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A remark on nonlocal symmetries for the Calogero–Degasperis–Ibragimov–Shabat Equation

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Abstract

We consider the Calogero–Degasperis–Ibragimov–Shabat depending on the local variables and on the integral of the only local conserved density of the equation in question. The resulting Lie algebra of these symmetries turns out to be a central extension of that of local symmetries.

1 Introduction

The existence of infinite-dimensional Lie algebra of commuting higher order symmetries for a system of PDEs is well known to be one of the most important signs of its integrability, at least in (1+1) dimensions [2, 10]. In (2+1) dimensions this algebra can be extended (see e.g. [18]) to a noncommutative algebra (which is sometimes referred as a hereditary algebra [18]) of time-dependent (and possibly nonlocal) symmetries being polynomials in time t of arbitrarily high degree.

For a long time the only known example of (1+1)-dimensional evolution system possessing a symmetry algebra of a similar kind was the Burgers equation (see [5] for the complete description of its symmetry algebra), so it is natural to ask whether there exist other (1+1)-dimensional evolution equations having the same property. In [1] we have answered this question in affirmative and shown that the Calogero–Degasperis–Ibragimov–Shabat (CDIS) [7, 8, 9, 10, 11, 12] equation also possesses a hereditary symmetry algebra which, exactly as in the case of Burgers equation, is its complete symmetry algebra in the class of *local* higher order symmetries.

The CDIS equation (2.1) has only one local conserved density $\rho = u^2$, so the natural next step in analyzing this equation is to consider its symmetries involving a nonlocal variable ω being the integral of this density. Theorem 1 below provides the complete characterization of symmetries which depend on this variable and on a finite number of local variables. The Lie algebra of these symmetries possesses a nontrivial one-dimensional center.

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2 Symmetries of the CDIS equation

The Calogero–Degasperis–Ibragimov–Shabat (CDIS) equation has the form [7, 8]

$$u_t = u_3 + 3u^2u_2 + 9uu_1^2 + 3u^4u_1 \equiv F, \tag{2.1}$$

Here $u_j = \partial^j u/\partial x^j$; see [1] for the further details on notation used. Let us mention that this is the only third order (1+1)-dimensional scalar polynomial λ-homogeneous evolution equation with $\lambda = 1/2$ possessing infinitely many x, t-independent symmetries [15].

Consider a nonlocal variable ω defined (cf. e.g. [9]) by the relations

$$\partial\omega/\partial x = u^2, \partial\omega/\partial t = 2uu_2 + 6u^3u_1 + u^6 - u_1^2. \tag{2.2}$$

Note that the CDIS equation is linearized into $v_t = v_3$ upon setting $v = \exp(\omega)u$.

The quantity $\rho = u^2$ is the only local conserved density (see e.g. [9, 10] and references therein) for (2.1), but (2.1) has a Hamiltonian operator $\exp(-2\omega)$ and infinitely many conserved densities explicitly dependent on ω [9].

We shall call a function $G(x, t, \omega, u, \ldots, u_k)$ a *symmetry* of CDIS equation, if

$$D_t(G) - F_*(G) = 0, \tag{2.3}$$

where $F_* = \sum\limits_{i=0}^{3} \partial F/\partial u_i D^i$, and $D \equiv D_x = \partial/\partial x + u^2\partial/\partial\omega + \sum_{i=0}^{\infty} u_{i+1}\partial/\partial u_i$ and $D_t = \partial/\partial t + (2uu_2 + 6u^3u_1 + u^6 - u_1^2)\partial/\partial\omega + \sum_{i=0}^{\infty} D^i(F)\partial/\partial u_i$ are the operators of total x- and t-derivatives. Note that our definition of nonlocal symmetries is a particular case of the usual one, cf. e.g. [18], but in terminology of [6] the solutions of (2.3) are referred as *shadows* of symmetries.

For any function $H = H(x, t, \omega, u, \ldots, u_q)$ we define its *order* $\operatorname{ord} H$ as a greatest integer m such that $\partial H/\partial u_m \neq 0$, and set

$$H_* = \partial H/\partial\omega D^{-1} \circ u^2 + \sum_{i=0}^{\operatorname{ord} H} \partial H/\partial u_i D^i.$$

Here $\circ$ denotes a composition law induced by 'generalized Leibnitz rule' (see e.g. [10, 14])

$$D^k \circ f = \sum_{j=0}^{\infty} \frac{k(k-1)\cdots(k-j+1)}{j!} D^j(f) D^{k-j}.$$

A function $H = H(x, t, \omega, u, \ldots, u_q)$ is called *local* if $\partial H/\partial\omega = 0$.

Let $S_{\text{CDIS}}^{(k)}$ be the set of nonlocal symmetries $G(x, t, \omega, u, \ldots, u_m)$ of order not higher than k for the CDIS equation, and let $S_{\text{CDIS}} = \bigcup\limits_{j=0}^{\infty} S_{\text{CDIS}}^{(j)}$, $S_{\text{CDIS},j} = S_{\text{CDIS}}^{(j)}/S_{\text{CDIS}}^{(j-1)}$ for $j \geq 1$, and $S_{\text{CDIS},0} = S_{\text{CDIS}}^{(j)}/\Theta_{\text{CDIS}}$, where $\Theta_{\text{CDIS}} = \{G(x, t, \omega) | G \in S_{\text{CDIS}}\}$.

The straightforward computation shows that any symmetry $G \in S_{\text{CDIS}}^{(2)}$ of the CDIS equation is a linear combination of u_1 and $W = \exp(-2\omega)u$.

Differentiating the left-hand side of (2.3) with respect to u_{k+2} and equating the result to zero, we find, in complete analogy with [2], that for a symmetry $G = G(x, t, \omega, u, \ldots, u_k)$ with $k \geq 1$ of the CDIS equation we have

$$\partial G/\partial u_k = c_k(t), \tag{2.4}$$

where $c_k(t)$ is a function of t.

Below we assume without loss of generality that any symmetry $G \in S_{\mathrm{CDIS},k}$, $k \geq 1$ vanishes if the relevant function $c_k(t)$ is identically equal to zero.

Further differentiating (2.3) with respect to u_{k+1} and u_k and then with respect to x shows that $\partial^2 G/\partial u_{k-j}\partial x = 0$ for $j = 0, 1$ and

$$\partial^2 G/\partial x \partial u_{k-2} = \dot{c}_k(t)/3. \tag{2.5}$$

Taking into account that $G \in S_{\mathrm{CDIS}}$ implies $\tilde{G} = \partial^r G/\partial x^r \in S_{\mathrm{CDIS}}$, and successively using (2.5), we find that $\mathrm{ord}\,\tilde{G} \leq k - 2r$ and

$$\partial \tilde{G}/\partial u_{k-2r} = (1/3)^r d^r c_k(t)/dt^r. \tag{2.6}$$

For $r = [k/2]$ we have $\mathrm{ord}\,\tilde{G} \leq 1$. As u_1 and $W = \exp(-2\omega)u$ are the only generalized symmetry of CDIS equation from $S^{(1)}_{\mathrm{CDIS}}$, and both of them are time-independent, the function $c_k(t)$ satisfies the equation $d^m c_k(t)/dt^m = 0$ for $m = [k/2] + 1$. Hence, $\dim S_{\mathrm{CDIS},k} \leq [k/2] + 1$ for $k \geq 1$ and the dimension of the quotient space of symmetries of the form $G = G(x, t, \omega, u, \ldots, u_k)$ modulo the space of symmetries of the form $G = G(x, t, \omega, u, \ldots, u_{k-1})$ does not exceed $[k/2] + 1$ for $k \geq 1$. From this it is immediate that all odd-order symmetries of the CDIS equation are exhausted by local ones, as we can exhibit exactly $[k/2] + 1$ such symmetries of order k for each odd k [1].

Furthermore, as all symmetries of the form $G(x, t, \omega, u, u_1, u_2)$ are exhausted by u_1 and W, in analogy with Theorem 2 of [19] we can show that all symmetries of the form $G(x, t, \omega, u, \ldots, u_k)$ of the CDIS equation are polynomial in time t for all $k \in \mathbb{N}$. Indeed, assume this result to be proved for the symmetries of order $k - 1$ and let us prove it for symmetries of order k. it readily follows from the above that the function $c_k(t) = \partial G/\partial u_k$ is a polynomial in t of degree not higher than $[k/2]$. Therefore, $\partial^m G/\partial t^m$, where $m = [k/2] + 1$, is a symmetry of CDIS of order not higher then $k - 1$ and thus is polynomial in t by assumption, whence we readily see that G is polynomial in t as well. The induction on k, starting from $k = 2$, completes the proof.

Now let us turn to the study of time-independent symmetries of the CDIS equation. This equation is well known to have infinitely many x, t-independent local generalized symmetries, hence it has [20] a formal symmetry of infinite rank of the form $\mathfrak{L} = D + \sum_{j=0}^{\infty} a_j D^{-j}$, where a_j are some x, t-independent local functions.

Taking the directional derivative of (2.3), we readily find that for any symmetry $G(x, t, \omega, u, \ldots, u_k)$ of the CDIS equation the quantity G_* is a formal symmetry of rank not lower than $k+1$ for the CDIS equation, and therefore (cf. e.g. [10, 14]), provided $k \geq 1$ and $\partial G/\partial t = 0$, we have

$$G_* = \sum_{j=1}^{k} c_j \mathfrak{L}^j + \mathfrak{B},$$

where c_j are some constants and $\mathfrak{B} = \sum_{j=-\infty}^{-1} b_j D^j$, b_j are some t-independent local functions.

From this equation we infer that (cf. [1]) any symmetry of the form $G(x, \omega, u, \ldots, u_k)$ can be represented as

$$G = G_0(u, \ldots, u_k) + Y(x, u, \omega). \tag{2.7}$$

It can be easily seen that $\partial Y/\partial x = \partial G/\partial x$ and $\partial Y/\partial \omega = \partial G/\partial \omega$ are time-independent symmetries of the CDIS equation. By the above we readily see that $\partial Y/\partial x = c_1 W$ and $\partial Y/\partial \omega = c_2 W$ for some constants c_1, c_2. As $\partial^2 Y/\partial x\partial\omega = \partial^2 Y/\partial\omega\partial x$, we find that $c_1 = 0$, and thus $\partial Y/\partial x = 0$, so any time-independent symmetry G of order $k \geq 2$ for the CDIS equation is x-independent as well, and

$$G = G_0(u, \ldots, u_k) + cW \tag{2.8}$$

for some constant c. Thus, any time-independent symmetry of the form $G(x, \omega, u, \ldots, u_k)$ is x-independent as well.

Using the symbolic method, it can be shown [15] that the CDIS equation has no even order t, x-independent local generalized symmetries, so its only even order time-independent symmetry is W.

Now let us show that the same result holds true for time-dependent symmetries as well. The CDIS equation is invariant under the scaling symmetry $K = 3tF + xu_1 + u/2$. Therefore, if a symmetry Q contains the terms of weight γ (with respect to the weighting induced by K, cf. [16], when the weight of u is 1/2, the weight of ω is 1, the weight of t is 3, the weight of x is -1 and the weight of $u_j = j+1/2$), there exists a homogeneous symmetry $\tilde{Q}$ of the same weight γ. We shall write this as $\mathrm{wt}(\tilde{Q}) = \gamma$. Note that we have $[K, \tilde{Q}] = \gamma\tilde{Q}$.

If $G \in S_{\mathrm{CDIS},k}$, $k \geq 1$, is a polynomial in t of degree m, then its leading coefficient $\partial G/\partial u_k = c_k(t)$ also is a polynomial in t of degree $m' \leq m$, i.e., $c_k(t) = \sum_{j=0}^{m'} t^j c_{k,j}$, where $c_{k,m'} \neq 0$. Consider $\tilde{G} = \partial^{m'} G/\partial t^{m'} \in S^{(k)}_{\mathrm{CDIS}}$. We have $\partial\tilde{G}/\partial u_k = \mathrm{const} \neq 0$, hence $\tilde{G}$ contains the terms of the weight $k + 1/2$. Let P be the sum of all terms of weight $k + 1/2$ in $\tilde{G}$. Clearly, P is a homogeneous symmetry of weight $k+1/2$ by construction, $\mathrm{ord}\, P = k$ and $\partial P/\partial u_k$ is a nonzero constant. Next, $\partial P/\partial t \in S_{\mathrm{CDIS}}$ is a homogeneous symmetry of weight $k + 7/2$. Obviously, $\mathrm{ord}\, \partial P/\partial t \leq k - 1$. By the above, all symmetries in S_{CDIS} are polynomial in t, and thus for any homogeneous $B \in S_{\mathrm{CDIS}}$, $b \equiv \mathrm{ord}\, B \geq 1$, we have $\partial B/\partial u_b = t^r c_b$, $c_b = \mathrm{const}$ for some $r \geq 0$. Hence, $\mathrm{ord}(B) = b - 3r \leq b$, and for $k \geq 1$ the set S_{CDIS} does not contain homogeneous symmetries B such that $\mathrm{ord}(B) = k + 7/2$ and $\mathrm{ord}\, B \leq k - 1$, so $\partial P/\partial t = 0$.

Thus, we conclude that the existence of a time-independent symmetry of order $k \geq 1$ from S_{CDIS} is a necessary condition for the existence of a polynomial-in-time symmetry $G \in S_{\mathrm{CDIS}}$ of the same order k. Moreover, by the above all symmetries from S_{CDIS} are polynomial in t. Hence, the fact that the CDIS equation has no time-independent symmetries $G(x, \omega, u, \ldots, u_k)$ of even order $k \geq 2$ immediately implies the absence of any *time-dependent* local generalized symmetries of even order $k \geq 2$.

Summing up the above results, we infer that the space S_{CDIS} is spanned by the symmetry $W = \exp(-2\omega)u$, and by *local* generalized symmetries of the CDIS equation. The latter were found in [1], and can be described in the following way.

Define the commutator of two functions f and g of $x, t, \omega, u, u_1, \ldots$ as (cf. e.g. [10, 14, 18]) as

$$[f, g] = g_*(f) - f_*(g).$$

Set $\tau_{m,0} = x^m u_1 + m x^{m-1} u/2$, $m = 0, 1, 2 \ldots$, and $\tau_{1,1} = x(u_3 + 3u^2 u_2 + 9uu_1^2 + 3u^4 u_1) + 3u_2/2 + 5u_1 u^2 + u^5/2$. The latter is the first nontrivial master symmetry for the CDIS equation, see e.g. [12, 15]. The quantities $\tau_{0,0}$, $\tau_{1,0}$, $\tau_{2,0}$, and $\tau_{1,1}$ meet the requirements of

Theorem 3.18 from [18], whence

$$[\tau_{m,j},\tau_{m',j'}] = ((2j'+1)(m+1)-(2j+1)(m'+1))\tau_{m+m'-1,j+j'}, \tag{2.9}$$

where $\tau_{m,j}$ with $j>0$ are defined inductively by means of (2.9), i.e. [18] $\tau_{0,j+1} = \frac{1}{2j+1}[\tau_{1,1},\tau_{0,j}]$, $\tau_{m+1,j} = \frac{1}{2+4j-m}[\tau_{2,0},\tau_{m,j}]$.

Thus, the CDIS equation, as well as the Burgers equation, represents a nontrivial example of a (1+1)-dimensional evolution equation possessing a hereditary algebra (2.9).

Using (2.9), it can be shown (cf. [18]) that $\mathrm{ad}^{m+1}_{\tau_{0,j}}(\tau_{m,j'}) = 0$, i.e. $\tau_{m,j}$ are master symmetries of degree $m+1$ for all equations $u_{t_j} = \tau_{0,j}$, $j=0,1,2,\ldots$. Here $\mathrm{ad}_B(G)\equiv[B,G]$ for any (smooth) local functions B and G.

Let $\exp(\mathrm{ad}_B)\equiv\sum_{j=0}^{\infty}\mathrm{ad}^j_B/j!$. As $\mathrm{ad}^{m+1}_{\tau_{0,j}}(\tau_{m,j'})=0$, it is easy to see (cf. [18]) that

$$G^{(k)}_{m,j}(t_k) = \exp(-t_k\,\mathrm{ad}_{\tau_{0,k}})\tau_{m,j} = \sum_{i=0}^{m}\frac{(-t_k)^i}{i!}\mathrm{ad}^i_{\tau_{0,k}}(\tau_{m,j}) = \sum_{i=0}^{m}\frac{((2k+1)t_k)^i m!}{i!(m-i)!}\tau_{m-i,j+ik}$$

are local time-dependent generalized symmetries for the equation $u_{t_k}=\tau_{0,k}$ and $\mathrm{ord}\,G^{(k)}_{m,j} = 2(j+mk)+1$. Note that $G^{(k)}_{m,j}$ obey the same commutation relations as $\tau_{m,j}$, that is

$$[G^{(k)}_{m,j},G^{(k)}_{m',j'}] = ((2j'+1)(m+1)-(2j+1)(m'+1))G^{(k)}_{m+m',j+j'}. \tag{2.10}$$

It is straightforward to verify that $\tau_{0,1} = F = u_3+3u^2u_2+9uu_1^2+3u^4u_1$ and thus $G_{m,j} = G^{(1)}_{m,j}(t) = \exp(-t\,\mathrm{ad}_F)\tau_{m,j}$ are time-dependent symmetries for the CDIS equation.

It is easy to see that the number of symmetries $G_{m,j}$ of given odd order $k=2l+1$ equals $[k/2]+1=l+1$. As $\dim S_{\mathrm{CDIS},k}\le[k/2]+1$, these symmetries exhaust the space $S_{\mathrm{CDIS},k}$. Thus, any local generalized symmetry of the CDIS equation of the symmetries $G_{m,j}$ for $m=0,1,\cdots$ and $j=0,1,2\ldots$.

Evaluating the commutator of W with $\tau_{m,0}=x^m u_1+mx^{m-1}u/2$, we readily see that it vanishes, if we assume that the result of application of D^{-1} to a homogeneous differential polynomial in $x,t,u,\ldots,u_k$ without u_j-independent terms is a differential polynomial of the same kind. Using the commutation relations between the local generalized symmetries of the CDIS equation, found in [1], and Jacobi identity, we conclude that the commutator of W with all these symmetries vanishes as well. As a result, we have

Theorem 1. *Any symmetry of the CDIS equation of the form $G(x,t,\omega,u,\ldots,u_s)$ is a linear combination of the symmetries $G_{m,j}$ for $m,j=0,1,\ldots$ and $j=0,1,2\ldots$, and of the symmetry $W=\exp(-2\omega)u$, which commutes with all other symmetries.*

This result can be generalized to the symmetries of 'higher CDIS equations' $u_{t_k}=\tau_{0,k}$. Recall [20] that because of existence of infinitely many x,t-independent local generalized symmetries the CDIS equation has a formal symmetry of infinite rank of the form $\mathfrak{L} = D+u^2+\sum_{j=-1}^{\infty}a_jD^{-j}$, where a_j are some x,t-indepedent local functions. Using Lemma 11 from [17], we can show that $\mathfrak{L}$ is a formal symmetry of rank at least $\mathrm{ord}\,\tau_{0,k}+3$ for the equation $u_{t_k}=\tau_{0,k}$, and hence $u^2=\mathrm{res}\ln\mathfrak{L}$ is a conserved density for all these equations: $D_{t_k}(u^2)=D(\sigma_k)$, where σ_k are some local functions. Note that σ_k can be choosen to be polynomials in u_j with x,t-independent coefficients and zero free term (in particular, we have chosen $\sigma_1=2uu_2+6u^3u_1+u^6-u_1^2$ for $\omega\equiv\omega_1$ in (2.2) to be exactly of this form).

Define the nonlocal variable ω_k, associated with the system $u_{t_k} = \tau_{0,k}$, by means of the equations

$$\partial\omega_k/\partial x = u^2, \partial\omega_k/\partial t_k = \sigma_k.$$

Theorem 2. *Any symmetry of the form $G(x, t_k, \omega_k, u, \dots, u_s)$ for the equation $u_{t_k} = \tau_{0,k}$ is a linear combination of the symmetries $G^{(k)}_{m,j}(t_k)$ for $m = 0, 1, \cdots$ and $j = 0, 1, 2 \dots$, and of $W_k = \exp(-2\omega_k)u$, which commutes with all other symmetries.*

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