On the topological structure of spaces of fuzzy compacta

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Abstract

We prove that the space of fuzzy compacta in a Peano continuum is homeomorphic to the Hilbert space \(\ell^2\). As a corollary we find that the spaces of fuzzy compacta in \(\mathbb{R}^n\) and \(\ell^2\) are also homeomorphic to Hilbert space.

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1. Introduction

All topological spaces in this paper are assumed to be separable metric. Let \((X, d)\) be such a space. We let the hyperspace \(K(X)\) denote the family of all nonempty compact subsets in \(X\). The natural topology on \(K(X)\) is the Vietoris topology which is generated by the Hausdorff metric \(d_H\) that is associated with \(d\). With respect to the Hausdorff metric two compacta \(A\) and \(B\) are \(\varepsilon\)-close if \(B\) is contained in the \(\varepsilon\)-neighbourhood of \(A\) and vice versa. Significant early results about the topological structure of \(K(X)\) were obtained by Wojdyslawski [13,14]. Let \(I\) be the closed interval \([0,1]\) and let \(Q\) be the Hilbert cube \(I^\mathbb{N}\), the countable infinite Cartesian product of copies of \(I\). Probably the most important theorem about \(K(X)\) is by Curtis and Schori [4] and states that the hyperspace \(K(X)\) is homeomorphic to \(Q\) if and only if \(X\) is a non-degenerate Peano continuum, that is, a connected and locally connected compactum. By the celebrated Hahn–Mazurkiewicz Theorem [6,9] the Peano continua are precisely the continuous images of \(I\).

Following Kloeden [7,8], who uses the concept to investigate fuzzy dynamical systems, we define a fuzzy compactum in \(X\) as an upper semi-continuous function \(f: A \to I\) such that \(A \in K(X)\) and \(\{x \in A : f(x) > 0\}\) is dense in \(A\). That is, the compact subset \(A\) of \(X\) comes equipped with a density function. Let \(K(X)\) denote the collection of all fuzzy compacta in \(X\). Note that \(K(X)\) is canonically imbedded in \(K(X)\) by choosing the density function equal to one at each point. A function \(f: A \to I\) is upper semi-continuous precisely if its hypograph \(\text{hyp } f = \{(x, t) : 0 \leq t \leq f(x)\}\) is closed.

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in the product $A \times I$. Thus for an $f \in \mathcal{K}(X)$ we have that $\text{hyp } f$ is a compact subset of $X \times I$ and we can topologize $\mathcal{K}(X)$ in a natural way by seeing it as a subspace of $\mathcal{K}(X \times I)$ with the Hausdorff metric. Using an approach similar to the one of Curtis–Schori we investigate the topology of $\mathcal{K}(X)$. Our main result is the following theorem.

**Theorem 1.** Let $X$ be compact. The space $\mathcal{K}(X)$ is homeomorphic to the separable Hilbert space $l^2$ if and only if $X$ is a non-degenerate Peano continuum.

Thus for Peano continua $X$ all the spaces $\mathcal{K}(X)$ are homeomorphic to each other. As an application of Theorem 1, we have:

**Corollary 2.** $\mathcal{K}(\mathbb{R}^n)$ and $\mathcal{K}(l^2)$ are also homeomorphic to $l^2$.

**2. Preliminaries**

For a metric space $(X, d)$, an $x \in X$ and a nonempty subset $A$ of $X$ we put $d(x, A) = \inf \{d(x, a) : a \in A\}$. If $\varepsilon \geq 0$ then we define the open sets $U_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ and $U_d(A, \varepsilon) = \{y \in X : d(y, A) < \varepsilon\}$ and the closed sets $\overline{U}_d(x, \varepsilon) = \{y \in X : d(x, y) \leq \varepsilon\}$ and $\overline{U}_d(A, \varepsilon) = \{y \in X : d(y, A) \leq \varepsilon\}$. As in the introduction let $\mathcal{K}(X)$ be the set of nonempty compacta in $X$ and let $2^X$ stand for the set of nonempty closed subsets of $X$. The Hausdorff distance between elements $A$ and $B$ of $2^X$ is given by

$$d_H(A, B) = \sup\{d(a, B), d(b, A) : a \in A, b \in B\}.$$  

If $d_H(A, B)$ is finite then it equals $\min\{\varepsilon : A \subseteq U_d, B \subseteq U_d\}$. The topology that $d_H$ generates on $\mathcal{K}(X)$ is independent of the choice of a compatible metric $d$ on $X$ because it coincides with the Vietoris topology; see [10, Lemma 1.11.11].

Let $f$ be a real-valued function $f$ on a space $X$. Denote the domain of $f$ by $\text{dom } f$. If $t \in \mathbb{R}$ then we put $f_{> t} = \{x \in \text{dom } f : f(x) > t\}$ and $f_{\geq t} = \{x \in \text{dom } f : f(x) \geq t\}$. If $f_{= 0}$ then the support of $f$, denoted by $\text{supp } f$, is the closure of $f_{> 0}$ in $\text{dom } f$. The function $f$ is called upper semi-continuous (USC) if $f_{> t}$ is closed in $\text{dom } f$ for every $t$. We let $\text{USC}(X)$ stand for the set of all USC functions from $X$ into $I$ and $\text{USCC}(X)$ is the set of all elements of $\text{USC}(X)$ with compact support. We define

$$\mathcal{Q}(X) = \{f \in \text{USC}(A) : A \in \mathcal{K}(X)\}$$

and

$$\mathcal{K}(X) = \{f \in \mathcal{Q}(X) : \text{supp } f = \text{dom } f\}.$$  

As in the Introduction we have for $f \in \text{USC}(X)$

$$\text{hyp } f = \{(x, t) \in X \times I : t \leq f(x)\}.$$  

If $f \in \text{USC}(X)$ respectively $f \in \mathcal{Q}(X)$, then $\text{hyp } f$ is closed respectively compact as a subspace of $X \times I$. Equip $X \times I$ with the metric $d'((x, t), (y, s)) = \max\{d(x, y), |t - s|\}$. Then we define a metric $D$ on $\text{USC}(X)$, $\text{USCC}(X)$, $\mathcal{Q}(X)$, and $\mathcal{K}(X)$ by $D(f, g) = d'_H(\text{hyp } f, \text{hyp } g)$. If $X$ is compact then of course $\text{USC}(X) = \text{USCC}(X)$. In Kloeden [7,8] the metric $D$ on $\text{USCC}(X)$ is called the endograph metric and on $\mathcal{K}(X)$ the metric $D$ corresponds to the sendograph metric. If $f, g \in \mathcal{Q}(X)$ then $\nu \vee g$ is the element of $\mathcal{Q}(X)$ that satisfies the equation $\text{hyp } (f \nu \vee g) = \text{hyp } f \cup \text{hyp } g$, so if $\text{dom } f = \text{dom } g$ then $\nu \vee g = \max\{f, g\}$. We have that $\nu$ is a continuous operation on $\mathcal{Q}(X)$ because $\cup$ is continuous on $\mathcal{K}(X \times I)$.

If $f \in \mathcal{Q}(X)$ then $\tilde{f} \in \text{USCC}(X)$ is defined by $\tilde{f}(x) = f(x)$ if $x \in \text{dom } f$ and $\tilde{f}(x) = 0$ otherwise. With this operation $\mathcal{K}(X)$ can be seen as a subset of $\text{USCC}(X)$, however, the topology on $\mathcal{K}(X)$ is much stronger than the topology it would inherit from $\text{USCC}(X)$. The following fact is easily verified.

**Proposition 3.** For $f, g \in \mathcal{Q}(X)$ we have

$$D(f, g) = \max\{D(\tilde{f}, \tilde{g}), d_H(\text{dom } f, \text{dom } g)\}.$$  

Thus $Q(X)$ and $\mathbb{K}(X)$ are isometrically imbedded in $USCC(X) \times K(X)$ with the metric $\rho((f, A), (g, B)) = \max\{D(f, g), d_H(A, B)\}$. From now on we will represent these spaces as subspaces of $(USCC(X) \times K(X), \rho)$:

$$Q(X) = \{(f, A) \in USCC(X) \times K(X) : \text{supp } f \subset A\}$$

and

$$\mathbb{K}(X) = \{(f, A) \in USCC(X) \times K(X) : \text{supp } f = A\}.$$  

As usual, the characteristic function of a set $A$ is given by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Note that $\chi$ is an imbedding of $K(X)$ in $USCC(X)$.

**Lemma 4.** $Q(X)$ is the closure of $\mathbb{K}(X)$ in $USCC(X) \times K(X)$.

**Proof.** Suppose that $(f, A) \in (USCC(X) \times K(X)) \setminus Q(X)$. Then $\text{supp } f \setminus A \neq \emptyset$ and there exists an $x \in X$ such that $f(x) > 0$ and $x \notin A$. Let $\varepsilon = \frac{1}{4} \min\{f(x), d(x, A)\}$ and let $(g, B) \in U_{\rho}(f, A, \varepsilon)$. We have that there exists a $(y, t) \in \text{hyp } g$ such that $d'((x, f(x)), (y, t)) < \varepsilon$, which means that $d(x, y) < \varepsilon$ and $g(y) > 0$. Note that $U_{\rho}(x, \varepsilon) \cap U_{\rho}(A, \varepsilon) = \emptyset$, so $y \in \text{supp } g \setminus B$. Hence $(g, B) \notin Q(X)$ and we may conclude that $Q(X)$ is closed in $USCC(X) \times K(X)$.

For each $(f, A) \in Q(X)$ and $\varepsilon > 0$, it is easy to see that $\rho((f, A), (f \vee \varepsilon \chi_A, A)) \leq \varepsilon$. Clearly, $\text{supp}(f \vee \varepsilon \chi_A) = A$ thus $(f \vee \varepsilon \chi_A, A) \in \mathbb{K}(X)$. Consequently, $\mathbb{K}(X)$ is dense in $Q(X)$.

A map $f : X \rightarrow \mathbb{R}$ is said to be Lipschitz if there exists some $k \geq 0$ such that $|f(x) - f(x')| \leq kd(x, x')$ for all $x, x' \in X$. The smallest such $k$ is called the Lipschitz constant of $f$ and denoted by $\text{lip } f$. If $\text{lip } f \leq k$, then $f$ is said to be $k$-Lipschitz. Let

$$\text{LIP}_k(X) = \{f \in USC(X) : f \text{ is } k\text{-Lipschitz}\}.$$  

Zhang and Yang [17] classify spaces of Lipschitz functions and the following result is a generalization of Lemma 2 in that paper.

**Lemma 5.** For each $k$ the topology on $\text{LIP}_k(X)$ generated by the hypograph metric $D$ coincides with the topology of uniform convergence.

**Proof.** It is evident that the topology of uniform convergence contains the hypograph topology. For the converse, let $\varepsilon > 0$, let $k \geq 1$, and let $f, g \in \text{LIP}_k(X)$ be such that $D(f, g) < \varepsilon/4k$. Consider an $x \in X$ and note that since $d'_H(\text{hyp } f, \text{hyp } g) < \varepsilon/4k$ there is a $y \in X$ such that $d(x, y) < \varepsilon/4k$ and $g(y) > f(x) - \varepsilon/4k$. Also there is a $z \in X$ such that $d(y, z) < \varepsilon/4k$ and $f(z) > g(y) - \varepsilon/4k$. Note that $d(x, z) < \varepsilon/2k$. Using the fact that $g$ is $k$-Lipschitz we find $g(z) > g(y) - ke/4k > f(x) - \varepsilon/4k - \varepsilon/4 \geq f(x) - \varepsilon/2$. On the other hand we have $g(x) < g(y) + ke/4k < f(z) + \varepsilon/4k + \varepsilon/4 \leq f(x) + ke/2k + \varepsilon/2 \leq f(x) + \varepsilon$. Thus $|f(x) - g(x)| < \varepsilon$ for every $x \in X$.

A space is called a Hilbert cube if it is homeomorphic to $Q = \mathbb{I}^\mathbb{N}$. Yang and Zhou [16] have shown that $USCC(X)$ is a Hilbert cube if and only if $X$ is an infinite compact metrizable space. Yang and Zhang [15] classify spaces of fuzzy convex sets with the topology inherited from $USCC(\mathbb{R}^n)$. A subset $A$ of a Hilbert cube $M$ is called a pseudointerior if the pair $(M, A)$ is homeomorphic to $(Q, s)$, that is, there is a homeomorphism $h : M \rightarrow Q$ such that $h(A) = s = (0, 1)^\mathbb{N}$. The Anderson Theorem [1] states that pseudointeriors are homeomorphic to the separable Hilbert space $\ell^2$. The complement of a pseudointerior in a Hilbert cube is called a pseudoboundary, $B(Q) = Q \setminus s$ being the standard example. Zhang and Yang [18] have proved that the space $USCC(X)$ is homeomorphic to $B(Q)$ if and only if $X$ is a non-discrete non-compact locally compact separable metric space.

If $f, g : Y \rightarrow X$ and $d$ is a metric on $X$, then we put $\hat{d}(f, g) = \sup\{d(f(y), g(y)) : y \in Y\}$. A space $X$ is called an absolute retract (AR) provided that $X$ is a retract of any space $Y$ that contains it as a closed subspace. Recall that a subspace $X$ is a retract of $Y$ if there exists a retraction $r : Y \rightarrow X$, that is, a continuous map such that $r|X$ is the
identity \(1_X\) on \(X\). A space is a compact AR if and only if it is a retract of a Hilbert cube. We say that a metric space \((X,d)\) has the disjoint cells property provided that for every natural number \(n\), every \(\varepsilon > 0\), and every pair of continuous functions \(f, g : \mathbb{I}^n \to X\) there exist continuous functions \(f', g' : \mathbb{I}^n \to X\) such that \(d(f,f') < \varepsilon\), \(d(g,g') < \varepsilon\), and \(f'(\mathbb{I}^n) \cap g'(\mathbb{I}^n) = \emptyset\). The following characterization of the Hilbert cube is due to Toruńczyk [12].

**Theorem 6.** A space is a Hilbert cube if and only if it is a compact AR with the disjoint cells property.

A closed subset \(A\) of a compact metric space \((X,d)\) is said to be a \(Z\)-set in \(X\) if for any \(\varepsilon > 0\) there exists a continuous map \(f : X \to X \setminus A\) such that \(d(f,1_X) < \varepsilon\). A countable union of \(Z\)-sets is called a \(\alpha Z\)-set, \(B(Q)\) is a standard example of a \(\alpha Z\)-set in \(Q\). A subset \(A\) of a space \(X\) is called homotopy dense if there exists a homotopy \(H : X \times \mathbb{I} \to X\) such that \(H_0 = \text{the identity and } H_1(X) \subset A\) for each \(t \in (0,1]\). Clearly, a closed set (respectively an \(F_\sigma\)-set) in \(X\) whose complement is homotopy dense is a \(Z\)-set (respectively a \(\alpha Z\)-set).

**Lemma 7.** If \(Z\) is a \(Z\)-set in a compact space \(X\), then the complement of \(Q(X \setminus Z)\) is a \(Z\)-set in \(Q(X)\).

**Proof.** Since \(Z\) is closed in \(X\), we have that \(U = \{A \in K(X) : A \cap Z = \emptyset\}\) is open in \(K(X)\). Thus \(Q(X \setminus Z) = Q(X) \cap (USCC(X) \times U)\) is open in \(Q(X)\). Let \(\varepsilon > 0\) and select a continuous map \(h : X \to X \setminus Z\) such that \(d(h(1_X),h) \leq \varepsilon\). Let \(H(x,t) = (h(x),t)\) for \(x \in X\) and \(t \in \mathbb{I}\). If \(f \in USCC(X)\) then clearly \(H(h(f) = \text{hyp } f = \text{hyp } g\) for a function \(g : h(X) \to \mathbb{I}\). Since \(H(\text{hyp } f)\) is compact \(g \in USCC(h(X))\). Define \(G(f) = \tilde{g} \in USCC(X)\). Since the operation \(A \mapsto H(A)\) is continuous on \(K(X \times \mathbb{I})\), we have that \(G : USCC(X) \to USCC(X)\) is continuous. Define \(G' : Q(X) \to Q(X \setminus Z)\) by \(G'(f,A) = (G(f), h(A))\) and note that this function is continuous. Since \(d(1_X,h) \leq \varepsilon\) it is evident that \(\rho((f,A), G'(f,A)) \leq \varepsilon\) for each \((f,A) \in Q(X)\).

A boundary set in a Hilbert cube is a \(\alpha Z\)-set whose complement is homeomorphic to \(\ell^2\). The following result is due to Curtis [3].

**Theorem 8.** A \(\alpha Z\)-set in a Hilbert cube is an AR boundary set if and only if it is homotopy dense.

3. The space \(Q(X)\)

Throughout this section we will assume that \(X\) is a non-degenerate Peano continuum. Recall that a Peano continuum is a compact metric space that is both connected and locally connected. According to Bing [2] and Moise [11] we may assume that \(X\) comes equipped with a convex metric \(d\), that is, for each \(x, y \in X\) there is an arc \([x,y]\) \(\subset X\) with endpoints \(x\) and \(y\) such that \([x,y]\) is isometric to the closed interval \([0,d(x,y)\] in the real line \(\mathbb{R}\). We also assume that \(d\) is bounded by \(1\).

**Lemma 9.** If \(A, B \in K(X)\) and \(\varepsilon, \delta > 0\), then \(d_H(U_d[A,\varepsilon], U_d[B,\delta]) \leq d_H(A, B) + |\varepsilon - \delta|\).

**Proof.** Let \(x \in U_d[B,\delta]\). By symmetry, it suffices to prove that \(d(x, U_d[A,\varepsilon]) \leq d_H(A, B) + |\varepsilon - \delta|\). By compactness there exist a \(b \in B\) and an \(a \in A\) such that \(d(x,b) \leq \delta\) and \(d(b,a) \leq d_H(A,B)\). If \(d(x,a) \leq d_H(A,B) + |\varepsilon - \delta|\) then \(d(x,U_d[A,\varepsilon]) \leq d_H(A,B) + |\varepsilon - \delta|\). If \(d(x,a) > d_H(A,B) + |\varepsilon - \delta|\) then by convexity there is a \(y \in X\) with \(d(x,y) = d_H(A,B) + |\varepsilon - \delta|\) and \(d(x,a) = d(x,y) + d(y,a)\). Then

\[
\begin{align*}
d(a,y) &= d(x,a) - d_H(A,B) - |\varepsilon - \delta| \\
&\leq d(x,b) + d(b,a) - d_H(A,B) - |\varepsilon - \delta| \\
&\leq \delta - |\varepsilon - \delta| \leq \varepsilon
\end{align*}
\]

so \(y \in U_d[A,\varepsilon]\) and \(d(x, U_d[A,\varepsilon]) \leq d_H(A,B) + |\varepsilon - \delta|\).\(\square\)

We show in this section that \(Q(X)\) is a Hilbert cube. Note that \(USCC(X) = USCC(X)\) is a Hilbert cube according to Yang and Zhou [16]. By Curtis and Schori [4] \(K(X)\) is also a Hilbert cube thus \(USCC(X) \times K(X)\) is a Hilbert cube with the metric \(\rho\).

Lemma 10. \( Q(X) \) is a retract of \( \text{USC}(X) \times K(X) \) and hence a compact AR.

Proof. We define the map \( r : \text{USC}(X) \times K(X) \to \text{USC}(X) \times K(X) \) as follows:
\[
r(f, A) = (\max\{f - \delta, 0\}, U_d[A, 2\delta]),
\]
for every pair \((f, A) \in \text{USC}(X) \times K(X)\), where \( \delta = \rho((f, A), Q(X)) \). If \( f, g \in \text{USC}(X) \) and \( t, s \in I \) then it is clear that
\[
D(\max\{f - t, 0\}, \max\{g - s, 0\}) \leq D(f, g) + |t - s|.
\]
So with Lemma 9 we have that \( r \) is continuous. It is also clear that if \((f, A) \in Q(X)\), then \( r(f, A) = (f, A) \).

Let \((f, A) \notin Q(X)\). To prove that \( r \) is a retraction onto \( Q(X) \) it remains to show that \( r(f, A) \in Q(X) \). By compactness there exists a pair \((g, B) \in Q(X)\) such that \( \rho((f, A), (g, B)) = \delta = \rho((f, A), Q(X)) \). Thus \( D(f, g) \leq \delta \) and \( d_H(A, B) \leq \delta \) so \( B \subset U_d[A, \delta] \). Let \( f' = \max\{f - \delta, 0\} \) and \( x \in f' \). Since \( D(f, g) \leq \delta \) there is a \( y \in X \) such that \( d(x, y) \leq \delta \) and \( g(y) \geq f(x) - \delta > 0 \). Since supp \( g \subset B \) we have \( y \in B \subset U_d[A, \delta] \). Consequently, \( x \in U_d[A, 2\delta] \) and hence \( f' \subset U_d[A, 2\delta] \). We may conclude that supp \( f' \subset U_d[A, 2\delta] \) and thus \( r(f, A) \in Q(X) \). \( \square \)

Lemma 11. \( \kappa(X) \) is homotopy dense in \( Q(X) \).

Proof. We define the map \( H : Q(X) \times I \to Q(X) \) as follows:
\[
H((f, A), t) = H_t(f, A) = (f \vee t \chi_A, A),
\]
for every pair \(((f, A), t) \in Q(X) \times I \). It is clear that \( H_0 \) is the identity map, that \( H_t(Q(X)) \subset \kappa(X) \) for each \( t > 0 \), and that \( H \) is continuous.

Lemma 12. \( Q(X) \setminus \kappa(X) \) is homotopy dense in \( Q(X) \).

Proof. According to Curtis [3] the finite subsets of \( X \) form an AR boundary set in \( K(X) \), which means that there is a homotopy \( F : K(X) \times I \to K(X) \) such that \( F_0 \) is the identity map and \( F_1(A) \) is a finite set for any \( t > 0 \); see Theorem 8. We can easily arrange that \( d_H(A, F(t, A)) \leq t^2 \) for each \( A \in K(X) \) and \( t \in I \).

We define the map \( G : \text{USC}(X) \times I \to \text{USC}(X) \) as follows:
\[
G(f, t)(x) = G_t(f)(x) = \begin{cases} 
\max \left\{ f(z) - \frac{1}{t} d(x, z) : z \in X \right\} & \text{if } t > 0, \\
 f(x) & \text{if } t = 0,
\end{cases}
\]
for every pair \((f, t) \in \text{USC}(X) \times I \) and \( x \in X \). By Claim 10 in [5], we have that \( G \) is a homotopy such that \( G_t(\text{USC}(X)) \subset \text{LIP}_{1/t}(X) \) for each \( t \in (0, 1] \) and \( G_0 \) is the identity. Let \( f \in \text{USC}(X), t \in (0, 1] \), and \( x, y \in X \). Note that \( G(f, t)(x) \geq f(x) = f(x) - d(x, x)/t \) so \( \text{hyp } f \subset \text{hyp } G(f, t) \). By compactness select a \( z \in X \) such that \( G(f, t)(x) = f(z) - d(x, z)/t \). Then \( d(x, z) = t(f(z) - G(f, t)) \leq t \) and hence \( d'(x, G(f, t)(x)) \leq t \). Thus we have \( D(f, G(f, t)) \leq t \).

Now we define the map \( H : Q(X) \times I \to Q(X) \) as follows:
\[
H((f, A), t) = (\min\{G(f, t), \chi_{F(A, t)}\}, U_d[F(A, t), t]),
\]
for every pair \(((f, A), t) \in Q(X) \times I \).

It is clear that \( H \) is well-defined and that \( H_0 \) is the identity. Note that if \( t > 0 \), supp(\( \min\{G'(f, t), \chi_{F(A, t)}\} \)) is a subset of the finite set \( F(A, t) \) and in a continuum \( F(A, t) \neq U_d[F(A, t), t] \) so we have \( H(Q(X) \times (0, 1]) \subset Q(X) \setminus \kappa(X) \).

It remains to show that \( H \) is continuous. With Lemma 9 it is clear that \( U_d[F(A, t), t] \) depends continuously on \( A \) and \( t \). Let \((f, A) \in Q(X)\), \( t \in I \), and let \( h_t(f, A) \) denote \( \min\{G(f, t), \chi_{F(A, t)}\} \). We first show that \( D(f, h_t(f, A)) \leq t \).

On the one hand we have \( \text{hyp } h_t(f, A) \subset \text{hyp } G(f, t) \subset U_d[\text{hyp } f, t] \) because \( D(f, G(f, t)) \leq t \). On the other hand, let \( x \in X \) such that \( f(x) > 0 \). Then \( x \in A \) and there is a \( y \in F(A, t) \) with \( d(x, y) \leq t^2 \). Since \( G(f, t) \) is \((1/t)\)-Lipschitz we have \( |G(f, t)(x) - G(f, t)(y)| \leq t \). Thus \( h_t(f, A)(y) = G(f, t)(y) \geq G(f, t)(x) - t \geq f(x) - t \).
and \( d'( (x, f(x)), \text{hyp } h_t(f, A) ) \leq \max\{t, r^2\} = t \). If \((f, A), (g, B) \in \mathcal{Q}(X) \) and \( t \in \mathbb{I} \), then \( D(h_0(g, B), h_t(f, A)) = D(g, h_t(f, A)) \leq D(g, f) + D(h_t(f, A)) \leq D(g, f) + t \), which proves continuity of \( h \) at points in \( \mathcal{Q}(X) \times \mathbb{I} \).

We now show that \( h \) is continuous on the open set \( \mathcal{Q}(X) \times (0, 1] \). Let \((f, A) \in \mathcal{Q}(X) \) and \( e, t \in (0, 1] \). If \( g \in \text{USC}(X) \) and \( s \in (t/2, 1] \), then \( G(g, s) \) is \( (2/t) \)-Lipschitz thus by Lemma 5 there exists a \( \delta > 0 \) such that \( |G(f, t)(x) - G(g, s)(x)| < e/2 \) for each \( g \in \text{USC}(X) \), \( x \in X \), and \( s \in (t/2, 1] \) with \( D(f, g) < \delta \) and \( |t - s| < \delta \). Moreover we can arrange that \( d_H(F(A, t), F(B, s)) < e/t + 4 \) whenever \( d_H(A, B) < \delta \) and \( |t - s| < \delta \). Let \((g, B) \in \mathcal{Q}(X) \) and \( s \in (t/2, 1] \) be such that \( D(f, g) < \delta \), \( d_H(A, B) < \delta \), and \( |t - s| < \delta \). Consider an \( x \in X \) such that \( h_t(f, A)(x) > 0 \). Then \( x \in F(A, t) \) and there is a \( y \in F(B, s) \) with \( d(x, y) < e/t + 4 \). Since \( G(g, s) \) is \( (2/t) \)-Lipschitz we have \( |G(g, s)(x) - G(g, s)(y)| \leq (2/t)d(x, y) < e/2 \). We now have

\[
|h_t(f, A)(x) - h_s(g, B)(y)| = |G(f, t)(x) - G(g, s)(y)| \\
\leq |G(f, t)(x) - G(g, s)(x)| + |G(g, s)(x) - G(g, s)(y)| \\
< e/2 + e/2 = e.
\]

Thus \( d'( (x, h_t(f, A)(x)), \text{hyp } h_t(g, B) ) < \max\{e, e/t + 4\} = e \) and \( \text{hyp } h_t(f, A) \subset U_{d'}/\text{hyp } h_s(g, B), e \). The same argument gives us the converse \( \text{hyp } h_s(g, B) \subset U_{d'}/\text{hyp } h_t(f, A), e \) and \( D(h_t(f, A), h_s(g, B)) < e \). We may conclude that \( H \) is continuous. \( \square \)

**Theorem 13.** \( \mathcal{Q}(X) \) is a Hilbert cube.

**Proof.** By Lemma 10, \( \mathcal{Q}(X) \) is a compact AR. Since both \( \mathcal{K}(X) \) and \( \mathcal{Q}(X) \setminus \mathcal{K}(X) \) are homotopy dense in \( \mathcal{Q}(X) \) by Lemmas 11 and 12, \( \mathcal{Q}(X) \) has the disjoint cells property. Thus, \( \mathcal{Q}(X) \approx \mathcal{Q} \) by Theorem 6. \( \square \)

4. The main results

In this section we assume merely that \((X, d)\) is a separable metric space.

**Lemma 14.** \( \mathcal{K}(X) \) is a \( G_\delta \)-set in \( \mathcal{Q}(X) \).

**Proof.** Let \( O_n = \{(f, A) \in \mathcal{Q}(X) : d_H(\text{supp } f, A) < 1/n\} \). Then \( \mathcal{K}(X) = \bigcap_{n=1}^{\infty} O_n \). We need only to verify that every \( O_n \) is open in \( \mathcal{Q}(X) \). Let \((f, A) \in O_n \) and select an \( e \in (d_H(\text{supp } f, A), 1/n) \) and put \( \delta = (1/n - e)/2 \). By compactness there is a finite subset \( F \) of \( f_{\neq 0} \) such that \( A \subset U_{\delta}(F, e) \). Choose a \( \gamma > 0 \) such that \( \gamma < \delta \) and \( f(x) > y \) for every \( x \in F \). Let \((g, B) \in \mathcal{Q}(X) \) be such that \( \rho((f, A), (g, B)) < \gamma \). Then \( B \subset U_{\gamma}(A, \gamma) \) so \( B \subset U_{\delta}(F, e + \gamma) \). If \( x \in F \) then there is a \( y \in X \) such that \( d(x, y) < y \) and \( g(y) > f(x) - y > 0 \). Thus \( y \in \text{supp } g \) and hence \( F \subset U_{\gamma}(\text{supp } g, \gamma) \). We have \( B \subset U_{\gamma}(\text{supp } g, \epsilon + 2\gamma) \) and since \( \text{supp } g \subset B \) by the definition of \( \mathcal{Q}(X) \) we have \( d_H(\text{supp } g, B) \leq \epsilon + 2\gamma < 1/n \) so \( g \in O_n \). \( \square \)

Thus if \( X \) is locally compact, then \( \mathcal{K}(X) \) has a compatible complete metric (that cannot be \( \rho \) as Kloeden [7, Section 4] notes). It is easily seen that \( \mathcal{K}(X) \) is locally compact if and only if \( X \) is discrete.

Theorem 1 is included in the following result.

**Theorem 15.** If \( X \) is a non-degenerate compact space, then the following statements are equivalent:

1. \( X \) is a Peano continuum.
2. \( \mathcal{Q}(X) \) is a Hilbert cube.
3. \( \mathcal{Q}(X) \) is connected and locally connected.
4. \( \mathcal{K}(X) \) is homeomorphic to \( \ell^2 \).
5. \( \mathcal{K}(X) \) is connected and locally connected.

**Proof.** The implication (1) \( \Rightarrow \) (2) is Theorem 13. (2) \( \Rightarrow \) (3) and (4) \( \Rightarrow \) (5) are trivial.

(1) \( \Rightarrow \) (4). By Lemmas 11 and 14 the complement of \( \mathcal{K}(X) \) is a \( \sigma Z \)-set in the Hilbert cube \( \mathcal{Q}(X) \). Using Lemma 12 and Theorem 8 we find that \( \mathcal{K}(X) \) is homeomorphic to \( \ell^2 \).

The proofs of (3) \( \Rightarrow \) (1) and (5) \( \Rightarrow \) (1) are analogous to Wojdysławski’s proof [14] for \( \mathcal{K}(X) \). We give the argument for \( \mathcal{K}(X) \). Assume (5). If \( C \) is a clopen proper subset of \( X \), then \( \mathcal{K}(C) \) is clearly also a proper clopen subset of \( \mathcal{K}(X) \).
thus $X$ must be connected. Now let $O$ be an open subset of $X$ and let $x \in O$. Note that $\mathcal{K}(O)$ is an open subset of $\mathcal{K}(X)$ that contains the element $(\mathcal{Z}(\{x\}), \{x\})$. By (5) let $P$ be the open subset of $\mathcal{K}(O)$ that is the component of $(\mathcal{Z}(\{x\}), \{x\})$ in $\mathcal{K}(O)$. Define $P' = \bigcup\{A : (f, A) \in P\}$. Let $y \in P'$ so $y \in A$ for some $(f, A) \in P$. Let $\varepsilon > 0$ be such that $\mathcal{K}(X) \cap U_{\varepsilon}(f, A) \subset P$. If $d(y, z) < \varepsilon$ then $(f \vee \varepsilon_{\mathcal{Z}(\{x\})}, A \cup \{z\}) \in P$ so $z \in P'$. We may conclude that $P'$ is open. If $C$ is a clopen proper subset of $P'$, then $\mathcal{K}(C) \cap P$ is also a proper clopen subset of $P$ thus $P'$ must be connected. We may conclude that $X$ is locally connected and a Peano continuum. □

Remark 16. Concerning statements (2) and (4) a slightly stronger statement can be proved, namely that $\mathcal{K}(X)$ is a pseudointerior of $Q(X)$.

Corollary 17. If $A$ is a $\sigma$-Z-set in a Peano continuum $X$, then $\mathcal{K}(X \setminus A)$ is homeomorphic to $\ell^2$.

Proof. By Lemma 7 we have that the complement of $Q(X \setminus A)$ is a $\sigma$-Z-set in $Q(X)$. As in the proof of Theorem 15 we have that the complement of $\mathcal{K}(X)$ is a homotopy dense $\sigma$-Z-set in $Q(X)$ and hence the complement of $\mathcal{K}(X \setminus A) = \mathcal{K}(X) \setminus Q(X \setminus A)$ is also a homotopy dense $\sigma$-Z-set in $Q(X)$. So by Theorem 8 we may conclude that $\mathcal{K}(X \setminus A)$ is homeomorphic to $\ell^2$. □

Note that $\mathbb{R}^n$ is homeomorphic to $(0, 1)^n$ which is the complement of a $Z$-set in $I^n$. Also $\ell^2$ is homeomorphic to a pseudointerior in $Q$ and therefore to the complement of a $\sigma$-Z-set. So Corollary 17 implies Corollary 2.

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References