A SPECTRAL SEQUENCE APPROACH TO NORMAL FORMS

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We formulate the concept of unique normal form in terms of a spectral sequence. As an illustration of this technique we reproduce some results of Baider and Churchill concerning the normal form of the anharmonic oscillator. The aim of this paper is to show that spectral sequences give us a natural framework in which to formulate normal form theory

1. Introduction

This paper is an informal version of [San02]. The reader is referred to loc. cit. for all references, exact statements and proofs.

The goal of this paper is to show that **spectral sequences** give us a natural language in which to describe the process of computing the **unique normal form.**

1.1. *What are spectral sequences?*

Not to answer this question, but to indicate the time and place of spectral sequences in the history of mathematics, we quote from [Wei94] and [Rot96], respectively.

> "Spectral sequences were invented by Jean Leray, as a prisoner of war during World War II, in order to compute the homology (or cohomology) of a chain complex. They were made algebraic by Koszul in 1945."

> "[] the most powerful method of computing homology groups uses spectral sequences. When I was a graduate student, I always

wanted to say, nonchalantly, that such and such is true ”by the usual spectral sequence argument,” but I never had the nerve.”

1.2. *A rough description*

The theory of spectral sequences has two necessary ingredients: a d operator, that is to say, $d^2 = 0$, and a filtration.

The idea is to study the action of the d operator using the filtration. One first looks what d does as a map

$$d^0 : \mathcal{F}^p/\mathcal{F}^{p+1} \to \mathcal{F}^p/\mathcal{F}^{p+1}.$$

Then one divides this knowledge out and studies

$$d^1 : \mathcal{F}^p/\mathcal{F}^{p+1} \to \mathcal{F}^{p+1}/\mathcal{F}^{p+2}.$$

&c.

1.3. *Where is the filtration in normal form theory?*

If we consider for example formal vector fields at equilibrium, the equation looks like

$$\dot{x} = Ax + F_1(x) + F_2(x) + \cdots,$$

where $F_i(x)$ denotes terms of degree $i+1$. We say that a filtration is given by letting $\mathcal{F}^i$ consist of all those vector fields with terms starting at degree $i+1$. One has

$$[\mathcal{F}^i, \mathcal{F}^j] \subset \mathcal{F}^{i+j},$$

which explains the $+1$ in the definition. One calls $\mathcal{F}^0$ a filtered Lie algebra and everything we say will hold for filtered Lie algebras in general.

1.4. *Where is the d in normal form theory?*

The d in normal form theory comes from the socalled *homological equation* (Cf. [Arn80])

$$[v, t] = f - \bar{f}, \quad t \in \mathcal{T}, v, f, \bar{f} \in \mathcal{N},$$

where v is the object to be normalized, $\mathcal{T}$ is the space of (higher order) transformations, and $\mathcal{N}$ denotes the space of vector fields. This equation is obtained by applying the transformation $exp(ad(t))$ to v and looking at the lowest order terms of $exp(ad(t)) - v$. For more details on normal form theory and it computational aspects, see for instance [Gae02]. Arnol'd does not explain the terminology and it may not immediately be clear how it should be interpreted. We can view this as a map

$$\mathcal{N} \oplus \mathcal{T} \xrightarrow{d} \mathcal{N} \oplus \mathcal{T}$$

with $d(n,t) = ([v,t],0)$. Now one trivially has $d^2 = 0$ and this at least gives an explanation of the terminology. One sees that dividing out the image of d gives one the normal form, and the kernel of d describes those transformations that commute with v (the symmetries) and which are useless in normal form theory.

In many problems one can think of $\mathcal{N}$ and $\mathcal{T}$ as essentially the same space, but if one considers for instance problems with time-reversal symmetry, then one should take for $\mathcal{T}$ the elements that are invariant under the group action, and for $\mathcal{N}$ those that change sign (which is then compensated for by simultaneous time-reversal).

2. Formal definitions

Now we get serious. Let $K_p = \mathcal{F}^p \oplus \mathcal{F}^p$. Since the normal form computation changes the vector field, it is not enough to work with one d. If the v^i are the normal forms of $v = v^0$ at stage i, we denote by d^i the map $d^i(n,t) = ([v^i,t],0)$. The difference between v^i and v^{i+1} lies in $\mathcal{F}^{i+1}$. We define $d^{-1} = 0$.
Now let Z_p^r be the set of all $x \in K_p$ such that $d^r x \in K_{p+r}$.
If $x \in Z_{p+1}^{r-1}$ then $x \in K_{p+1} \subset K_p$ and $d^{r-1}x \in K_{p+r}$.
Since $v^r = v^{r-1} + w_r$, with $w_r \in K_r$, $d^r x = d^{r-1}x + [w_r, x] \in K_{p+r}$.
In other words $x \in Z_p^r$.
It follows that $Z_{p+1}^{r-1} \subset Z_p^r$. On the other hand, if $x \in d^{r-1}Z_{p-r+1}^{r-1}$ then $x = d^{r-1}y$ with $y \in Z_{p-r+1}^{r-1}$, implying that $y \in K_{p-r+1}$ and $d^{r-1}y \in K_p$.
It follows that $x \in K_p$ and $d^r x = 0$; thus $x \in Z_p^r$.
It follows that $d^{r-1}Z_{p-r+1}^{r-1} \subset Z_p^r$.
We now put, for $r \geq 0$,

$$E_p^r = Z_p^r/(d^{r-1}Z_{p-r+1}^{r-1} + Z_{p+1}^{r-1}), E_\cdot^r = \sum_p E_p^r.$$

Theorem 2.1. *There exists on the graded module $E_\cdot^r$ a differential $\partial_\cdot^r$ such that $H^\cdot(E_\cdot^r)$ is canonically isomorphic to $E_\cdot^{r+1}$, $r \geq 0$, where $H^p(E_p^r)$ stands for the cohomology induced by ∂_p^r.*

This enables us to translate the normal form problem into a cohomology problem.

2.1. *What is E_p^1?*

Before we answer that, let us first observe that $E_p^0 = K_p/K_{p+1}$.
By definition,

$$E_p^1 = Z_p^1/(d^0 Z_p^0 + Z_{p+1}^0).$$

Thus E_p^1 consists of pairs n, t with $n, t \in \mathcal{F}^p$, such that n is in first order normal form and t_p, the lowest order term in t commutes with v_0, the linear term of the vector field.
In other words, E_p^1 codifies the first order normal form and the still allowable transformations to compute higher order normal forms. This is true in general:
E_p^r codifies the rth order normal form and the still allowable transformations to compute higher order normal forms.

3. The anharmonic oscillator

There are not many problems where the unique normal form can be completely determined. The anharmonic oscillator has been completely normalized by Baider and Churchill[BC88, Bai89].
Here I will sketch the spectral sequence approach for this problem, a bit more general than it is done in the literature. We will take our coefficients from a local ring R containing $\mathbb{Q}$. [a] Then the noninvertible elements are in the maximal ideal, say $\mathfrak{m}$, and although subsequent computations are going to affect terms that we already consider as fixed in the normal form calculation, they will not affect their equivalence class in the residue field $R/\mathfrak{m}$. We denote the invertible elements in $R/\mathfrak{m}$ by $R^\star = R \setminus \mathfrak{m}$. So the convergence of the spectral sequence is with respect to the residue field. The actual normal form will contain formal power series which converge in the $\mathfrak{m}$-adic topology. To keep things simple, think of $R = \mathbb{R}$ and $\mathfrak{m} = 0$. When in the sequel we say that something is in the kernel of a coboundary operator, this means that the result has its coefficents in $\mathfrak{m}$. When we compute the image then this is done first in the residue field to check invertibility, and then extended to the whole of R. This gives us more accurate information than simply listing the normal form with coefficients in a field, since it allows for terms which are have nonzero coefficients, through which we do not want to divide, either because they are very small or because

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[a] One can think for instance of formal power series in a deformation parameter λ, which is the typical situation in bifurcation problems. Then a term $\lambda x^2 \frac{\partial}{\partial x}$ has coefficient λ which is neither zero nor invertible, since $\frac{1}{\lambda}$ is not a formal power series.

they contain a deformation parameter in such a way that the coefficient is zero for one or more values of this parameter. One might try to ignore the parameter problem by simply making the parameters into dynamical variables, adding equations

$$\frac{d\lambda}{dt} = 0,$$

but this would force one to treat these parameters polynomially, if they are given as such in the original variables, while in practice one would like to be able to work with expressions of the type

$$\frac{1}{1-\lambda}.$$

Expressions of this type arise naturally in versal deformation problems. This is treated extensively in [San94].

3.1. *Planar vector fields*

Let, with $k \geq -1, l \geq 0, q \in \mathbb{Z}/4$,

$$A_{k+l}^{k-l,q} = i^q (x^{k+1} y^l \frac{\partial}{\partial x} + i^{2q} x^l y^{k+1} \frac{\partial}{\partial y}).$$

Since $A_k^{l,q+2} = -A_k^{l,q}$, a basis is given by

$$\langle A_k^{l,q} \rangle_{k=-1,\cdots,l=0,\cdots,q=0,1},$$

but we have to compute in $\mathbb{Z}/4$. The commutation relation is

$$[A_{k+l}^{k-l,p}, A_{m+n}^{m-n,q}] =$$
$$= (m-k) A_{k+m+l+n}^{k-l+m-n,p+q} + n A_{k+m+l+n}^{m-n-(k-l),q-p} - l A_{k+m+l+n}^{k-l-(m-n),p-q}.$$

Then the anharmonic oscillator is of the form

$$v = A_0^{0,1} + \sum_{q=0}^{1} \sum_{k+l=1}^{\infty} \alpha_{k+l}^{k-l,q} A_{k+l}^{k-l,q}, \alpha_k^l \in R$$

Since

$$[A_0^{0,1}, A_{k+l}^{k-l,q}] = (k-l) A_{k+l}^{k-l,q+1}$$

we see that the kernel of $ad(A_0^{0,1})$ consists of those $A_{k+l}^{k-l,q}$ with $k = l$ and the image of those with $k \neq l$. We are now in a position to compute E_p^1:

$$E_p^1 = Ker\ ad(v_0^0)|_{G_p} \oplus Ker\ ad(v_0^0)|_{G_p},$$

where $G_p = \mathcal{F}_p / \mathcal{F}_{p+1}$. In general we have

$$\mathcal{T}_{2p}^1 = \mathcal{N}_{2p}^1 = \langle A_{2p}^{0,0}, A_{2p}^{0,1} \rangle_R.$$

and $\mathcal{T}^1_{2p+1} = \mathcal{N}^1_{2p+1} = 0$. One has the following commutation relations

$$[A^{0,p}_{2k}, A^{0,q}_{2m}] = (m-k)A^{0,p+q}_{2k+2m} + mA^{0,q-p}_{2k+2m} - kA^{0,p-q}_{2k+2m}.$$

For later use we write out the three different cases:

$$[A^{0,0}_{2k}, A^{0,0}_{2m}] = 2(m-k)A^{0,0}_{2k+2m},$$
$$[A^{0,0}_{2k}, A^{0,1}_{2m}] = 2mA^{0,1}_{2k+2m},$$
$$[A^{0,1}_{2k}, A^{0,1}_{2m}] = 0.$$

It follows that $\mathcal{A} = \langle A^{0,1}_{2m}\rangle_{m\in\mathbb{N}} \oplus \langle A^{0,1}_{2m}\rangle_{m\in\mathbb{N}}$, is an abelian Lie algebra ideal in $E^1_\cdot$, which itself is a $\mathbb{N}\times\mathbb{Z}/2$-graded Lie algebra.

We have $\partial^1_p = 0$, since either domain or image is zero. This implies $E^2_\cdot = E^1_\cdot$.
We now continue our normal form calculations until we hit a term $v^{2r}_{2r} = \beta^0_{2r}A^{0,0}_{2r} + \beta^1_{2r}A^{0,1}_{2r}$ with either β^0_{2r} or β^1_{2r} invertible. One has $E^{2r}_\cdot = E^1_\cdot$.
A general element in $\mathcal{T}^{2r}_{2p}$ is given by

$$t^{2r}_{2p} = \sum_{q=0}^{1} \gamma^q_{2p} A^{0,q}_{2p}$$

We have, with $p > r$,

$$\partial^{2r}_{2p}(0, t^{2r}_{2p}) =$$
$$= (2(p-r)\beta^0_{2r}\gamma^0_{2p}A^{0,0}_{2p+2r} + 2p\beta^0_{2r}\gamma^1_{2p}A^{0,1}_{2p+2r} - 2r\beta^1_{2r}\gamma^0_{2p}A^{0,1}_{2p+2r}, 0)$$

We view this as a map from the coefficients at G_{2p} to those at G_{2r+2p} with matrix representation

$$\begin{pmatrix} 2(p-r)\beta^0_{2r} & 0 \\ -2r\beta^1_{2r} & 2p\beta^0_{2r} \end{pmatrix}.$$

We see that for $0 < p \neq r$ the map is surjective if β^0_{2r} is invertible; if it is not, it has a one-dimensional image since we assume that in this case β^1_{2r} is invertible.

3.1.1. β^0_{2r} *is invertible.*

We have already shown that

$$Im\ \partial^{2r}_{2p} = \langle A^{0,0}_{2p+2r}, A^{0,1}_{2p+2r}\rangle \oplus 0$$

for $0 < p \neq r$ and $Im\ \partial^{2r}_{2r} = \langle A^{0,1}_{4r}\rangle \oplus 0$. Furthermore $Ker\ \partial^{2r}_{2r} = G_2 \oplus \langle v^{2r}_{2r}\rangle$ and $Ker\ \partial^{2r}_{2p} = G_2 \oplus 0$ for $0 < p \neq r$.
We are now in a position to compute

$$E^{2r+1}_{2p} = H^{2p}(E^{2r}_\cdot).$$

One has

$$E_{2p}^{\infty} = \begin{cases} A_{2p}^{0,0} & A_{2p}^{0,1} & \oplus A_{2p}^{0,0} & A_{2p}^{0,1} & p \\ \\ \mathfrak{m} & R^{\star} & R & R & 0 \\ \mathfrak{m} & \mathfrak{m} & 0 & 0 & 1,\cdots,r-1 \\ R^{\star} & R & R & 0 & r \\ 0 & 0 & 0 & 0 & r+1,\cdots,2r-1 \\ R & 0 & 0 & 0 & 2r \\ 0 & 0 & 0 & 0 & 2r+1,\cdots \end{cases}$$

This means the following. We can reconstruct the normal form out of this result. Here $c_{2p} \in \mathfrak{m}$ at position $A_{2p}^{0,\cdot}$ means that the coefficient of $A_{2p}^{0,\cdot}$ in c_{2p} cannot be invertible. And $c_{2p} \in R^{\star}$ means that it should be invertible. While $c_{2p} \in R$ indicates that the coefficient could be anything in R. By ignoring the $\mathfrak{m}A_{2p}^{0,\cdot}$ terms we obtain Baider's results. The $R^{\star}$-terms indicate the organizing center of the corresponding bifurcation problem.

Since we have no more effective transformations at our disposal, all cohomology after this will be trivial and we have reached the end of our spectral sequence calculation.

3.1.2. β_{2r}^{0} *is not invertible,but* β_{2r}^{1} *is.*

We can remove all terms $A_{2p+2r}^{0,1}$ for $p > 0$ by taking $\gamma_{2p}^{1} = 0$. This only contributes terms in $\mathfrak{m}A_{2p+2r}^{0,0}$. We obtain

$$E_{2p}^{2r+1} = \begin{cases} A_{2p}^{0,0} & A_{2p}^{0,1} & \oplus A_{2p}^{0,0} & A_{2p}^{0,1} & p \\ \\ \mathfrak{m} & R^{\star} & R & R & 0 \\ \mathfrak{m} & \mathfrak{m} & 0 & R & 1,\cdots,r-1 \\ \mathfrak{m} & R^{\star} & 0 & R & r \\ R & 0 & 0 & R & r+1,\cdots \end{cases}$$

3.1.3. β_{2q}^{0} *is invertible.*

We now continue our normal form calculation until at some point we hit on a term $\beta_{2q}^{0}A_{2q}^{0,0}$ with β_{2q}^{0} invertible and $q > r$.
We view $ad(A_{0}^{0,1} + \beta_{2r}^{1}A_{2r}^{0,1})$ as one coboundary operator $\partial_{\cdot}^{2r,1}$ and $ad(\beta_{2q}^{0}A_{2q}^{0,0})$ as another, $\delta_{\cdot}^{2q,0}$. Both operators act completely homogeneous with respect to the grading induced by the filtering and the $\mathbb{Z}/2$-grading and allow us to consider the bicomplex spanned by $E_{\cdot,0}^{1}$ and $E_{\cdot,1}^{1}$, where

$E^1_{2p,0} = \langle A^{0,0}_{2p}\rangle \oplus \langle A^{0,0}_{2p}\rangle$ and $E^1_{2p,1} = \langle A^{0,1}_{2p}\rangle \oplus \langle A^{0,1}_{2p}\rangle$. Since

$$\begin{aligned} A^{0,0}_{2q}, A^{0,0}_{2p}] &= 2(p-q)A^{0,0}_{2p+2q} \\ [A^{0,0}_{2q}, A^{0,1}_{2p}] &= 2pA^{0,1}_{2p+2q} \end{aligned}$$

we see that the only nontrivial cohomology is $H^{4q}_{\delta}(E^1_{\cdot,0}) \oplus H^{2q}_{\delta}(E^1_{\cdot,0})$.

To compute the image of $\partial_\cdot^{2r,1} + \delta_\cdot^{2q,0}$ we start with the E^{2r+1}_{2s}-term. Take $t^{(1)}_s = A^{0,1}_{2s}$. Then $\delta^{2q,0}_{2s} A^{0,1}_{2s} \in E^{2r+1}_{2q+2s,1}$, that is, we can write

$$\delta^{2q,0}_{2s} A^{0,1}_{2s} + \partial^{2r,1}_{2q+2s-2r} t^{(2)}_s = 0,$$

with $t^{(2)}_s \in E^1_{2q+2s-2r}$. If we now compute $(\partial_\cdot^{2r,1} + \delta_\cdot^{2q,0})(t^{(1)}_s + t^{(2)}_s)$, we obtain

$$(\partial_\cdot^{2r,1} + \delta_\cdot^{2q,0})(t^{(1)}_s + t^{(2)}_s) = \delta^{2q,0}_{2q+2s-2r} t^{(2)}_s.$$

Looking at the δ-cohomology we see that this gives us a nonzero result under the condition $0 < s \neq r$.

The image of $\partial_\cdot^{2q} = \partial_\cdot^{2r,1} + \delta_\cdot^{2q,0}$ in $E^1_{\cdot,0}$ is spanned by

$$\prod_{\substack{j=1\\ j\neq r}}^{\infty} \langle A^{0,0}_{4q-2r+2j}\rangle \oplus 0$$

and in $E^1_{\cdot,1}$ by $\prod_{j=1}^{\infty}\langle A^{0,1}_{2r+2j}\rangle \oplus 0$. Thus $H^{2p}(E^{2q}_{\cdot,0}) = \langle A^{0,0}_{2p}\rangle \oplus 0$ for $p = 0, \cdots, 2q-r, 2q$ and zero for other p. Thus

$$E^{\infty}_{2p} = \left\{ \begin{array}{ccccl} A^{0,0}_{2p} & A^{0,1}_{2p} & \oplus\, A^{0,0}_{2p} & A^{0,1}_{2p} & p \\ & & & & \\ \mathfrak{m} & R^\star & R & R & 0 \\ \mathfrak{m} & \mathfrak{m} & 0 & 0 & 1, \cdots, r-1 \\ \mathfrak{m} & R^\star & 0 & R & r \\ \mathfrak{m} & 0 & 0 & 0 & r+1, \cdots, q-1 \\ R^\star & 0 & 0 & 0 & q \\ R & 0 & 0 & 0 & q+1, \cdots, 2q-r \\ 0 & 0 & 0 & 0 & 2q-r+1, \cdots, 2q-1 \\ R & 0 & 0 & 0 & 2q \\ 0 & 0 & 0 & 0 & 2q+1, \cdots \end{array} \right.$$

4. Concluding remarks

The theoretical framework of spectral sequences is natural to describe the concept of normal form. This theoretical framework will not immediately solve any open problems, but one can hope that it will give us a good organizational tool to attack the unique normal form problem for higher dimensions.

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