Integrable Systems in Symplectic Geometry

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In this article we show that if one writes down the structure equations for the evolution of a curve embedded in an $4n-$dimensional symplectic manifold with zero curvature, this leads to a Nijenhuis operator for an integrable scalar-vector evolution equation generalizing the known cases of the vmKdV equation and the noncommutative scalar KdV. The procedure also gives us the symplectic and Hamiltonian operators.

Keywords: Integrable system, Cartan structure, Lie algebra, symplectic geometry

1. Introduction

We study the connection between the motion of a curve and the theory of integrable equations.

It was shown recently in[1] that if one writes down the structure equations for the evolution of a curve embedded in an n-dimensional Riemannian manifold with constant curvature, this leads to a symplectic, a Hamiltonian and a Nijenhuis (or hereditary) operator. This gives us a natural connection among finite dimensional geometry, infinite dimensional geometry and integrable systems.

The goal of the present paper is to generalize this analysis to symplectic geometry. We do this by replacing $\mathbb{R}$ by $\mathbb{H}$, the skew field of the quaternions. The Riemannian and symplectic geometries are formally similar, in the sense that we can identify any quaternionic number with a 4×4 real orthogonal matrix, so that we can still work with a natural moving frame. Since we work over a skew field, we derive as a byproduct noncommutative integrable system. It turns out that a Nijenhuis operator, a symplectic operator and a Hamiltonian operator naturally come out of the analysis just like in the Riemannian case. This confirms our working hypothesis that the approach is applicable to any Cartan geometry.

The paper is organized as follows. In section 2 we introduce symplectic geometry. In section 3 we compute the structure equations in the symplectic case while we choose a connection and obtain a Hamiltonian operator, symplectic operator and a Nijenhuis operator, leading to the integrable equations. In section 4 we express all the geometric operators in terms of Lie algebra bracket, Killing form and projections. In section 5 we compare our equation with some known results.

Remark 1.1. Complete proofs of the results in this paper can be found in the PhD thesis which is being prepared by the first author (E.A.) under supervision of second author (J.A.S.).

2. Moving frame method in symplectic geometry

We define the symplectic Lie group over a quaternionic algebra by

$$G = Sp(n+1) = \{A \in GL(n+1, \mathbb{H}) | A^* A = I\},$$

in which $A^*_{ij} = \overline{A_{ji}}, A_{ij} \in \mathbb{H}$. Then the Lie subgroup $H = Sp(1) \times Sp(n)$ of G will be closed so that the pair (G, H) will define a Klein geometry. The Lie algebras of G and H are

$$\mathfrak{g} = \mathfrak{sp}_{n+1} = \{A \in GL(n+1, \mathbb{H}) | A^* + A = 0\}, \quad \mathfrak{h} = \mathfrak{sp}_1 \times \mathfrak{sp}_n.$$

As is known in the literature, $M = G/H$ is then a smooth manifold. In fact M is the quaternionic projective space $\mathbb{H}P^{n-1}$. For a comprehensive reference, see.[2] Similar to the situation in Riemannian manifold,[1] given a curve in M, we know its tangent vectors D_x and want to compute all possible D_t. Let ω be Cartan 1−form with its values in the Lie algebra $\mathfrak{g}$. We make a specific choice of $\omega(D_x)$ and leave $\omega(D_t)$ as a general element of $\mathfrak{g}$. We see that the dimension of M is equal to the dimension of $\mathfrak{g}/\mathfrak{h}$ which is easily computed to be $4n$. With x taken to be the arc length parameter, the dimension of the space of differential invariants in $\omega(D_x)$ describing the curve must be one less than the dimension of the manifold, that is, $4n - 1$.

3. Cartan structure equation in symplectic geometry

Now let us choose a Cartan matrix $\omega(D_x)$ similar to that of the parallel coframe in Riemannian geometry with proper dimension counting as follows:

$$\omega(D_x) = \begin{pmatrix} 0 & 1 & \mathbf{0}^T \\ -1 & u & -\overline{\mathbf{u}}^T \\ \mathbf{0} & \mathbf{u} & \mathbf{0} \end{pmatrix}.$$

Here u is purely imaginary, and $\mathbf{u} \in \mathbb{H}^{n-1}$.

Remark 3.1. Other choices of coframe tend to destroy the scalar-vector character of the analysis and complicate matters tremendously, which seems to be one of the main reasons why the n-dimensional analysis using Frenêt frames never took of.

We see that this matrix is parametrized by $4n-1$ real parameters. Notice that here we have taken the curvature and torsion part of Cartan form in one picture.

Now $\omega(D_t)$ must be a typical element of $\mathfrak{g}$ which we write as follows:

$$\omega(D_t) = \begin{pmatrix} m_{11} & m_{12} & -\overline{\mathbf{m}}_1^T \\ m_{21} & m_{22} & -\overline{\mathbf{m}}_2^T \\ \mathbf{m}_1 & \mathbf{m}_2 & \mathbf{M} \end{pmatrix}.$$

In the Riemannian case, if we use a parallel frame and assume constant curvature $\varkappa$, this can be taken zero and we still can derive all the geometric quantities. Therefore we have taken the curvature equal to zero. Hence the Cartan structure equation evaluated at the evolutionary vector fields D_x, D_t is as follows:

$$D_x\omega(D_t) - D_t\omega(D_x) + [\omega(D_t), \omega(D_x)] = 0.$$

Before we explore the Cartan structure equation, let us define some notation. Commutators of vectors and scalars are defined by

$$C_{\mathbf{u}}\mathbf{m}_2 := \langle \mathbf{u}, \mathbf{m}_2\rangle - \langle \mathbf{m}_2, \mathbf{u}\rangle, \quad C_u m_{22} := u m_{22} - m_{22} u,$$

where the inner product $\langle\cdot,\cdot\rangle$ is the Hermitian inner product. Right multiplication by scalar u on vector $\mathbf{h}$ and left multiplication by vector $\mathbf{u}$ on scalar h are defined respectively by

$$R_u\mathbf{h} = \mathbf{h}u, \quad L_{\mathbf{u}}h = \mathbf{u}h.$$

On the other hand, the anti-commutator on vector and scalar are defined by

$$A_{\mathbf{u}}\mathbf{h} = \langle \mathbf{u}, \mathbf{h}\rangle + \langle \mathbf{h}, \mathbf{u}\rangle, \quad A_u h = uh + hu.$$

Now we explicitly write the components of the Cartan structure equation. Among these equations, the four first equations are concerned with the curvature and the last three with the torsion. These equations lead to evolution of the scalar invariant u and the vector invariant $\mathbf{u}$ as combination of geometric operators applied on the proper torsion variables of $\omega(D_t)$ according to the proposition below, in which we have defined $\mathfrak{H}_1$ as the operator acting on vectors by $\mathfrak{H}_1\mathbf{h} = \left(D_x^{-1}(\mathbf{h}\overline{\mathbf{u}}^t - \mathbf{u}\overline{\mathbf{h}}^t)\right)\mathbf{u}$, where, for instance, $\mathbf{h}\overline{\mathbf{u}}^t$ is the outer product of a vector and a covector, that is, a matrix. Hence, for instance, we can write $\mathbf{M}\mathbf{u} = \mathfrak{H}_1\mathbf{m}_2$.

$$D_x m_{11} - m_{12} - m_{21} = 0 \tag{1a}$$
$$D_x m_{22} - D_t u - C_u m_{22} + C_{\mathbf{u}}\mathbf{m}_2 + m_{12} + m_{21} = 0 \tag{1b}$$
$$D_x \mathbf{m}_2 - D_t \mathbf{u} + R_u \mathbf{m}_2 + \mathfrak{H}_1 \mathbf{m}_2 - L_{\mathbf{u}} m_{22} + \mathbf{m}_1 = 0 \tag{1c}$$
$$D_x \mathbf{M} - \mathbf{m}_2 \overline{\mathbf{u}}^t + \mathbf{u}\overline{\mathbf{m}}_2{}^t = 0 \tag{1d}$$
$$D_x \mathbf{m}_1 - \mathbf{m}_2 - \mathbf{u} m_{21} = 0 \tag{1e}$$
$$D_x m_{12} + m_{11} - m_{22} + m_{12} u - \langle \mathbf{m}_1, \mathbf{u} \rangle = 0 \tag{1f}$$
$$D_x m_{21} + m_{11} - m_{22} - u m_{21} + \langle \mathbf{u}, \mathbf{m}_1 \rangle = 0 \tag{1g}$$

Solving these equations we obtain

Proposition 3.1. *The evolution of differential invariants can be written in the form*

$$\begin{pmatrix} D_t u \\ D_t \boldsymbol{u} \end{pmatrix} = \mathfrak{H}\mathfrak{I} \begin{pmatrix} m_{12} + m_{21} \\ \boldsymbol{m}_1 \end{pmatrix} + \mathfrak{A} \begin{pmatrix} m_{12} + m_{21} \\ \boldsymbol{m}_1 \end{pmatrix}, \tag{2}$$

where

$$\mathfrak{H} = \begin{pmatrix} D_x - C_u & C_{\boldsymbol{u}} \\ -L_{\boldsymbol{u}} & D_x + R_u + \mathfrak{H}_1 \end{pmatrix}, \quad \mathfrak{A} = \begin{pmatrix} (2D_x - C_u)D_x^{-1} & 0 \\ -L_{\boldsymbol{u}} D_x^{-1} & I \end{pmatrix},$$

and

$$\mathfrak{I} = \begin{pmatrix} \frac{1}{2}D_x - \frac{1}{4}C_u - \frac{1}{4}A_u D_x^{-1}\frac{1}{2}A_u & \frac{1}{2}C_{\boldsymbol{u}} + \frac{1}{2}uD_x^{-1}A_{\boldsymbol{u}} \\ \\ -\frac{1}{2}L_{\boldsymbol{u}}D_x^{-1}\frac{1}{2}A_u - \frac{1}{2}L_{\boldsymbol{u}} & D_x + \frac{1}{2}L_{\boldsymbol{u}}D_x^{-1}A_{\boldsymbol{u}} \end{pmatrix}.$$

Remark 3.2. If we subtract the equation (1f) from (1g), then we obtain that

$$m_{21} - m_{12} = D_x^{-1}(\frac{1}{2}A_u(m_{12} + m_{21}) - A_{\mathbf{u}}\mathbf{m}_1).$$

In fact the expression on the right is the Killing form of two proper matrices in the Lie algebra $\mathfrak{g}$ and this indeed appeared in the Lie algebra form of $\mathfrak{I}$ as described here and in the next section. Compare this equation with the equation (13.1) (the equation after (13) in that paper) in the

$n-$dimensional Riemannian case using the parallel coframe in[1] and equation (12) in $3-$dimensional Riemannian geometry using the Frenêt frame in.[3] The difference with the later paper is that we use D_x^{-1} instead of dividing by curvature κ in there.

Let us define $\begin{pmatrix} h \\ \mathbf{h} \end{pmatrix} = \mathfrak{A} \begin{pmatrix} m_{12}+m_{21} \\ \mathbf{m}_1 \end{pmatrix}$. Then we obtain

$$\begin{pmatrix} m_{12}+m_{21} \\ \mathbf{m}_1 \end{pmatrix} = \mathfrak{A}^{-1} \begin{pmatrix} h \\ \mathbf{h} \end{pmatrix}, \quad \mathfrak{A}^{-1} = \begin{pmatrix} D_x(2D_x - C_u)^{-1} & 0 \\ L_{\mathbf{u}}(2D_x - C_u)^{-1} & I \end{pmatrix}.$$

Hence the evolution in the proposition takes the following form:

$$\begin{pmatrix} D_t u \\ D_t \mathbf{u} \end{pmatrix} = \mathfrak{R} \begin{pmatrix} h \\ \mathbf{h} \end{pmatrix} + \begin{pmatrix} h \\ \mathbf{h} \end{pmatrix}, \quad \mathfrak{R} = \mathfrak{H}\mathfrak{I}\mathfrak{A}^{-1}. \tag{3}$$

If we make the specialization $\begin{pmatrix} h \\ \mathbf{h} \end{pmatrix} = \begin{pmatrix} u_1 \\ \mathbf{u}_1 \end{pmatrix}$, where $u_1, \mathbf{u}_1$ are the derivatives of u and $\mathbf{u}$ with respect to x, respectively, then we obtain the non-commutative evolution equations:

$$\begin{cases} u_t = \frac{1}{4}u_3 + \frac{3}{8}(-uu_1u - uu_2 + u_2u) + \frac{3}{2}\langle \mathbf{u}, \mathbf{u}\rangle u_1 + \langle \mathbf{u}, \mathbf{u}_1\rangle u + \frac{1}{2}u\langle \mathbf{u}, \mathbf{u}_1\rangle \\ \qquad +2u\langle \mathbf{u}_1, \mathbf{u}\rangle - \frac{1}{2}\langle \mathbf{u}_1, \mathbf{u}\rangle u + \frac{3}{2}C_{\mathbf{u}}\mathbf{u}_2, \\ \mathbf{u}_t = \mathbf{u}_3 + \frac{3}{2}\mathbf{u}_2 u + \frac{3}{4}\mathbf{u}_1(u_1 + \frac{1}{2}u^2 + 2\langle \mathbf{u}, \mathbf{u}\rangle). \end{cases}$$

Definition 3.1. The *pairing* between $\begin{pmatrix} h \\ \mathbf{h} \end{pmatrix}$ and $\begin{pmatrix} g \\ \mathbf{g} \end{pmatrix}$ is defined by

$$\langle \begin{pmatrix} h \\ \mathbf{h} \end{pmatrix}, \begin{pmatrix} g \\ \mathbf{g} \end{pmatrix} \rangle = \int K(\sigma \begin{pmatrix} h \\ \mathbf{h} \end{pmatrix}, \sigma \begin{pmatrix} g \\ \mathbf{g} \end{pmatrix}).$$

in which σ is a section of $\mathfrak{g}/\mathfrak{h}$ subject to the zero constant curvature condition into the subvector space of the Lie algebra $\mathfrak{g}$ generated just as the Cartan matrix $\omega(D_x)$. For instance, one can take

$$\sigma \begin{pmatrix} h \\ \mathbf{h} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & h & -\overline{\mathbf{h}}^T \\ 0 & \mathbf{h} & 0 \end{pmatrix}.$$

The adjoint of the operator P is defined by

$$\langle \begin{pmatrix} h \\ \mathbf{h} \end{pmatrix}, P \begin{pmatrix} g \\ \mathbf{g} \end{pmatrix} \rangle = \langle P^* \begin{pmatrix} h \\ \mathbf{h} \end{pmatrix}, \begin{pmatrix} g \\ \mathbf{g} \end{pmatrix} \rangle.$$

Since the pairing is nondegenerate, P^* is well-defined. Using the explicit formula for the Killing form, one can prove that the operators $\mathfrak{H}$ and $\mathfrak{I}$ are indeed skew-symmetric, that is, $\mathfrak{H}^* = -\mathfrak{H}$ and $\mathfrak{I}^* = -\mathfrak{I}$. The following lemma shows that in fact $\mathfrak{A}\mathfrak{H}$ is also a skew-symmetric operator.

Lemma 3.1. *We have that* $\mathfrak{A}\mathfrak{H} = \mathfrak{H}\mathfrak{A}^*$.

Hence we can write the operator $\mathfrak{R}$ in the form of

$$\mathfrak{R} = \Big(\mathfrak{H}\mathfrak{A}^*\Big)\Big(\mathfrak{A}^{-1*}\mathfrak{I}\mathfrak{A}^{-1}\Big).$$

Using the pairing we defined and the fact that the Killing form is invariant under the adjoint action:

$$K([X,Y],Z) + K(Y,[X,Z]) = 0 \quad \text{for} \quad X,Y,Z \in \mathfrak{g},$$

and that $d^2K = 0$ where d is the boundary operator of forms on the Lie algebra, one can, although not easily, prove:

Theorem 3.1. *The operators* $\mathfrak{H}\mathfrak{A}^*, \mathfrak{A}^{-1*}\mathfrak{I}\mathfrak{A}^{-1}$ *and* $\mathfrak{R}$ *are Hamiltonian, symplectic and Nijenhuis operator, respectively. Furthermore, the operators* $\mathfrak{A}$ *and* $\mathfrak{A}^{-1}$ *are Nijenhuis operators, that is, their Nijenhuis tensor is zero.*

4. The geometric operators in terms of Lie bracket, Killing form and projections

In the method we are using, the only tools we have are the Lie algebra and the Cartan geometry, hence we expect to be able to write the geometric operators $\mathfrak{H}$ and $\mathfrak{I}$ in terms of the Lie bracket, the Killing form and proper projections.

Let us define the projections π_0 and π_1 as follows:

$$\pi_0 \begin{pmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & -\bar{\mathbf{m}}_2^t \\ 0 & \mathbf{m}_2 & \mathbf{M} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{M} \end{pmatrix},$$

and

$$\pi_1 \begin{pmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & -\bar{\mathbf{m}}_2^t \\ 0 & \mathbf{m}_2 & \mathbf{M} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & m_{22} & -\bar{\mathbf{m}}_2^t \\ 0 & \mathbf{m}_2 & 0 \end{pmatrix}.$$

Let $\hat{u}$ and $\hat{m}_2$ be the projection of $\omega(D_x)$ and $\omega(D_t)$ over the Lie subalgebra $\mathfrak{h}$, respectively and $\hat{m}_1$, the projections of $\omega(D_t)$ over the vector space $\mathfrak{g}/\mathfrak{h}$ which itself is indeed the dual orthogonal of $\mathfrak{h}$ with respect to the Killing form. Then we simply find that

$$\hat{u}_t = \hat{\mathfrak{H}}(\pi_1\hat{m}_2) + \hat{m}_0, \quad \hat{m}_0 = \pi_1 ad_A \hat{m}_1,$$

where

$$\hat{\mathfrak{H}} = D_x - \pi_1 ad_{\hat{u}} - ad_{\hat{u}} D_x^{-1} \pi_0 ad_{\hat{u}}.$$

This is exactly the Poisson operator [1.13] in[4] which in general is defined on symmetric spaces.

Now the torsion part gives in fact the following matrix equation:

$$ad_{\hat{a}} \hat{m}_2 = (D_x + ad_{\hat{u}})\hat{m}_1, \quad \hat{a} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4}$$

Since $ad_{\hat{a}}^2 \neq \lambda I$ for any $\lambda \in \mathbb{R}$, we can not solve equation (4) in the usual way. Therefore the existence of the Nijenhuis operator $\mathfrak{A}$ plays a crucial rule in the symplectic case. Notice that in the Riemannian case we do have $ad_{\hat{a}}^2 = -I$.

In order to get rid of this difficulty, we define two projections ρ_1, ρ_0 which in fact split off the scalar and vector part:

$$\rho_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & m_{22} & -\bar{\mathbf{m}}_2^t \\ 0 & \mathbf{m}_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & m_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho_0 \begin{pmatrix} 0 & 0 & 0 \\ 0 & m_{22} & -\bar{\mathbf{m}}_2^t \\ 0 & \mathbf{m}_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\bar{\mathbf{m}}_2^t \\ 0 & \mathbf{m}_2 & 0 \end{pmatrix}.$$

Now one can show that

$$\hat{u}_t = \hat{\mathfrak{H}} \hat{\mathfrak{I}} \hat{m}_0 + \hat{\mathfrak{A}} \hat{m}_0,$$

in which the Lie algebra form $\hat{\mathfrak{I}}$ of geometric operator $\mathfrak{I}$ and $\hat{\mathfrak{A}}$ of Nijenhuis operator $\mathfrak{A}$ appears as

$$\hat{\mathfrak{I}} = \frac{1}{2} \hat{u} D_x^{-1} K(\hat{u}, .) + (\frac{1}{2}\rho_1 - \rho_0)\pi_1 ad\hat{a}(D_x + ad_{\hat{u}}) ad_{\hat{a}} \pi_1 (\frac{1}{2}\rho_1 + \rho_0),$$
$$\hat{\mathfrak{A}} = \rho_0 + 2\rho_1 - ad_{\hat{u}} D_x^{-1} \rho_1.$$

5. Reduction to scalar part

If we take vector part $\mathbf{u} = 0$, the equation becomes:

$$u_t = -\frac{1}{4} u_3 + \frac{3}{8}(uu_1u - uu_2 + u_2u),$$

which is the noncommutative mKdV equation and the Nijenhuis operator becomes:

$$\mathfrak{R} = (D_x - C_u)(D_x - L_u)(2D_x - C_u)^{-1}(D_x + R_u)D_x(2D_x - C_u)^{-1},$$

as found by V.V. Sokolov e.a.[5] using the Lax method. From our results it also follows that the Hamiltonian operator is:

$$\mathfrak{A}\mathfrak{R}_1 = (2D_x - C_u)D_x^{-1}(D_x - C_u),$$

and the symplectic operator

$$\begin{aligned}&\mathfrak{A}^{-1*}\mathfrak{R}_2\mathfrak{A}^{-1}\\ &= (2D_x - C_u)^{-1}D_x(D_x - L_u)(2D_x - C_u)^{-1}(D_x + R_u)D_x(2D_x - C_u)^{-1}.\end{aligned}$$

Remark 5.1. In[6] the quaternion version of the reduced scalar equation is given in equation (3.17). If we identify $u \in \mathfrak{H}$ with a vector in $\vec{u} \in \mathbb{R}^3$, then

$$\mathfrak{R} = \frac{1}{4}(D_x - 2\vec{u} \times \cdot)(D_x - \vec{u} \times \cdot + \vec{u}D^{-1}\langle \vec{u}, \cdot\rangle)(1 - \vec{u} \times D_x^{-1}\cdot)^{-1}.$$

The operator $(1-\vec{u}\times D_x^{-1}\cdot)^{-1}$, when applied to symmetries of the equation, only contains on expansion as many terms as the order of the symmetry.

Remark 5.2. The Lax operator in the symplectic case is

$$L = D_x + \lambda\hat{a} + \hat{u}.$$

Then the Sokolov-Drinfeld method can be applied to the Kac-Moody algebra, as a $\mathbb{Z}-$graded Lie algebra, based on the symplectic Lie algebra $\mathfrak{g} = \mathfrak{sp}_{n+1}$.

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