_c = \{ z \in \ell^1 : z_i \in C \text{ for } i \in \omega \}$$

and

$$\mathcal{E} = \{ z \in \ell^1 : z_i \in Q \text{ for } i \in \omega \}.$$

The subscript $c$ refers to the fact that $\mathcal{E}_c$ is a complete space. We let $Z_c$ and $Z$ stand for $\mathcal{E}_c$ respectively $\mathcal{E}$ with the witness topologies that these spaces
inherit from $\mathbb{R}^\omega$. Let $\nu: \omega \times \omega \to \omega$ be a bijection such that $\nu(i, j) \geq j$ for all $i, j \in \omega$. We define an imbedding $\zeta: \mathcal{E}_c \rightarrow E_4$ by the rule $(\zeta(z))_{\nu(i,j)} = (\gamma(z_i))_j$ for $z \in \mathcal{E}_c$ and $i, j \in \omega$. It is clear from the definition and the fact that the norm and product topology coincide on the compactum $C'$ that $\zeta: Z_c \rightarrow Z_4$ is a closed imbedding. Note that $\|\zeta(z)\| = \|z\|$ for each $z \in \mathcal{E}_c$ which implies that $\zeta$ is also a closed imbedding with respect to the norm topologies (recall that the norm topology is generated by the product topology together with the norm function).

We select a null sequence of nonempty open sets $V_0, V_1, \ldots$ such that their closures are disjoint subsets of $O$. Put $V = \bigcup_{k=0}^{\infty} V_k$. Using Proposition 3.4 we can find for every $k \in \omega$ a closed imbedding $G^k: E_4 \rightarrow \mathcal{H}_{V_k}(M_{n+1}^0)$, a copy $\hat{\mathbb{R}}_k$ of $\mathbb{R}$ in $V_k$ and a sequence $p^0_k, p^1_k, \ldots \in V_k \setminus \hat{\mathbb{R}}_k$ such that the conditions (a)--(d) of Proposition 3.4, with $\hat{\mathbb{R}}_k$ replaced by $\hat{\mathbb{R}}_k$ and $p_i$ replaced by $p^i_k$, are satisfied for $G^k$. If $x \in \mathbb{R}$ we write $x_k$ for the representation of $x$ in $\hat{\mathbb{R}}_k$. Let $A_k = \{\infty, p^0_k, p^1_k, \ldots\}$ and let $\beta_k: \mathcal{H}(M_n^0) \rightarrow (M_{n+1}^0)^{A_k}$ be given by $\beta_k(h) = h\mid A_k$. Then we have that condition (d) of Proposition 3.4 is satisfied for $G^k$ with the set $A_k$ and the map $\beta_k$.

We now define $H: \mathcal{E}_c \rightarrow \mathcal{H}_V(M_{n+1}^0)$ by

$$H_z(x) = \begin{cases} G^0_{\zeta(z)}(x), & \text{if } x \in V_0; \\ G^k_{\gamma(z_{k-1})}(x), & \text{if } x \in V_k \text{ for some } k \in \mathbb{N}; \\ x, & \text{if } x \in M_{n+1} \setminus V, \end{cases}$$

for $z \in \mathcal{E}_c$. Since the $V_k$’s form a null sequence it is clear that every $H_z$ is a homeomorphism of $M_{n+1}^0$ and that $H_z$ depends continuously on $z \in \mathcal{E}_c$. Let $\Pi: \mathcal{H}_V(M_{n+1}^0) \rightarrow \mathcal{H}_{V_0}(M_{n+1}^0)$ be the continuous map that is defined by $\Pi(h) = (h\mid V_0) \cup e_{M_{n+1} \setminus V_0}$. Since $\zeta$ and $G^0$ are closed imbeddings and $\Pi \circ H = G^0 \circ \zeta$ we have by Lemma 2.1 that $H: \mathcal{E}_c \rightarrow \mathcal{H}_V(M_{n+1}^0)$ is also a closed imbedding. Now we consider the three cases of Claim 3.2 separately.

Case (i). In this case we have that $D \cap O$ consists entirely of interior points of $M_{n+1}^0$. Choose a $k \in \omega$. With Proposition 3.4 we can choose the imbedding $G^k$ in (1) such that $A_k$ and $\mathbb{R}_k$ consist of interior points of $M_{n+1}^0$. Note that $\mathbb{R}_k$ is a nowhere dense subset of $V_k$. This means that we can find a countable dense subset $D_k$ of $V_k$, consisting of interior points of $M_{n+1}^0$, with $D_k \cap \mathbb{R}_k = \emptyset$ and $A_k \subset D_k$. Since $P$ is countable and $G^k_\zeta(\mathbb{R}_k) = \mathbb{R}_k$ for all $z \in E_4$, see property (b) of Proposition 3.4, we may assume that $G^k_\zeta(D_k) = D_k$ for each $z \in P$. Let $\mathbb{Q}_4$ be the additive group $\{i4^j : i, j \in \mathbb{Z}\}$ and note that $C \cap \mathbb{Q}_4 = Q$. Let $\mathbb{Q}_4^L$ be the copy of $\mathbb{Q}_4$ that lies in $\mathbb{R}_k$, so $\mathbb{Q}_4^L$ entirely consists of interior points of $M_{n+1}^0$. As observed in Remark 2.10 we
may assume that the set $D$ has the properties
\[
D \cap V_0 = D_0,
\]
\[
D \cap V_k = D_k \cup Q^k_k \quad \text{for } k \in \mathbb{N}.
\]

We verify that
\[
\mathcal{E} = \{ z \in \mathcal{E}_c : H_z(D) = D \}
\]
and hence that $H \mathcal{E}$ is a closed embedding of $\mathcal{E}$ into $\mathcal{H}_U(M_{n+1}^n, D)$ for $n \in \mathbb{N}$. If $H_z \in \mathcal{H}_U(M_{n+1}^n, D)$ and $k \in \mathbb{N}$ then by property (b) of Proposition 3.4 we have $H_z(0_k) = \|\gamma(z_k-1)\| = z_{k-1} \in \mathbb{Q}_4$. Since $z \in \mathcal{E}_c$ we also have $z_{k-1} \in C$ and hence $z_{k-1} \in Q$. Thus $z \in \mathcal{E}$. Consider now a $z \in \mathcal{E}$. If $x \in V_k \setminus \mathbb{R}_k$ for some $k \in \omega$ then by property (c) of Proposition 3.4 there is a $z' \in P$ such that $H_z(x) = G^k_{z'}(x)$. Since $G^k_{z'}(D_k) = D_k$ we have that $x \in D_k = D \cap V_k \setminus \mathbb{R}_k$ if and only if $H_z(x) \in D_k$. Note that $H_z(\mathbb{R}_0) = \mathbb{R}_0$ and that this set is disjoint from $D$. Consider finally the case that $x \in \mathbb{R}_k$ for $k \in \mathbb{N}$. Then $z_{k-1} \in Q \subset \mathbb{Q}_4$ and $H_z(x) = G^k_{\gamma(z_k-1)}(x) = x + \|\gamma(z_k-1)\| = x + z_{k-1}$ which is in $Q_4$ if and only if $x \in \mathbb{Q}_4$.

Remember that $\mathcal{T}$ is the topology on $\mathcal{H}_U(M_{n+1}^n, D)$ that it inherits from $D^D$. Let $\mathcal{T}'$ be the topology that $\mathcal{H}(M_{n+1}^n)$ inherits from the product space $(M_{n+1}^n)^D$ and note that $\mathcal{T}'$ restricts to $\mathcal{T}$ on $\mathcal{H}_U(M_{n+1}^n, D)$. We first verify that $H : Z_c \to (\mathcal{H}(M_{n+1}^n), \mathcal{T}')$ is continuous. Let $x \in D$. If $x \not\in V$ or if $x \in V_k$ for some $k \in \mathbb{N}$, then $H_z(x)$ depends on at most a single coordinate of $z$, so continuity with respect to the product topology is obvious. Let $x \in V_0$ and thus $x \in D_0 \subset V_0 \setminus \mathbb{R}_0$. Then by property (c) of Proposition 3.4, $G^0_{z'}(x)$ depends on only finitely many coordinates of $z' \in E_4$ and hence $H_z(x) = G^0_{\gamma(z)}(x)$ depends also on only finitely many coordinates of $z \in Z_c$. This shows that $H$ is continuous with respect to the product topologies. With property (d) of Proposition 3.4 we find that $\beta_0 \circ H = \beta_0 \circ G^0 \circ \zeta$ is a closed imbedding of $Z_c$ into $\beta_0(\mathcal{H}(M_{n+1}^n))$. Since $A_0 \subset D$ we have that $\beta_0 : (\mathcal{H}(M_{n+1}^n), \mathcal{T}') \to (M_{n+1}^n)^A_0$ is continuous. Thus with Lemma 2.1 we may conclude that $H : Z_c \to (\mathcal{H}(M_{n+1}^n), \mathcal{T}')$ is a closed imbedding. Since $Z = H^{-1}(\mathcal{H}_U(M_{n+1}^n, D))$ we also have that $H \mathcal{E}$ is a closed imbedding of $Z$ in $(\mathcal{H}_U(M_{n+1}^n, D), \mathcal{T})$.

Consider the point $0_1 \in \mathbb{Q}_4 \subset \mathbb{R}_1$. For every $a \in D$ we define $\Gamma_a = \{ h \in \mathcal{H}_U(M_{n+1}^n, D) : h(0_1) = a \}$. Note that every $\Gamma_a$ is closed with respect to $\mathcal{T}$ and that $\bigcup_{a \in D} \Gamma_a = \mathcal{H}_U(M_{n+1}^n, D)$. Let for $i \in \mathbb{N}$, $z^i = (4^{-i}, 0, 0, \ldots) \in \mathcal{E}$ and let $h \in \Gamma_{a_i}$. Since $\lim_{i \to \infty} z^i = 0$, where $0$ denotes the zero vector in $\mathbb{R}^\omega$, we have that $\lim_{i \to \infty} h \circ H_{0}^{-1} \circ H_{z^i} = h$ in $\mathcal{H}_U(M_{n+1}^n, D)$. However, $h \circ H_{0}^{-1} \circ H_{z^i} \not\in \Gamma_{a_i}$. To see this, note that it follows from Proposition 3.4, property (b), that $H_0 \mathcal{R}_1 = e_{\mathcal{R}_1}$ and $H_{z^i}(0_1) = (4^{-i})_1$. This implies that
$h(H_0^{-1}(H_z(0_1))) = h((4^{-i})_1) \neq h(0_1) = a$. Thus $\Gamma_a$ is nowhere dense in $\mathcal{H}_U(M_n^{n+1}, D)$ and condition $(5')$ of Definition 2.20 is satisfied.

We now make an observation which will be the key to satisfying conditions $(2')$ and $(4')$ of Definition 2.20.

**Claim 3.5.** If $A$ is an unbounded subset of $E$ then

$$\text{diam}_\rho\{H_z : z \in A\} \geq \rho(-\infty_0, \infty_0).$$

**Proof.** Let $z \in A$ and let $n \in \mathbb{N}$ arbitrary. Select a $z^n \in A$ such that $\|z^n\| > \|z\| + 2n$. It follows from $(1)$, condition (b) of Proposition 3.4 and the fact that $\|\zeta(z)\| = \|z\|$ for all $z \in \mathcal{E}_c$ that

$$H_z((-\|z\| - n)_0) = G_\zeta(z)((-\|z\| - n)_0) = -n_0.$$  

Similarly, we see that

$$H_{z^n}((-\|z\| - n)_0) = (\|z^n\| - \|z\| - n)_0.$$  

We conclude that

$$\text{diam}_\rho\{H_z : z \in A\} \geq \limsup_{n \to \infty} \rho(H_z, H_{z^n}) \geq \lim_{n \to \infty} \rho(-n_0, (\|z^n\| - \|z\| - n)_0) = \rho(-\infty_0, \infty_0),$$

proving Claim 3.5. \qed

Let $T = Q^{<\omega}$ and define for $s = q_1 \ldots q_k \in T$ with $k \in \omega$ the subspace $\mathcal{E}_s$ of $E$ by

$$\mathcal{E}_s = \{z \in E : z_{i-1} = q_i \text{ for } 1 \leq i \leq k\}.$$  

With the same arguments as given after Theorem 2.21 we see that the spaces $\mathcal{E}_s$ satisfy the conditions of Definition 2.20, every bounded subset of $E$ is an anchor for $(\mathcal{E}_s)_{s \in T}$ in $Z$, and every nonempty clopen subset of any $\mathcal{E}_s$ is unbounded. Let $J = \{f_q : q \in Q\}$ be a countable dense subset of $\mathcal{H}_U(M_n^{n+1}, D)$. Since $H : Z \to (\mathcal{H}_U(M_n^{n+1}, D), \mathcal{T})$ is a closed map we have that $X_s = \{H_z : z \in \mathcal{E}_s\}$ is closed with respect to $\mathcal{T}$ for each $s \in T$. We define $(E_s)_{s \in T}$ as follows:

$$E_\lambda = X_\lambda \circ J$$

and if $s = q_0 \ldots q_k \in T \setminus \{\lambda\}$ then

$$E_s = X_{q_1 \ldots q_k} \circ f_{q_0}.$$
Note that if \( f \in \mathcal{H}_U(M_{n+1}^n, D) \) then the map \( h \mapsto h \circ f \) is a homeomorphism of \( (\mathcal{H}_U(M_{n+1}^n, D), \mathcal{T}) \) as well as of \( \mathcal{H}_U(M_{n+1}^n, D) \). So every \( E_s \) is closed with respect to \( \mathcal{T} \) provided \( s \neq \lambda \).

It remains to show that \((E_s)_{s \in T}\) satisfies conditions (1')–(4') of Definition 2.20. Since \( X_\lambda \neq \emptyset \) we have that \( E_\lambda \) is, just as \( J \), dense in \( \mathcal{H}_U(M_{n+1}^n, D) \). The other part of condition (1') follows with the same ease. Since \( H : E \rightarrow \mathcal{H}_U(M_{n+1}^n, D) \) is an imbedding we have that condition (3') is satisfied. Now let \( W \) be an arbitrary set in \( \mathcal{H}_U(M_{n+1}^n, D) \) such that \( \text{diam}(W) < \rho(\infty_0, \infty_0) \). We show that \( W \) works for condition (2') as well as (4'). Let \( \sigma = q_0q_1 \ldots \in [T] \) be such that \( E_{\sigma|k} \cap W \neq \emptyset \) for each \( k \in \omega \).

Putting \( \tau = q_1q_2 \ldots \in [T] \) we have that \( X_\tau|k \cap (W \circ f_{q_0}^{-1}) \neq \emptyset \) for each \( k \in \omega \).

Since \( \rho \) is right invariant it follows that \( \text{diam}_\rho(W \circ f_{q_0}^{-1}) < \rho(\infty_0, \infty_0) \) and hence \( F = \{ z \in E : H_z \in W \circ f_{q_0}^{-1} \} \) is bounded by Claim 3.5. Thus \( F \) is an anchor for \((E_s)_{s \in T}\) in \( Z \) and obviously \( E_{\tau|k} \cap F \neq \emptyset \) for each \( k \in \omega \). Thus \( E_{\tau|0}, E_{\tau|1}, \ldots \) converges to an element \( z \) in \( Z \). Then \( X_{\tau|0}, X_{\tau|1}, \ldots \) converges to \( H_z \) and \( E_{\sigma|0}, E_{\sigma|1}, \ldots \) converges to \( H_z \circ f_{q_0} \), both with respect to \( \mathcal{T} \). Thus condition (2') is satisfied. Now let \( C \) be a nonempty clopen subset of some \( E_s \) such that \( C \subset W \). We may assume that \( |s| \geq 1 \) and we put \( q = s|1 \) and \( q \leftarrow t = s \).

So \( \text{diam}_\rho(C \circ f_{q}^{-1}) < \rho(\infty_0, \infty_0) \) and \( C \circ f_{q}^{-1} \) is a nonempty clopen subset of \( X_t \). This means that \( \{ z \in E : H_z \in C \circ f_{q}^{-1} \} \) is a nonempty, clopen, bounded subset of \( E_t \). As mentioned above, this contradicts Corollary 2.15, so we conclude that (4') is satisfied and \( \mathcal{H}_U(M_{n+1}^n, D) \in \mathcal{E}' \). Now apply Theorem 2.21 to see that \( \mathcal{H}_U(M_{n+1}^n, D) \) is homeomorphic to \( \mathcal{E} \).

**Case (ii).** In this case we have that \( D \cap \partial U_i \cap O \) is dense in \( \partial U_i \cap O \) for every \( i \) and the interior points of \( M_{n+1}^n \) contained in \( D \cap O \) are dense in \( O \). We use the same method as in case (i). Take \( k \in \omega \). By means of Proposition 3.4 we choose the imbedding \( G^k \) in (1) again such that the sets \( A_k \) and \( \mathbb{R}_k \) both consist of interior points of \( M_{n+1}^n \). Noting that \( \mathbb{R}_k \) is a nowhere dense subset of \( M_{n+1}^n \) we can find a countable dense subset \( D_k \) of \( V_k \) such that \( A_k \subset D_k \), \( D_k \cap \mathbb{R}_k = \emptyset \), \( D_k \cap \partial U_i \) is dense in \( \partial U_i \cap V_k \) for every \( i \) with \( \partial U_i \cap V_k \neq \emptyset \) and the interior points of \( M_{n+1}^n \) in \( D_k \) are also dense in \( V_k \).

Furthermore, we may assume that \( G^k(\mathbb{R}_k) = \mathbb{R}_k \) for all \( z \in E_4 \). It follows from Lemma 2.9 that we may assume that \( D \) has the properties mentioned in (2). We continue in precisely the same way as in case (i) to conclude that \( \mathcal{H}_U(M_{n+1}^n, D) \in \mathcal{E}' \) and hence \( \mathcal{H}_U(M_{n+1}^n, D) \) is homeomorphic to \( \mathcal{E} \) according to Theorem 2.21.

**Case (iii).** In this case \( D \cap \partial U_i \cap O \) is dense in \( \partial U_i \cap O \) for every \( i \) and \( D \cap O \) contains no interior points of \( M_{n+1}^n \). Again, we want \( D \) to
have the properties as mentioned in (2) for appropriate sets $D_k$ so that in the same way as in case (i) (and (ii)) we can conclude that $\mathcal{H}_U(M_n^{n+1}, D)$ is homeomorphic to $\mathcal{C}$. We have to treat the cases $n = 1$ and $n > 1$ separately.

First we consider the case $n = 1$. We want that $D \cap V_0 = D_0$, with $D_0$ a countable dense subset of $V_0$ with $A_0 \subset D_0$ and $D_0 \cap \mathbb{R} = \emptyset$. Since $D$ only contains boundary points of $M^2$ we want that $D_0$ consists of boundary points of $M^2$. Furthermore, since we are aiming towards Remark 2.10 again, we also want that $D_0$ is dense in $\partial U_i \cap V_0$ for every $i$ with $\partial U_i \cap V_0 \neq \emptyset$. This means that $\mathbb{R}_0$ cannot be contained in the boundary of some component $U_i$ of the complement of $M^2$. Therefore, we choose $G^0$ in (1) such that $A_0$ consists of boundary points of $M^2$ and $\mathbb{R}_0$ consists of interior points of $M^2$. This is possible according to Proposition 3.4. It is clear then that we can find a set $D_0$ as required and with Remark 2.10 we may indeed conclude that $D \cap V_0 = D_0$.

Now take $k \in \mathbb{N}$. Just as in (2) we want that $D \cap V_k = Q_k^4 \cup D_k$, where $D_k$ is a countable dense subset of $V_k$ with $D_k \cap \mathbb{R}_k = \emptyset$ and $A_k \subset D_k$. Since $D$ consists entirely of boundary points of $M^2$ we choose $G^k$ in (1) such that both $A_k$ and $\mathbb{R}_k$ contain only boundary points of $M^2$. This can be done according to Proposition 3.4. Suppose that $\mathbb{R}_k \subset \partial U_{i_k}$ for some component $U_{i_k}$ of the complement of $M^2$. Noting that $\mathbb{R}_k$ is a nowhere dense subset of $M^2$ we can choose the set $D_k$ such that it is made out of boundary points of $M^2$, it is dense in $\partial U_i \cap V_k$ for every $i \in \omega \setminus \{i_k\}$ with $\partial U_i \cap V_k \neq \emptyset$ and it is dense in $(\partial U_{i_k} \setminus \mathbb{R}_k) \cap V_k$. We see that $D_k \cup Q_k^4$ is a countable dense subset of $V_k$, entirely consisting of boundary points of $M^2$, that is dense in $\partial U_i \cap V_k$ for every $i$ with $\partial U_i \cap V_k \neq \emptyset$. It then follows from Remark 2.10 that we may assume that indeed $D \cap V_k = Q_k^4 \cup D_k$.

We conclude that we may assume that $D$ satisfies (2). As before, we may assume that $G^k(D_k) = D_k$ for all $k \in \omega$ and each $z \in P$, so we can continue in the same way as in case (i) to conclude that $\mathcal{H}_U(M_n^{n+1}, D)$ is homeomorphic to $\mathcal{C}$.

Now consider the case that $n \in \mathbb{N} \setminus \{1, 3\}$. This is easier than the one-dimensional case. Take $k \in \omega$. Using Proposition 3.4 we choose the imbedding $G^k$ in (1) such that both the sets $A_k$ and $\mathbb{R}_k$ consist of boundary points of $M_n^{n+1}$. Note that if $\mathbb{R}_k \subset \partial U_{i_k}$ then $\mathbb{R}_k$ is, in contrast to the case $n = 1$, nowhere dense in $\partial U_{i_k}$. This means that we can find a countable dense subset $D_k$ of $V_k$, consisting of boundary points of $M_n^{n+1}$, such that $A_k \subset D_k$, $D_k \cap \mathbb{R}_k = \emptyset$ and $D_k \cap \partial U_i$ is dense in $\partial U_i \cap V_k$ for all $i$ such that $V_k \cap \partial U_i \neq \emptyset$. With Remark 2.10 it follows that we may assume that $D \cap V_0 = D_0$ if $k = 0$. 


and $D \cap V_k = Q_k^k \cup D_k$ if $k \in \mathbb{N}$, so we may assume (2). Again, without loss of generality we have that $G_z^k(D_k) = D_k$ for all $k \in \omega$ and each $z \in P$, so the same reasoning as in case (i) shows that $\mathcal{H}_U(M_n^{n+1}, D)$ is homeomorphic to $\mathcal{E}$. \hfill $\Box$

In analogy to [5, Theorem 10.4] and [5, Remark 10.7] we can adapt the proof of Theorem 3.1 to produce the following slight generalization.

**Theorem 3.6.** Let $X$ be a locally compact space and let $D'$ be a countable dense subset of $X$. Suppose that $X$ contains an open subset $O'$ that is homeomorphic to an open $O \subseteq M_n^{n+1}$ for some $n \in \mathbb{N} \setminus \{3\}$, such that $D' \cap O'$ corresponds to a countable dense subset $D$ of $O$ that satisfies the conditions of Theorem 3.1. Then $\mathcal{H}_U(X, D')$ is homeomorphic to $\mathcal{E}$ for every open set $U$ that contains $O'$.

**References**


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