TRANSVECTION AND DIFFERENTIAL INVARIANTS OF PARAMETRIZED CURVES

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Abstract. In this paper we describe an $\mathfrak{sl}_2$ representation in the space of differential invariants of parametrized curves in homogeneous spaces. The representation is described by three operators, one of them being the total derivative D. We use this representation to find a basis for the space of differential invariants of curves in a complement of the image of D, and so generated by transvection. These are natural representatives of first cohomology classes in the invariant bicomplex. We describe algorithms to find these basis and study most well-known geometries.

1. Introduction

In classical invariant theory, that is in the study of the invariants of the action of $SL_2(\mathbb{R})$ on binary forms, the basic computational tool is that of transvection. Given any two covariants (or finite dimensional irreducible $\mathfrak{sl}_2$-representations in modern language) one can construct a number of (possibly) new covariants by computing the transvectants. The simplest example is given by two linear forms, $a_oY + a_1X$ and $b_0Y + b_1X$. Their first transvectant is the determinant

$$\begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix}$$

of the coefficients. Another example is the discriminant $a_0a_2 - a_1^2$ of a quadratic form $a_0Y^2 + 2a_1XY + a_2X^2$, which is the second transvectant of the quadratic form with itself.

The process of transvection generates the kernel of a certain operator F. This operator is obtained as follows. Differentiation can be thought of as a map of $\langle u, u_1, u_2, \cdots \rangle$ onto itself using the rule $D : u_k \mapsto u_{k+1}$, combined with linearity. This map can be extended to (tensor) products by prescribing the usual product rule, which makes D a derivation. In the u_i-coordinates, D is part of an $\mathfrak{sl}_2$

1991 *Mathematics Subject Classification.* Primary: 37K; Secondary: 53A55.

Key words and phrases. Differential invariants of curves, classical invariant theory, representations, transvectants, Clebsch-Gordan .

representation given by the three operators

$$
\begin{aligned}
F &= \sum_{k=1}^{\infty} k(k-1)u_{k-1}\frac{\partial}{\partial u_k} \\
E &= \sum_{k=0}^{\infty} 2ku_k\frac{\partial}{\partial u_k} \qquad (1.1)\\
D &= \sum_{k=0}^{\infty} u_{k+1}\frac{\partial}{\partial u_k}.
\end{aligned}
$$

As we will see later, the kernel of F is generated by recurrent transvection of u_1, first with itself, then with the produced transvectants, as is done in classical invariant theory of $SL_2(\mathbb{R})$.

In late 1999 and motivated by modular function theory, John McKay asked the second author of this paper if the Schwarzian derivative, $S(u) = \frac{u_3}{u_1} - \frac{3}{2}\frac{u_2^2}{u_1^2}$ was in the kernel of F (the original question was if $S(u)$ was in the kernel of a related operator part of a representation of the Heisenberg algebra). Clearly the answer is yes. The fact that the Schwarzian derivative is in the kernel of F hints to a possible connection between transvection and the differential invariants of parametrized curves. Indeed, the Schwarzian derivative is the generator of projective differential invariants of $u : \mathbb{R} \to \mathbb{RP}^1$. That is, any other differential invariant can be written as a function of the Schwarzian and its derivatives. The Schwarzian lies in the kernel of the operator F and can be generated by transvection of u_1. After a natural generalization to several dependent variables given by

$$F = \sum_{\alpha=1}^{n}\sum_{k=1}^{\infty} k(k-1)u_{k-1}^{\alpha}\frac{\partial}{\partial u_k^{\alpha}},$$

a simple calculation shows that, if $u : \mathbb{R} \to \mathbb{R}^n$, then $\frac{u_2 \cdot u_2}{u_1 \cdot u_1} - \left(\frac{u_1 \cdot u_2}{u_1 \cdot u_1}\right)^2$ is in the kernel of F. The expression is the square of the Euclidean curvature of u, expressed in terms of an arbitrary parametrization. It can be generated by transvection of u_1 (in fact, of its components u_i^{α}) with itself and its transvectants. The Euclidean curvature is also one of the generating differential invariants for Euclidean curves. A slightly longer calculation shows that torsion of u, when written in terms of an arbitrary parametrization, also lies in the kernel of the operator F and it is also a function of transvectants of u_1 with themselves. We found other examples of this situation for curves in the Möbius sphere, the local model for flat conformal manifolds. These examples strongly suggest a connection between transvection and differential invariants of parametrized curves.

This paper is based on a basic property. The operators F, D and E all commute with the prolongation of vector fields, insofar the action of the group does not affect the parameter. Therefore, they define a representation of $\mathfrak{sl}_2$ in the space of differential invariants of curves. In this paper we show how such a representation can be used to show that transvection can take the role of differentiation in the process of finding differential invariants of parametrized curves. By doing so we are capable of finding a basis of differential invariants for most well-known geometries which are always in the kernel of F. In some simple cases (including polynomials in u_k), the kernel of F is the complement of the image of D, so in that sense we are producing generators that do not include the derivative of lower order invariants. As it is shown in the examples, one can also consider simple fractional expressions in u_i, rather than polynomials. In affine cases one can also follow a process that ensures the result is invariant under reparametrizations. It is not clear the method works in general for fractional expressions or how far one can go weakening this assumption. In Section 2 we introduce basic notions and describe the theorems that allow us to use transvection to generate differential invariants in general and show that, under certain technical conditions, all differential invariants are produced this way. In Section 3 we study the case of affine geometries. We describe an algorithm to find a system of independent relative invariants in the kernel of F using iterative transvection of u_1 with an appropriately chosen initial differential invariant. Group invariant contractions of these relative invariants produce a basis of differential invariants. We also show how transvection produces results which are invariant under reparametrization.

Section 4 is perhaps a more surprising one. There we show how to apply this same algorithm to some non affine geometries. In the case of differential invariants of projective curves Wilczynski proved in [17] that one could lift a curve in $\mathbb{RP}^n$ to a curve in $\mathbb{R}^{n+1}$ in the standard way, multiply the lift by a factor and define a different vector $\mu \in \mathbb{R}^{n+1}$. He then proceeded to recurrently differentiate this vector and to produce a basis of differential invariants for projective curves by taking determinants of the derivatives (this was not his original idea, but that is what the process boils down to). It just happens that the vector μ is a relative invariant of the prolonged projective action with constant weight and Wilczynski's process mirrors the transvection process we describe in Section 3. In Section 5 we show that transvection can replace differentiation in Wilczynski's original method. Furthermore, we show that in many other geometric manifolds G/H the method works identically for both differentiation and transvection. That is, one can lift the curve to a curve in a higher dimensional $\mathbb{R}^m$ where the group acts linearly. We can modify the curve to produce a relative differential invariant and we can then apply recurrent transvection of it with a properly chosen differential invariant. By doing so we can produce enough relative invariants and combine them to generate a basis for the space of differential invariants of curves in G/H all of them in the kernel of the operator F. We show explicitly the projective, conformal and Lagrangian Grasmannian cases. Transvection has been previously used to generate invariants of differential operators in [11] (see page 216) and [12]. Our is a different approach and the invariants we generate are different.

As a conclusion, it is not at all surprising that Schwarzian and Euclidean curvature lie in the kernel of the operator F and can be written as transvectants of the derivative u_1. In fact, being differential invariants of lowest order within their corresponding geometrical background (projective and Euclidean respectively) they were very likely to be so.

This research was funded by a grant from the Netherlandse Organisatie voor Wetenschappelijk Onderzoek (NWO). The first author would like to thank the Vrije Universiteit in Amsterdam for their support during a very pleasant visit.

2. Infinite dimensional representation theory of $\mathfrak{sl}_2$

The infinite dimensional representation theory of $\mathfrak{sl}_2$ is described in [4], to which we refer at the appropriate places.

Definition 1 ($\mathfrak{sl}_2$-module, constant)**.** A module $\mathcal{V}$ is called an **$\mathfrak{sl}_2$-module** when $\mathcal{F}, \mathcal{E}, \mathcal{D} \in End(\mathcal{V})$ and

$$[\mathcal{F}, \mathcal{D}] = \mathcal{E} \qquad [\mathcal{E}, \mathcal{F}] = -2\mathcal{F} \qquad [\mathcal{E}, \mathcal{D}] = 2\mathcal{D}. \tag{2.1}$$

If a nonzero $v \in \mathcal{V}$ is annihilated by $\mathcal{D}$, it will be called a **constant**.

Assumption 1. *Since we are always thinking of $\mathcal{D}$ as a differentiation, we will restrict ourselves to $\mathfrak{sl}_2$-modules for which $\mathcal{F}v = 0$ whenever v is constant.*

Definition 2 (nondegenerate)**.** We say that an $\mathfrak{sl}_2$-module is **nondegenerate** if the constants form a direct summand as an $\mathfrak{sl}_2$-module. This means that if for instance $\mathcal{F}w$ is a constant, then w itself must be a constant.

Example 1. The space of polynomials (or tensors) in $u_k^\alpha, \alpha = 1, \dots, n$, is an $\mathfrak{sl}_2$-module, with $\mathcal{F} = F, \mathcal{E} = E, \mathcal{D} = D$ as in the introduction (1.1) and its natural extension to several variables.

Example 2. Another example is given by the following construction. Let $u_1 = \begin{pmatrix} u_1^1 \\ \vdots \\ u_1^n \end{pmatrix}$, with $u_1^\alpha \in \ker F$ for $\alpha = 1, \dots, n$. Then

$$F_v u_1 = F_v \begin{pmatrix} u_1^1 \\ \vdots \\ u_1^n \end{pmatrix} = \begin{pmatrix} F u_1^1 \\ \vdots \\ F u_1^n \end{pmatrix} = 0,$$

where we denote by F_v the induced operator on the space of vectors in $\mathbb{R}^n$ when applying F to each entry. Furthermore, computing in the same way,

$$E_v u_1 = 2u_1 \text{ and } D_v u_1 = u_2.$$

Taking $\mathcal{F} = F_v, \mathcal{E} = E_v$ and $\mathcal{D} = D_v$ we see that the space of polynomial vectorfields of the form $\begin{pmatrix} p_1(u_1^1, \dots, u_1^n, u_2^1, \dots) \\ \vdots \\ p_n(u_1^1, \dots, u_1^n, u_2^1, \dots) \end{pmatrix}$ is an $\mathfrak{sl}_2$-module.

Definition 3 (Lowest Weight Module)**.** For each $\lambda \in \mathbb{C}$ we define an abstract space $\mathbf{V}_\lambda = \langle \mathbf{v}_0, \mathbf{v}_1, \cdots \rangle$, ($\langle , \rangle$ represents the linear span) to be an irreducible $\mathfrak{sl}_2$-module on which $\mathfrak{sl}_2$ acts as follows

$$\mathcal{F}\mathbf{v}_k = k(\lambda + k - 1)\mathbf{v}_{k-1}, \quad \mathcal{E}\mathbf{v}_k = (\lambda + 2k)\mathbf{v}_k, \quad \mathcal{D}\mathbf{v}_k = \mathbf{v}_{k+1}. \tag{2.2}$$

and where $\mathbf{v}_0$ is a given element in the kernel of $\mathcal{F}$ with $\mathcal{E}\mathbf{v}_0 = \lambda \mathbf{v}_0$. The $\mathfrak{sl}_2$-module $\mathbf{V}_\lambda$ is called a **Lowest Weight Module**, and $\mathbf{v}_0$ the lowest weight vector, which would be the the starting vector for the $\mathfrak{sl}_2$-representation $\{\mathcal{F}, \mathcal{E}, \mathcal{D}\}$.

For general λ, one has $\dim \ker \mathcal{F}|_{\mathbf{V}_\lambda} = 1$. Only for the special values $\lambda = 0, -1, -2, \dots$ one has $\dim \ker \mathcal{F}|_{\mathbf{V}_\lambda} = 2$. The one-dimensional case is easy to control algoritihmically, the two-dimensional case is causing difficulties as we shall see.

In our examples, a fractional λ might arise by dividing by fractional powers of the length of the vector u_1, as is sometimes done to control the $\mathcal{E}$-eigenvalue of an element.

The following examples give concrete representations of the abstract definition.

Example 3. Take $\mathbf{v}_0 = u_0^1$, $\mathcal{F} = F, \mathcal{E} = E, \mathcal{D} = D$ as defined in (1.1) (or its extension to several variables) and $\mathbf{v}_i = D^i\mathbf{v}_0$. Then $\mathbf{V}_0 = \langle \mathbf{v}_0, \cdots, \mathbf{v}_i, \cdots \rangle$ is a Lowest Weight Module.

Example 4. Take $\mathbf{v}_0 = u_1^1$, $\mathcal{F} = F, \mathcal{E} = E, \mathcal{D} = D$ as defined in (1.1) and $\mathbf{v}_i = D^i\mathbf{v}_0$. Then $\mathbf{V}_2 = \langle \mathbf{v}_0, \cdots, \mathbf{v}_i, \cdots \rangle$ is a Lowest Weight Module.

Example 5. Take $\mathbf{v}_0 = u_1^1 u_1^2$, $\mathcal{F} = F, \mathcal{E} = E, \mathcal{D} = D$ as defined in (1.1) and $\mathbf{v}_i = D^i\mathbf{v}_0$. Then $\mathbf{V}_4 = \langle \mathbf{v}_0, \cdots, \mathbf{v}_i, \cdots \rangle$ is a Lowest Weight Module.

Example 6 (Continuing Example 2, the vector case)**.** We take $\mathbf{v}_0 = u_1$. We see that $\mathbf{v}_0$ is a lowest weight vector that generates an example of $\mathbf{V}_2$ different from Example 4.

The following is a list of properties that hold for any representation of the Lie algebra $\mathfrak{sl}_2$. We will use these properties in the next section.

Lemma 1. *The following relations hold true:*

- $[\mathcal{E}, \mathcal{D}^n] = 2n\mathcal{D}^n, \quad [\mathcal{E}, \mathcal{F}^n] = -2n\mathcal{F}^n$;
- $[\mathcal{F}, \mathcal{D}^n] = n\mathcal{D}^{n-1}(\mathcal{E} + n - 1)$;
- $[\mathcal{F}^n, \mathcal{D}] = n\mathcal{F}^{n-1}(\mathcal{E} - n + 1)$.

If $\mathcal{E}p = \lambda p$, we denote λ by ω_p and we call it the *weight* or *eigenvalue* of p.

Definition 4 (depth)**.** Let $\mathcal{V}$ be an $\mathfrak{sl}_2$-module. For $p \in \mathcal{V}$ we define its **depth** $\mathfrak{d}(p)$ as the lowest $m \in \mathbb{N}$ such that $\mathcal{F}^m p \neq 0$, but $\mathcal{F}^{m+1}p = 0$. If there is no such m, we say that p is of infinite depth.

Definition 5 (nonresonant)**.** We say that $q \in \mathcal{V}$ is **nonresonant** if its weight $\omega_q \notin 0, -1, -2, \cdots$. Let $\mathcal{V}$ be an $\mathfrak{sl}_2$-module such that every element in $\mathcal{V}$ can be written as $\sum v^\iota$, where the v^ι are nonresonant $\mathcal{E}$-eigenvectors and the sum is finite. Then we say that $\mathcal{V}$ is a nonresonant $\mathfrak{sl}_2$-module.

Example 7. The Lowest Weight Module $\mathbf{V}_\lambda$ with $\lambda \notin 0, -1, -2, \cdots$ is nonresonant

Definition 6 (Casimir operator)**.** The **Casimir operator** $\mathcal{C}$ is defined by

$$\mathcal{C} = \mathcal{E}^2 - 2(\mathcal{D}\mathcal{F} + \mathcal{F}\mathcal{D}).$$

The Casimir operator is characterized by the property that it commutes with $\mathfrak{sl}_2$.

Notice that $\mathcal{C}$ acts on $\mathbf{V}_\lambda$ by multiplication with $\lambda(\lambda - 2)$.

Definition 7 (quasisimple)**.** An $\mathfrak{sl}_2$-module is called **quasisimple** if the Casimir operator acts on it as a multiple of the identity. (For an example of a non-quasisimple situation, see Example 12.)

The following theorems formulate the main results in this section. The theorems, and the comments that follow them, will describe under which circumstances one can conclude that the kernel of $\mathcal{F}$ is a complement to the image of $\mathcal{D}$. This is important since our basis of differential invariants will be generated by transvection and, according to this theorem, it will not include (in most cases) total derivatives of other differential invariants. The proof of the theorem also describes an algorithm to construct explicitly the splitting of an element into a component in the kernel of $\mathcal{F}$ and a component in the image of $\mathcal{D}$.

Theorem 1. *Let $\mathcal{V}$ be a nondegenerate $\mathfrak{sl}_2$-module. Suppose $p \in \mathcal{V}$ is an $\mathcal{E}$- and a $\mathcal{C}$-eigenvector, of finite depth but not a constant. (The $\mathcal{C}$-eigenvector condition ensures that the module generated by p is quasisimple.) Let $q = \mathcal{F}^{\mathfrak{d}(p)}p$ and assume q is nonresonant. Then p can be written uniquely (up to constants for the v^j with $j > 0$) as $p = p^{(0)} + \mathcal{D}\sum_{j=1}^{\mathfrak{d}(p)} \mathcal{D}^{j-1}p^{(j)}$, where $p^{(j)}$ are $\mathcal{E}$-eigenvectors in the kernel of $\mathcal{F}$ for all j. That is, $p - p^{(0)} \in \operatorname{im} \mathcal{D}$.*

Notice that that if q is nonresonant, then so is p. But if p is nonresonant, it does not follow that q is nonresonant.

Proof. Let Q be the module generated by $q = \mathcal{F}^{\mathfrak{d}(p)}p$. From the structure theorem of indecomposable, quasisimple, $\mathcal{E}$-multiplicity free $\mathfrak{sl}_2$-modules in [4, Theorem 1.1.13] we can deduce that, since Q does not contain a constant, it must be a Lowest Weight Module, say $\mathbf{V}_\lambda$ (where we have no a priori knowledge of the value of $\lambda \in \mathbb{C}$). Since $q \in \ker \mathcal{F}$, this implies that either $q = \mathbf{v}_0$ and $\lambda = \omega_q$ or $q = \mathbf{v}_{1-\lambda}$ and $\lambda = 2-\omega_q$. The last condition only makes sense if $\lambda = 1, 0, -1, -2, \cdots$, and coincides with the first case if $\lambda = 1$. So for this case to occur we assume $\lambda = 0, -1, -2, \cdots$, that is $\omega_q = 2, 3, 4, \cdots$.

If $q = \mathbf{v}_0$ then it is not in $\operatorname{im} \mathcal{D}$.

If $\omega_q \notin 2, 3, 4, \cdots$ we take $\lambda = \omega_q$ and $\sigma = 0$. Otherwise let $\lambda = 2 - \omega_q \leq 0$ and $\sigma = 2(\omega_q - 1) \geq 2$. We work in $\mathbf{V}_\lambda$. In $\mathbf{V}_\lambda$ we let $\mathbf{v}_\sigma = q$. Let

$$p^{(m)} = \frac{\Gamma(\lambda+\sigma)\mathbf{v}_\sigma}{(m+\sigma)!\Gamma(\lambda+\sigma+m)},$$

where $m = \mathfrak{d}(p)$. Notice that this is well defined since we assume q to be nonresonant. If we would not assume this, $\lambda + m$ might become zero. Then, since $\mathcal{F}\mathbf{v}_j = j(\lambda + j - 1)\mathbf{v}_{j-1}$, and $\mathcal{D}^m p^{(m)} = \frac{\Gamma(\lambda+\sigma)\mathbf{v}_{m+\sigma}}{(m+\sigma)!\Gamma(\lambda+\sigma+m)}$, we have $\mathcal{F}^m\mathcal{D}^m p^{(m)} = q$ and therefore

$$\mathcal{F}^m(p - \mathcal{D}^m p^{(m)}) = 0$$

and hence $\mathfrak{d}(p - \mathcal{D}^m p^{(m)}) < \mathfrak{d}(p)$. We can now replace p by $p - \mathcal{D}^m p^{(m)}$ and repeat the process. Iterative application of this method results in the proof of the existence of the splitting. The $p^{(m)}$ are in $\ker \mathcal{F}$ by construction.

Thus, we only need to prove uniqueness to conclude the proof of Theorem 1. If the depth is zero, uniqueness is obvious. If the depth is not zero, one can apply induction concentrating on the highest depth term. The difference of two different splitting choices would lie in the kernel of $\mathcal{D}$ (and in $\ker \mathcal{F}$). Therefore it is constant, that is, 0 under the $\mathfrak{sl}_2$-action by Assumption 1. □

Theorem 2. *If in addition to the assumptions in Theorem 1 we require $\mathcal{V}$ to be nonresonant, then $p^{(j)} \notin \operatorname{im} \mathcal{D}$ for $j = 0, \ldots, \mathfrak{d}(p)$.*

Notice that, since every element in $\mathcal{V}$ can be written as a finite sum of $\mathcal{E}$-eigenvectors, the decomposition shown in the Theorem also holds for any element in $\mathcal{V}$, with each $p^{(j)}$ being a finite sum of eigenvectors.

Proof. For $p^{(m)}$ to be in $\operatorname{im} \mathcal{D}$ we need $\lambda = 0, -1, -2, \cdots$. But this is excluded by the nonresonance condition. □

It is of some practical importance to be able to relax the nondegeneracy condition in Theorem 1. We illustrate here how to do this in a concrete example. It is not clear how this could be done abstractly.

Take, for example, $\frac{1}{2}u_2/u_1$, which goes to 1 under F. (This is in contradiction with the nondegeneracy condition.) Clearly there is no way back in the algorithm used to create the splitting since we obtain zero when we apply D. The $\mathfrak{sl}_2$-module is of the type $U(0,0)$ as defined in [4, Chapter II, section 1.2]. In this case, since the constant $F^m p$ happens to be the last element before zero, one can conclude that the element before that is (a multiple of) $\frac{\tau_1}{\tau}$, with $\tau \in \ker F$ (that is, a multiple of $\mathcal{D}\log(\tau) \in \operatorname{im} D$). We then proceed as before by defining $p^{(m)} = \frac{\log(\tau)}{2m!(m-1)!}F^m p$. Here $\log(\tau)$ is defined to be $D^{-1}\frac{\tau_1}{\tau}$, we can extend the differential module to include D^{-1} in the standard way. This makes the module into a standard $\mathbf{V}_0$ and one can use the theorem.

The following lemma justifies the correction factor.

Lemma 2.

$$F^m D^m \log(\tau) = 2m!(m-1)!.$$

Proof. We show this by induction on m. For $m = 1$ the statement is easy to verify. Then, using Lemma 1 we see that

$$\begin{aligned} F^{m+1}D^{m+1}\log(\tau) &= (m+1)F^m(E-m)D^m\log(\tau) \\ &= m(m+1)F^m D^m \log(\tau) \\ &= 2m!(m+1)!, \end{aligned}$$

and this proves the statement. □

Corollary 1. *If $r = F^m p \in \mathbb{R}$, then*

$$F^m\left(p - \frac{r}{2m!(m-1)!}D^m\log(\tau)\right) = 0,$$

and so one has $\mathfrak{d}(p - p^{(m)}) < \mathfrak{d}(p)$

As in the previous case, one can replace p by $p - p^(m)$ and repeat the process.

In a different concrete situation, the vector case, the analogous element $\frac{1}{2}\frac{u_2}{\|u_1\|}$ goes to $\frac{u_1}{\|u_1\|}$ under the action of $\tilde{F}$, which is in the kernel of $\tilde{F}$ and $\tilde{H}$ in the notation of Example 2, but not in the kernel of D. Therefore it does not create a problem.

The following are three examples that illustrate the practical application of the algorithm, one for which the method works, and two for which it does not.

Example 8. Let $p = \frac{u_4}{u_1}$. Then

$$Fp = 12\frac{u_3}{u_1}, \quad F^2 p = 72\frac{u_2}{u_1}, \quad F^3 p = 144.$$

So $\mathfrak{d}(p) = 3$ and $\omega_{F^3 p} = 0$. Then $v_3 = 6\log(u_1)$ and

$$\begin{aligned} Dv_3 &= 6\frac{u_2}{u_1}, \quad D^2 v_3 = 6\frac{u_3}{u_1} - 6\frac{u_2^2}{u_1^2} \\ D^3 v_3 &= 6\frac{u_4}{u_1} - 18\frac{u_2 u_3}{u_1^2} + 12\frac{u_2^3}{u_1^3} \end{aligned}$$

Let $q = p - D^3 v_3 = -5\frac{u_4}{u_1} + 18\frac{u_2 u_3}{u_1^2} - 12\frac{u_2^3}{u_1^3}$. Then

$$Fq = -24(\frac{u_3}{u_1} - \frac{3}{2}\frac{u_2^2}{u_1^2}), \quad F^2 q = 0.$$

so $\mathfrak{d}(q) = 1$ and $\omega_{Fq} = 4$, $v_1 = -6(\frac{u_3}{u_1} - \frac{3}{2}\frac{u_2^2}{u_1^2})$ and $Dv_1 = -6\frac{u_4}{u_1} + 24\frac{u_2 u_3}{u_1^2} - 18\frac{u_2^3}{u_1^3}$. Then $v_0 = q - Dv_1 = \frac{u_4}{u_1} - 6\frac{u_2 u_3}{u_1^2} + 6\frac{u_2^3}{u_1^3} \in \ker F$. So

$$p = \frac{u_4}{u_1} - 6\frac{u_2 u_3}{u_1^2} + 6\frac{u_2^3}{u_1^3} - 6D\left(\frac{u_3}{u_1} - \frac{3}{2}\frac{u_2^2}{u_1^2}\right) + 6D^3 \log(u_1).$$

Example 9. Take $p = \frac{1}{u_1}$. Then $Dp = -\frac{u_2}{u_1^2}$, $D^2 p = -\frac{u_3}{u_1^2} + 2\frac{u_2^2}{u_1^3}$, and $D^3 p = -\frac{u_4}{u_1^2} + 6\frac{u_2 u_3}{u_1^3} - 6\frac{u_2^3}{u_1^4} \in \ker F$. Thus if one starts with $\frac{u_5}{u_1^2}$, the algorithm in the proof of Theorem 1 would give a division by zero.

Example 10. Take $p = \frac{u_5}{u_1^2}$. It has a positive eigenvalue, so why doesn't Theorem 1 apply? This is because after applying F four times, one ends up with $\frac{1}{u_1}$, see the previous Example 9. This has a negative eigenvalue, so the module generated by p is resonant and Theorem 1 does not apply. Any correction factor $D^m p^{(m)}$ computed along the lines of the given algorithm, will vanish on the next step, so that the depth is not decreasing, and the algorithm fails. Notice that the module generated by the monomials in the terms of the orbit of p under $\mathfrak{sl}_2$ does not split into irreducible $\mathfrak{sl}_2$-modules, so this is a good example to study the abstract theory with.

The following follows from Theorem 1

Theorem 3. *Consider $p = P(u_1, \cdots, u_k)/u_1^m$, where P is a polynomial. Clearly, $\mathfrak{d}(p) < \infty$. If the minimal u-degree of the monomials in P is m, Theorem 1 applies as in Example 8, with the possible extension by a logarithmic term. If the degree is less than m it does not as in Example 9. If the minimal u-degree of the monomials in P is $\geq m+1$, Theorem 2 applies.*

Proof. If we require that the minimal u-degree of the polynomial P is $> m$, it follows that after applying F enough times on p, we end up with a polynomial. Once we have a polynomial, applying F will not change that fact, and so the final eigenvalue $\omega_{F^{\mathfrak{d}(p)}p}$ will be ≥ 2. If the u-degree of the polynomial P is equal to m, then we are in the situation of Lemma 2. $\square$

The following definition is nonstandard, see Comment 1.

Definition 8 (external transvectant)**.** Let $\mathcal{V}^1$ and $\mathcal{V}^2$ be $\mathfrak{sl}_2$-modules. We define the **external n-transvectant** $\tau^{(n)} : \mathcal{V}^1 \otimes \mathcal{V}^2 \to \mathcal{V}^1 \otimes \mathcal{V}^2$ as follows. Let $f \in \mathcal{V}^1$ and $g \in \mathcal{V}^2$ be eigenvectors of $\mathcal{E}$. We denote $f_i = \mathcal{D}^i f, g_i = \mathcal{D}^i g$. Then

$$\tau^{(m)} f \otimes g = \sum_{i+j=m} (-1)^i \binom{\omega_f + m - 1}{j} f_i \otimes \binom{\omega_g + m - 1}{i} g_j. \tag{2.3}$$

The definition is extended to the span of the eigenvectors by bilinearity.

(See Example 14.)

Lemma 3. *If f and g are both in $\ker\mathcal{F}$, then If f and g are both in $\ker\mathcal{F}$, then $\tau^{(m)} f\otimes g \in \ker\mathcal{F}$, where $\mathcal{F}(f\otimes g)=\mathcal{F}f\otimes g + f\otimes\mathcal{F}g$, as usual. Furthermore, for general f and g, $w_{(f,g)^{(m)}} = w_f + w_g + 2m$.*

Proof.

$$
\begin{aligned}
\mathcal{F}\tau^{(m)} f\otimes g &= \mathcal{F}\sum_{i+j=m}(-1)^i\binom{w_f+m-1}{j} f_i \otimes \binom{w_g+m-1}{i} g_j \\
&= \sum_{i+j=m}(-1)^i\binom{w_f+m-1}{j}(\mathcal{D}^i F + i\mathcal{D}^{i-1}(\mathcal{E}+i-1))f \otimes \binom{w_g+m-1}{i} g_j \\
&+ \sum_{i+j=m}(-1)^i\binom{w_f+m-1}{j} f_i \otimes \binom{w_g+m-1}{i}(\mathcal{D}^j\mathcal{F} + j\mathcal{D}^{j-1}(\mathcal{E}+j-1))g \\
&= \sum_{i+j=m}(-1)^i\binom{w_f+m-1}{j} i(w_f+i-1) f_{i-1} \otimes \binom{w_g+m-1}{i} g_j \\
&+ \sum_{i+j=m}(-1)^i\binom{w_f+m-1}{j} f_i \otimes \binom{w_g+m-1}{i} j(w_g+j-1) g_{j-1} \\
&= \sum_{i+j=m}(-1)^i\binom{w_f+m-1}{j+1} i(j+1) f_{i-1} \otimes \binom{w_g+m-1}{i} g_j \\
&+ \sum_{i+j=m}(-1)^i\binom{w_f+m-1}{j} f_i \otimes \binom{w_g+m-1}{i+1} j(i+1) g_{j-1} \\
&= -\sum_{i+j=m-1}(-1)^i\binom{w_f+m-1}{j+1}(i+1)(j+1) f_i \otimes \binom{w_g+m-1}{i+1} g_j \\
&+ \sum_{i+j=m-1}(-1)^i\binom{w_f+m-1}{j+1} f_i \otimes \binom{w_g+m-1}{i+1}(j+1)(i+1) g_j \\
&= 0.
\end{aligned}
$$

The eigenvalue computation is similar but easier.

$$
\begin{aligned}
\mathcal{E}\tau^{(m)} f\otimes g &= \mathcal{E}\sum_{i+j=m}(-1)^i\binom{w_f+m-1}{j} f_i \otimes \binom{w_g+m-1}{i} g_j \\
&= \sum_{i+j=m}(-1)^i\binom{w_f+m-1}{j}(\mathcal{D}^i\mathcal{E} + 2i\mathcal{D}^i) f \otimes \binom{w_g+m-1}{i} g_j \\
&+ \sum_{i+j=m}(-1)^i\binom{w_f+m-1}{j} f_i \otimes \binom{w_g+m-1}{i}(\mathcal{D}^j\mathcal{E} + 2j\mathcal{D}^j) g \\
&= \sum_{i+j=m}(-1)^i\binom{w_f+m-1}{j}(w_f+2i) f_i \otimes \binom{w_g+m-1}{i} g_j \\
&+ \sum_{i+j=m}(-1)^i\binom{w_f+m-1}{j} f_i \otimes \binom{w_g+m-1}{i}(w_g+2j) g_j \\
&= (w_f+w_g+2m)\sum_{i+j=m}(-1)^i\binom{w_f+m-1}{j} f_i \otimes \binom{w_g+m-1}{i} g_j.
\end{aligned}
$$

Notice that for the last computation f and g need not be in $\ker \mathcal{F}$, see Theorem 4. $\square$

Definition 9 (C-transvectant)**.** Let $\mathcal{V}^1, \mathcal{V}^2, \mathcal{V}^3$ be $\mathfrak{sl}_2$-modules and $C \in \mathrm{Hom}_{\mathfrak{sl}_2}(\mathcal{V}^1 \otimes \mathcal{V}^2, \mathcal{V}^3)$. Define $(f,g)_C^{(n)} = C(\tau^n f \otimes g)$ as the n C**-transvectant** on $\mathcal{V}^1$.

If the choice of C is clear from the context or it has been stated, for simplicity we will refer to the n C-transvectant simply as $(f,g)^{(n)}$, the n-transvectant. Notice that the external n-transvectant is also an n-transvectant, take $\mathcal{V}^3 = \mathcal{V}^1 \otimes \mathcal{V}^2$ and for C the identity map of $\mathcal{V}^1 \otimes \mathcal{V}^2$ to itself.

Corollary 2. *If $f, g \in \ker \mathcal{F}$, then $(f,g)^{(n)} \in \ker \mathcal{F}$.*

Example 11. *If $\mathcal{V}$ is a ring, and $C : \mathcal{V} \otimes \mathcal{V} \to \mathcal{V}$ is given by $C(f \otimes g) = fg$, then Corollary 2 shows among other things that the product of two elements in $\ker \mathcal{F}$ is again in $\ker \mathcal{F}$, since $fg = C(\tau^0 f \otimes g)$.*

Comment 1. In the classical definition of transvectant, one uses the fact that the coefficients $a_0, \cdots, a_n$ of the groundform span the ring $P[a_0, \cdots, a_n]$. By using the ring multiplication, one defines a transvectant of two covariants in such a way that the result is again a covariant. There are now two observations to be made.

- One can restrict ones attention to the leading terms of covariants, corresponding to the elements in $\ker \mathcal{F}$.
- The multiplication is not necessary. One can see this as a tensor operation. This should be interpreted as follows. In the classical Clebsch-Gordan formula
$$V_n \otimes V_m = \bigoplus_{i=0}^{\min(m,n)} V_{m+n-2i}$$
(where V_n is the $(n+1$-dimensional irreducible representation of $\mathfrak{sl}_2$) the ith transvectant is the projection of $V_n \otimes V_m$ on a subset, which is isomorphic with V_{m+n-2i} (the isomorphism is the C in Definition 9). One can either read this as computing the leading term of a covariant by multiplication, or, using our definition of external transvectant, as a way to identify certain tensors within $V_n \otimes V_m$. Of course, this was implicit in the classical literature too: if one computes joint invariants, one does not write explicit tensor products, but one is careful to assign different symbols to different groundforms, as in $a_0 Y + a_1 X$ and $b_0 Y + b_1 X$, where the first transvectant is the familiar $a_0 b_1 - a_1 b_0$. We would write $a_0 \otimes b_1 - a_1 \otimes b_0$, so it is just a matter of notation. The usual contraction operator is replacing the tensor by ordinary commutative multiplication. At that point it matters a great deal whether we compute the first transvectant of two different groundforms, or the first transvectant of a form with itself, since in the latter case the result will be zero on contraction (in tensor terms, it is the symmetrization of an antisymmetric tensor).

Theorem 4. *Let $\mathbf{V}_\lambda$ and $\mathbf{V}_\mu$ be lowest weight $\mathfrak{sl}_2$-modules and assume $\lambda + \mu \notin \mathbb{Z}_-$. Then*
$$\mathbf{V}_\lambda \otimes \mathbf{V}_\mu \cong \bigoplus_{i=0}^{\infty} \mathbf{V}_{\lambda+\mu+2i},$$

where the projections are given by transvections, that is, by $(\cdot,\cdot)^{(i)} : \mathbf{V}_\lambda \otimes \mathbf{V}_\mu \to \mathbf{V}_{\lambda+\mu+2i}$. *If f is the lowest weight vector of $\mathbf{V}_\lambda$, and g of $\mathbf{V}_\mu$, then $(f,g)^{(i)}$ is the lowest weight vector of* $\mathbf{V}_{\lambda+\mu+2i}$.

Proof. See [4]. □

The following are the different results that will be used in our last two sections. They describe under which conditions the kernel of F is generated by transvection.

Proposition 1. *Let $p \in \ker \mathcal{F}$ be an $\mathcal{E}$-eigenvector in $\mathbf{V}_\lambda \otimes \mathbf{V}_\mu$ with $\mathcal{E}$-eigenvalue ν and with $\lambda + \mu \in \mathbb{N}$. Then p is a $\frac{1}{2}(\nu - \lambda - \mu)$-transvectant of elements in* $\ker \mathcal{F}$.

Proof. Through the relation in Theorem 4 we have that p corresponds to the generator of one of the $\mathbf{V}_{\lambda+\mu+2i}$, for some i, via transvection of an element in $\mathbf{V}_\lambda$ and an element in $\mathbf{V}_\mu$. Therefore $\nu = \lambda + \mu + 2i$ and we see that $\frac{1}{2}(\nu - \lambda - \mu) \in \mathbb{N}$. The result follows. □

Proposition 2. *Let $p(u_1, \cdots, u_k)$ be a polynomial of degree l in $\ker F$. Then p can be written as a the sum of repeated transvectants of* u_1.

Proof. Consider u_1 as the generator of the space $\mathbf{V}_2$. Then the polynomial is an element in the symmetrization of

$$\mathbf{V}_2^{\otimes l}.$$

Applying Theorem 4 repeatedly, we can express p as the sum of repeated transvectants of u_1. □

This result is of theoretical importance since it tells us that later on when we want to compute $\ker F$, it is enough to compute all transvectants. Once we know that, we can just compute transvectants, which is a lot easier than writing given expressions in terms of transvectants. See, however, Examples 14 and 15 for an illustration of the following proposition.

Proposition 3. *Consider $p = P(u_1, \cdots, u_k)/(u_1)^m \in \ker F$, where P is a polynomial. Clearly, $\mathfrak{d}(p) < \infty$. If the u-degree l of the polynomial P is $> m$, p can be written as a polynomial in transvectants of u_1 and $\frac{1}{u_1}$ (the generator of $\mathbf{V}_{-2}$).*

Proof. We consider p as an element of the symmetrization of

$$\mathbf{V}_{-2}^{\otimes m} \otimes \mathbf{V}_2^{\otimes l} = \bigoplus_{i=0}^{\infty} \mathbf{V}_{2l-2m+2i}.$$

Since $l > m$, one can always apply Theorem 4 such that the sums of eigenvalues is strictly positive. The proposition follows. □

Corollary 3. *If $l = m$ and the eigenvalue of p is strictly positive, then p has to be a repeated transvectant.*

Proof. If $l = m$, then

$$\mathbf{V}_{-2}^{\otimes l} \otimes \mathbf{V}_2^{\otimes l} \cong \mathbf{V}_{-2} \otimes \bigoplus_{i=0}^{\infty} \mathbf{V}_{2i+2} \cong (\mathbf{V}_{-2} \otimes \mathbf{V}_2) \oplus \bigoplus_{i=1}^{\infty} \mathbf{V}_{2i}.$$

This implies that if the eigenvalue of p is strictly positive, then p is a repeated transvectant. □

Notice that if the eigenvalue is zero we are in the situation $p \in \mathbb{R}$.

The equivalency does not always hold true. When $w_f + w_g = 0$, one has $\tau^0 f \otimes g = f \otimes g$ and $\tau^1 f \otimes g = w_f(f_0 \otimes g_1 + f_1 \otimes g_0) = w_f \mathcal{D}\tau f \otimes g$. This contradicts the picture

$$\mathbf{V}_{w_f} \otimes \mathbf{V}_{-w_f} \cong \bigoplus_{i=0}^{\infty} \mathbf{V}_{2i},$$

since the term generating $\mathbf{V}_2$ is already generated in $\mathbf{V}_0$. Therefore one cannot have the direct sum decomposition generalizing the Clebsch-Gordan formula to the infinite dimensional case as given by Theorem 4. This problem arises in general whenever one has $\mathbf{V}_\lambda \otimes \mathbf{V}_\mu$ with $\lambda + \mu$ an integer ≤ 0. Compare [4, Chapter 2, Exercise 6].

Example 12. Let $w_f = 1$ and $w_g = -1$. The Casimir operator $\mathcal{C} = \mathcal{E}^2 - 2(\mathcal{D}\mathcal{F} + \mathcal{F}\mathcal{D})$ has on the basis $\langle f_0 \otimes g_1, f_1 \otimes g_0 \rangle$ the matrix

$$\begin{bmatrix} 4 & -4 \\ -4 & 0 \end{bmatrix},$$

which is not a multiple of the identity. This means that the representation of $\mathfrak{sl}_2$ on $\mathbf{V}_1 \otimes \mathbf{V}_{-1}$ is not quasisimple.

3. A generating set of differential invariants created by transvection

Let $\mathcal{M}$ be a manifold of dimension n and let G be a group acting on $\mathcal{M}$. Let $J^{(k)}(\mathbb{R}, \mathcal{M})$ be the kth order jet bundle or set of equivalence classes of curves under the equivalence relation of kth order contact. If we introduce coordinates $u = (u^\alpha)$ on $\mathcal{M}$, x is the parameter and we denote by $u_r^\alpha = \frac{d^r u^\alpha}{dx^r}$, we can introduce coordinates in $J^{(k)}(\mathbb{R}, \mathcal{M})$ given by $(x, u^{(k)}) = (x, (u^\alpha), (u_1^\alpha), (u_2^\alpha), \ldots, (u_k^\alpha)) = (x, u, u_1, \ldots, u_k)$. Very often one works in the infinite jet $J^{(\infty)}(\mathbb{R}, \mathcal{M})$ as a way to avoid specifying at each step the highest order involved. See [11] for more details. We will work with parametrized curves, i.e., we are assuming that G leaves the parameter invariant (x is always an invariant, so it will usually be discounted from our list).

Definition 10 (prolonged action, (relative) differential invariant)**.** If a transformation group G acts on $\mathcal{M}$, the action preserves the order of contact between curves and so there is an induced action in $J^{(k)}(\mathbb{R}, \mathcal{M})$ called the **prolonged action** or kth prolongation. Since the parameter x is preserved, the prolonged action is locally given by

$$g \cdot (x, u^{(k)}) = g \cdot (x, u, u_1, \ldots, u_k) = (x, g \cdot u, (g \cdot u)_1, \ldots, (g \cdot u)_k).$$

where by $(g \cdot u)_k$ we denote the formula on u_r obtained after differentiating k times. A (kth order) **differential invariant** is a local scalar function $I : J^{(k)}(\mathbb{R}, \mathcal{M}) \to \mathbb{R}$ invariant under the prolonged action of G, that is, such that $I(g \cdot (x, u^{(k)})) = I(x, u^{(k)})$ for all $g \in G$.

A kth order **relative differential invariant with Jacobian weight** is a vector function $V : J^{(k)}(\mathbb{R}, \mathcal{M}) \to \mathbb{R}^n$ such that

$$V\left(g \cdot (x, u^{(k)})\right) = J_g(u) V(x, u^{(k)})$$

where J_g is the Jacobian matrix of the map $\phi_g : \mathcal{M} \to \mathcal{M}$ given by $\phi_g(u) = g \cdot u$ (see [11] for more details on the Jacobian multiplier representation).

Classical moving frames (that is, an invariant curve in the frame bundle over the curve u) and relative differential invariants with Jacobian weight are the same concept, the first one emphasizes geometric properties while the second emphasizes the invariance under the prolonged action (see [11]).

Given a group acting on a manifold, let

$$v = \xi \frac{\partial}{\partial x} + \sum_{\alpha=1}^{n} \phi^\alpha \frac{\partial}{\partial u^\alpha},$$

be a infinitesimal generator of the action, where $\xi = \xi(x, u)$ and $\phi^\alpha = \phi^\alpha(x, u)$. The infinitesimal generators of the prolonged action are well known. They are given by the *prolongation formula*

$$v^{(n)} = \xi \frac{\partial}{\partial x} + \sum_{\alpha=1}^{n} \sum_{k=0}^{\infty} \left(D^k(\phi^\alpha - \xi u_1^\alpha) + \xi u_{k+1}^\alpha \right) \frac{\partial}{\partial u_k^\alpha}, \tag{3.1}$$

where D is the total derivative with respect to the parameter x. Infinitesimally, a differential invariant is a local scalar function in the kernel of the infinitesimal generators of the prolonged action.

Definition 11 ($J^{(\infty)}(\mathbb{R}, \mathcal{M})$ as $\mathfrak{sl}_2$-module)**.** Consider the following operators defined on $J^{(\infty)}(\mathbb{R}, \mathcal{M})$:

$$\begin{aligned} F_\lambda &= \sum_{\alpha=1}^{n} \sum_{k=1}^{\infty} k(\lambda + k - 1) u_{k-1}^\alpha \frac{\partial}{\partial u_k^\alpha}, \quad \lambda \in \mathbb{C} \\ E_\lambda &= \sum_{\alpha=1}^{n} \sum_{k=0}^{\infty} (\lambda + 2k) u_k^\alpha \frac{\partial}{\partial u_k^\alpha} \\ D &= \sum_{\alpha=1}^{n} \sum_{k=0}^{\infty} u_{k+1}^\alpha \frac{\partial}{\partial u_k^\alpha}. \end{aligned} \tag{3.2}$$

In our geometric context, $\lambda = 0$ will be the choice that corresponds to the representation introduced in (1.1). In Theorem 5, we show that if the action leaves the parameter invariant these three operators (with $\lambda = 0$) will form an $\mathfrak{sl}_2$ representation in the space of differential invariants of parametrized curves. But other values of λ are relevant since they produce models of $\mathbf{V}_\lambda$: each u_0^α is the lowest weight vector for a $\mathbf{V}_\lambda$.

Since the definition of the operators corresponds to the definition of the Lowest Weight Module $\mathbf{V}_\lambda$, to check the following commutation relations is straightforward

$$[F_\lambda, D] = E_\lambda \qquad [E_\lambda, F_\lambda] = -2F_\lambda \qquad [E_\lambda, D] = 2D. \tag{3.3}$$

Recall that the space of differential invariants for curves on a manifold $\mathcal{M}$ of dimension n is generated by a set of n functionally independent differential invariants. If the order is minimal, we call such a set a *minimal basis*.

A straightforward calculation shows that

$$[D, v^{(n)}] = \xi_1 D.$$

So we assume ξ_1 to be zero in the sequel.

Theorem 5. *Assume v is a vector field on $\mathcal{M}$ given by*

$$v = \xi \frac{\partial}{\partial x} + \sum_{\alpha=1}^{n} \phi^\alpha \frac{\partial}{\partial u^\alpha} \tag{3.4}$$

then, if $\xi_1 = 0$, the commutator of F_λ and $v^{(n)}$ is given by

$$[F_\lambda, v^{(n)}] = \lambda \sum_{\alpha=1}^{n} \sum_{k=1}^{\infty} k D^{k-1} (u \frac{\partial \phi}{\partial u} - \phi) \frac{\partial}{\partial u_k^\alpha}.$$

We will call $F = F_0$ and $E = E_0$. In particular, F and $v^{(n)}$ commute. This singles out F as the *geometric* choice to complement D.

Proof. Indeed, straightforward calculation gives us

$$\begin{aligned} [F_\lambda, v^{(n)}] &= \sum_{\alpha=1}^{n} \sum_{k=0}^{\infty} \left(F_\lambda (D^k(\phi - \xi u_1^\alpha) + \xi u_{k+1}^\alpha) \right) \frac{\partial}{\partial u_k^\alpha} \\ &- \sum_{\alpha=1}^{n} \sum_{k=0}^{\infty} (\lambda + k)(k+1)(D^k(\phi^\alpha - \xi u_1^\alpha) + \xi u_{k+1}) \frac{\partial}{\partial u_{k+1}^\alpha}, \end{aligned}$$

where we used the fact that

$$[F_\lambda, \frac{\partial}{\partial u_s}] = -(s+1)(\lambda + s) \frac{\partial}{\partial u_{s+1}}$$

Using Lemma 1 this can be rewritten as

$$\begin{aligned} [F_\lambda, v^{(n)}] &= \sum_{\alpha=1}^{n} \sum_{k=0}^{\infty} \left(k D^{k-1} (\lambda u^\alpha \frac{\partial \phi}{\partial u^\alpha} + (k-1)\phi) \right) \frac{\partial}{\partial u_k^\alpha} \\ &- \sum_{\alpha=1}^{n} \sum_{k=0}^{\infty} D^k \phi (k+1)(\lambda + k) \frac{\partial}{\partial u_{k+1}^\alpha}. \end{aligned}$$

From here one obtains the expression in the theorem. □

These two formulas result in the following Corollary.

Corollary 4. *Assume that the action of G leaves the parameter x invariant. Then, the operators F, D and E take differential invariants to differential invariants.*

Notice that the fact that D and $v^{(n)}$ commute if x is left invariant by the action follows from invariant theory since D is the invariant differentiation. From now on we will assume the parameter x is invariant.

First of all, the following lemma.

Lemma 4. *Given $u : I \to G/H$ generic. Assume a minimal basis of differential invariants can be found as rational functions of u_k^α's. Then, there exists a basis of differential invariants of u formed by invariants which are eigenvectors of the operator E.*

Proof. Notice that, if the action of G is rational, a minimal basis of differential invariants can be found as rational functions of u_k, using the algebraic formulation of the moving frame method found in [5]. Notice also that such a basis can always be decomposed in terms that are eigenvectors of E.

Assume k is any differential invariant for u, and assume $k = \nu + \mu$ where ν and μ are eigenvectors of E, that is, $E(\nu) = \alpha\nu$ and $E(\mu) = \beta\mu$, $\alpha \neq \beta$. Since

E and $v^{(r)}$ commute for any r, we conclude that $v^{(r)}(\nu)$ and $v^{(r)}(\mu)$ will also be eigenvectors with the same eigenvalue as ν and μ. Since k is a differential invariant, $v^{(r)}(\nu+\mu) = 0$ and this implies that $v^{(r)}(\nu) = -v^{(r)}(\mu)$ and so

$$E(v^{(r)}(\nu)) = \alpha v^{(r)}(\nu) = -E(v^{(r)}(\mu)) = -\beta v^{(r)}(\mu) = \beta v^{(r)}(\nu).$$

If $\alpha \neq \beta$ we have $v^{(r)}(\nu) = v^{(r)}(\mu) = 0$. Both ν and μ are differential invariants. This is also clearly the case if instead of two terms we have any number of them.

Let $\{k^1, \dots, k^n\}$ be a minimal basis of differential invariants. Splitting these into eigenvectors, we can then produce a system of generators which are eigenvectors $\{\nu^1, \dots, \nu^m\}$. Since the dimension of the space of differential invariants is n, and, being minimal, the number of generators at each degree r is determined by the rank of the r prolonged action, following a standard procedure we can extract a subset of n elements from $\{\nu^1, \dots, \nu^m\}$ formed by independent and generating invariants. Notice that after splitting $k^1, \dots, k^n$ we will always have enough differential invariants at each order r to cover the needed number of generators, it suffices to choose the terms with the highest order, for example. The lemma follows. □

Using Theorem 2, Proposition 1 and Lemma 4, we obtain the following result.

Theorem 6. *Let $\mathcal{M} = G/H$ be a homogeneous space. Given a curve $u : I \to \mathcal{M}$, assume that we can find a generating set of differential invariants forming a nonresonant $\mathfrak{sl}_2$-module, and assume that each member of this generating set can also be found to be elements on the image of a contraction map defined on some $V_\lambda \otimes V_\mu$ with $\lambda + \mu \in \mathbb{N}$.*

Then, there exists a generating set for differential invariants of curves that lies in the kernel of F. This set is generated by transvection of elements in the kernel of F.

Proof. The proof of this theorem is the direct application of Theorem 2 and its consequences (modified for degeneracy as in the comments that follow its proof). Notice that in our case the Casimir $E^2 - 2(DF + FD) = 0$ and so any vector is its eigenvector. If we start with a basis $\{k^i\}$ of differential invariants, we can choose a basis which are also E-eigenvectors following Lemma 4. Therefore, without loss of generality we can assume they are eigenvectors with nonnegative eigenvalues. Therefore, applying Theorem 2, we can find a set $\{v^i_j\}$, all of them in the kernel of F, and such that

$$k^i = \sum_{k=0}^{\mathfrak{d}(k^i)} D^j v^i_j.$$

Since k^i are all differential invariants, the algorithm shown in the proof of Theorem 1 to produce v^i_j ensures that v^i_j are also differential invariants. Indeed, they are linear combinations of elements obtained after applying F and D repeatedly to k^i and these operators take differential invariants to differential invariants. Since $\{k^i\}$ generate all differential invariants, $\{v^i_j\}$ also do. Observe that since the v^i_j are computed using the elements of $\mathfrak{sl}_2$, they are part of the nonresonant $\mathfrak{sl}_2$-module, and therefore have nonnegative eigenvalues. This implies that if they lie in the contraction image of a tensor product $V_\lambda \otimes V_\mu$, one must have $\lambda + \mu \geq 0$. Finally, also due to the process followed to generate v^i_j, they are guaranteed to belong to tensor products of V_α factors. Applying Proposition 1 we obtain that v^i_j are generated by transvection of elements in the kernel of F. □

Notice that, according to Theorem 2 the weight of the invariants will tell us whether or not the invariant can be split into a transvectant component and an invariant in the image of D. In the examples that follow this section the component in the image of D vanishes. Notice also that, in general, one cannot guarantee that a minimal basis of differential invariants can be extracted from $\{v_k^i\}$. Indeed, if $p = \sum_{k=0}^{\mathfrak{d}(p)} D^k v_k$, the differential order of v_k could be higher than that of p.

For example, if $p = u_2^3$, the process in Theorem 1 results in $p = v_0 + Dv_1 + D^3 v_3$, where

$$v_0 = \frac{3}{5}\left(u_2^3 - u_1u_2u_3\right) + \frac{1}{10}u_1^2u_4, \qquad v_1 = -\frac{6}{35}\left(u_1^2u_3 - \frac{3}{2}u_1u_2^2\right), \qquad v_3 = \frac{1}{42}u_1^3.$$

In view of this example, if we apply the process in Theorem 1 to a basis of differential invariants one might generate sets of differential invariants that are not minimal in order, even though they generate all other invariants and are independent. They might have more elements that the minimal basis. As above, no linear combination of these might produce a minimal set. For that we would have to use functional combinations that might turn them into non-tramsvectants. This fact gives the impression that generating a minimal basis using transvection is not possible, it is possible only to generate large sets of generating invariants. But that is far from the real situation in specific cases, and in fact this situation does not happen in any of the known geometric cases, where lower order differential invariants are always in the kernel of F.

Example 13. Assume that $\mathcal{M} \cong O(2) \ltimes \mathbb{R}^2/O(2)$ is the Euclidean plane. A system of generators for the differential invariants of a planar Euclidean curve is given by $\|u_1\|^2 = u_1 \cdot u_1$ and the curvature $k = p_{22} - p_{12}^2$ where $p_{ij} = \frac{u_i \cdot u_j}{u_1 \cdot u_1}$. It is trivial to see that both $\|u_1\|^2$ and k are in the kernel of F.

Example 14. Assume that $\mathcal{M} \cong PSL(2)/H \cong \mathbb{RP}^1$. It is known that any differential invariant for curves in $\mathbb{RP}^1$ can be written as a function of the *Schwarzian derivative* of u,

$$S(u) = \frac{u_3}{u_1} - \frac{3}{2}\left(\frac{u_2}{u_1}\right)^2$$

and its derivatives with respect to x. It is also trivial to check that $S(u)$ is in the kernel of F. We compute, following the procedure in Corollary 3,

$$\begin{aligned} \tau^{(2)} u_1 \otimes \frac{1}{u_1} &= \sum_{i+j=2} \binom{3}{j} u_{i+1} \otimes D^j \frac{1}{u_1} \\ &= 3u_1 \otimes \left(2\frac{u_2^2}{u_1^3} - \frac{u_3}{u_1^2}\right) - 3u_2 \otimes \frac{u_2}{u_1^2} + u_3 \otimes \frac{1}{u_1} \end{aligned}$$

and this contracts by symmetrization to

$$3\frac{u_2^2}{u_1^2} - 2\frac{u_3}{u_1} = -2S(u).$$

Example 15. Alternatively,

$$\tau^{(2)} u_1 \otimes u_1 = 6u_3 \otimes u_1 - 9u_2 \otimes u_2, \quad \tau^{(0)} u_1 \otimes u_1 = u_1 \otimes u_1$$

and we can write

$$S(u) = \tfrac{1}{6}\frac{C\tau^{(2)} u_1 \otimes u_1}{C\tau^{(0)} u_1 \otimes u_1}$$

where $Cp\otimes q = p\cdot q$ is an example of a *contraction operator* (and thus $C\tau^{(0)}u_1\otimes u_1 = u_1^2$).

It follows that there is no unique way to see differential invariants as transvectants, and that one can try and make the choice optimal with respect to the type of general result one wants to prove.

Finally, the restriction of having a system of generating differential invariants that form a nonresonant module *and* are elements of tensor products might seem very restrictive. In reality it is not. Traditional ways of generating differential invariants for curves in many geometric manifolds include the use of bilinear forms or group invariant operators (like the determinant in the case of $\mathrm{SL}(n)$) and their applications to combination of derivatives of the curve. This method naturally produces systems of differential invariants which are in the image of contraction operators and form nonresonant modules. Even for non-affine geometries for which the use of these group invariant forms is not so apparent, it is still possible to effectively apply the method generating a basis of differential invariants directly by transvection of u_1 and basic invariants. In fact, it will be interesting to find a situation in which this method is not applicable.

In our next sections we will try to show how one can effectively use this representation to generate differential invariants by recurrent transvection of u_1 starting with a lowest weight invariant which is guaranteed to be generated by transvection. We will do first the simplest case, that of affine geometries. We will also show how the process can be minimally adapted to ensure the result is also invariant under reparametrization.

4. Differential invariant of curves in $(G \ltimes \mathbb{R}^n)/G$

Let $\mathcal{P}$ be the R-module of elements of the form

$$\sum P^i u_i$$

where the $P^i \in R$ are polynomials in the u_i^α, $i = 1, 2, \dots$, $\alpha = 1, \dots n$, divided by elements in the kernel of F. Notice that the elements of $\mathcal{P}$ are vector functions. Define F_v on $\mathcal{P}$ by applying F as in (3.2) to each entry of this vector to produce another vector. Likewise we define E_v and D_v by applying E and D to each entry. We can also extend these operators to tensor products of $\mathcal{P}$ using the standard product rule. We will denote the resulting operators with the same notation, F_v, E_v and D_v. Using Proposition 3 (and its obvious generalization to several dependent variables where one divides by inner products or other combinations of u_1^α) and its preceding comments, we know that under certain verifiable conditions the kernel of F_v is generated by transvectants of the form (2.3).

Assume next that we have a bilinear map (or *contraction*)

$$C : \mathcal{P}\otimes\mathcal{P} \to R$$

and assume that

$$X(C(f\otimes g)) = C(X_v(f\otimes g)), \quad \forall X \in \mathfrak{sl}_2, \tag{4.1}$$

that is, $C \in \mathrm{Hom}_{\mathfrak{sl}_2}(\mathcal{P}\otimes\mathcal{P}, R)$ (as in Definition 9).

Consider the special case of $\mathcal{M} \cong \mathbb{R}^n \cong (G \ltimes \mathbb{R}^n)/G$ with action given by $g\cdot u = Au + b$ for any $g = (A, b) \in G \ltimes \mathbb{R}^n$. These Klein geometries include Euclidean, symplectic, affine and equi-affine, for example. For these special geometries we will see how the $\mathfrak{sl}_2$ representation produced above will describe a method to find differential invariants by recursive transvection. Furthermore, we will show that by making appropriate choices the resulting differential invariants can also be made to be invariant under reparametrizations. We will use the first transvectant to generate all relative differential invariants of Jacobian weight (a classical moving frame) by recurrent transvection of one initial invariant with u_1. Differential invariants of classical groups were previously classified in [18], with no connection to transvectants.

The following are basic observations. The expression $\phi^* f$ represents the change of parameter x in f by a diffeomorphism ϕ. The proofs are straightforward.

Proposition 4. *If $\phi^* f = (\phi_1^p f) \circ \phi^{-1}$, then $Ef = 2pf$.*

Notice that the reciprocal statement is not true: take $f = u_2$.

Proposition 5. *Assume f and h are real functions defined on $J^{(k)}(\mathcal{M})$ and assume they hold $\phi^* f = (\phi_1^p f) \circ \phi^{-1}$, $\phi^* g = (\phi_1^r g) \circ \phi^{-1}$ for any change of x-variable $\phi : I \subset \mathbb{R} \to \mathbb{R}$.*

Then, $(f, g)^{(1)}$ is transformed according to the rule $\phi^*(f, g)^{(1)} = (\phi_1^{r+p+1}(f, g)^{(1)}) \circ \phi^{-1}$. Likewise, if $E(f) = \omega_f f$, then $\omega_{(f,g)^{(1)}} = \omega_f + \omega_g + 2$.

The basis for the generation process is the following observation.

Proposition 6. *Let k be a differential invariant, and let R be a relative differential invariant with Jacobian weight. Then $(k, R)^{(1)}$ is also a relative differential invariant with Jacobian weight.*

Proof. If k is a differential invariant and R is a relative invariant of Jacobian weight, then $v^{(n)}(k) = 0$ and $v^{(n)}(R) = J_v R$, where by $v^{(n)}(R)$ we mean the application of the prolongation vector to each one of the entries of R. Since the action is the composition of a linear action and a translation, J_v will be a constant matrix for any $v \in \mathfrak{g}$ (given in fact by the element in G generating v). Therefore, since D and the prolongation commute,

$$v^{(n)}((k, R)^{(1)}) = J_v (k, R)^{(1)}.$$

□

Consider now the case where G is a classical group . If a Lie group G is defined as the group preserving a bi-linear form,

$$G = \{g \in GL(n), \text{ such that } g^T J g = J\}$$

then we can define C to be $C(v \otimes w) = v^T J w \in R$. The one instance for which this does not exist is the equi-affine case $G = \mathrm{SL}(n)$. In this case we will need to use a slightly different approach that will be described separately.

We start by finding a differential invariant for the action. The lowest order differential invariants, let's call it k, will be in the kernel of F and hence it will be written in terms of transvectants. One can then analyze the lower order transvectants to search for a first invariant. The simplest one will be

$$k = \left[C\tau^0 u_1 \otimes u_1\right]^{\frac{1}{2}} = \left[u_1^T J u_1\right]^{\frac{1}{2}}.$$

This is indeed a nonzero differential for $G = \mathrm{SO}(p, q)$ and $E(k) = 2k$. On the other hand, this vanishes when $G = \mathrm{Sp}(2n)$. In that case we would choose

$$k = \left[C\tau^1 u_1 \otimes u_1\right]^{\frac{1}{3}} = \left[u_2^T J u_1\right]^{\frac{1}{3}}.$$

Both choices are in the kernel of F. In the case at hand there is a basic relative differential invariant of Jacobian weight, namely u_1, which trivially holds $E_v(u_1) = 2u_1$. We then define

$$V^2 = \tilde{C}\tau^1 k^r \otimes u_1$$

and choose r so that $E_v(V^2) = 2V^2$, namely $r = -1$. The contraction $\tilde{C}$ is defined by $\tilde{C}(k^r \otimes u_1) = k^r u_1$. We repeat the process and define V^3 to be given by

$$V^3 = \tilde{C}\tau^1 k^r \otimes V^2$$

where $r = -1$ is chosen so that $E_v(V^3) = 2V^3$.

And so on. If we assume that $u_1, \ldots, u_n$ are all independent for all x (we will then say that u is *nondegenerate*) this process generates relative invariants that are independent; if one wishes the system to be also independent under reparametrizations, then we merely need to divide the result by k and apply Proposition 4.

Theorem 7. *If $G = \mathrm{SO}(p,q)$ and u is nondegenerate, any differential invariant for curves in $(G \ltimes \mathbb{R}^n)/G$ can be written as a function of k, $k^i = C\tau^0 V^i \otimes V^i$, $i = 2, 3, \ldots, n$ and their derivatives. If $G = \mathrm{Sp}(n)$ the same holds for the choices of k, and $k^i = C\tau^0 V^i \otimes V^{i+1}$, $i = 2, \ldots, n+1$.*

Furthermore, in either case k and k^i lie in the kernel of the operator F and k and $\frac{k^i}{k}$ are additionally invariant under reparametrizations.

Proof. We can use standard differential invariant theory to conclude that for $G = \mathrm{SO}(p,q)$ one has r independent differential invariants at each one of the orders $r = 1, 2, \ldots, n$. For the case $G = \mathrm{Sp}(n)$, we have $r-1$ of order r for $r = 2, \ldots, n-1$. That means we have one first (or second for $\mathrm{Sp}(n)$) order invariant, the differential of this plus an extra invariant of order 2 (or 3 for $\mathrm{Sp}(n)$), the differential of these two plus an extra invariant at the next order and so on. Thus, to prove the Theorem it suffices to show that k, k^i, $i = 2, \ldots, n$ are all functionally independent. The other two properties (invariant under reparametrizations and in the kernel of F) are clearly true since they hold for each V^i.

Indeed, if $G = \mathrm{SO}(p,q)$, up to terms and factors depending of k and its derivatives, k^2 is equivalent to $u_2^T J u_2$. Up to terms and factors depending on k, k^2 and their derivatives, k^3 is equivalent to $u_3^T J u_3$ and, in general, up to the previous invariants and their derivatives, k^i is equivalent to $u_i^T J u_i$. If the curve is nondegenerate, $u_i^T J u_i$ are all functionally independent ([18]) and the proof in this case is concluded.

In the case $G = \mathrm{Sp}(n)$ a similar argument is valid. Up to k and derivatives of k, k^2 is equivalent to $u_2^T J u_3$, up to k, k^2 and their derivatives, k^3 is equivalent to $u_3^T J u_4$. In general, up to derivatives of k and k^s, $s \leq r$, k^r is equivalent to $u_r^T J u_{r+1}$. These are all independent in the non degenerate case ([18]), and so are the original differential invariants. □

In the case of $G = \mathrm{SL}(n,\mathbb{R})$ classical invariant theory tells us that there is one independent differential invariant of order n plus $n-1$ (other than the derivative of the n order one) independent ones of order $n+1$. These can be obtained as above, with some changes. As before, the key is to make an initial choice of differential invariant and relative invariant, but the contraction, since it needs to be invariant under the group, is now different.

Theorem 8. *If $G = \mathrm{SL}(n,\mathbb{R})$, then we can choose $k = \det(u_n, \ldots, u_1)^{\frac{1}{r}}$, with $r = \binom{n}{2}$ and produce V^i as above, $i = 1, \ldots n+1$ with $V^1 = u_1$. Then, a basis for the* *space of differential invariants of equi-affine curves is given by* k *and the differential invariants* $k^i = \det(V^{n+1}, V^n, \dots, V^{i+1}, V^{i-1}, \dots, V^1)$, $i = 1, 2, \dots, n-1$.

Note : notice that we are using here the contraction map

$$C : \hat{V}_0^{\otimes n} \to R = S(V_0^{\otimes n})$$

given by $C(v_1, \dots, v_n) = \det(v_1, \dots, v_n)$.

Proof. One can see that, up to factors of k, k^{n-1} is equivalent to

$$\det(V^{n+1}, V^n, u_{n-2}, \dots, u_1).$$

Therefore, again up to functions of k and its derivatives, k^{n-1} will be determined by $\det(u_{n+1}, u_n, u_{n-2}, \dots, u_1)$ and $\det(u_{n+1}, u_{n-1}, u_{n-2}, \dots, u_1)$. This last one is the derivative of k^r and so k^{n-1} is equivalent to

$$\nu^{n-1} = \det(u_{n+1}, u_n, u_{n-2}, \dots, u_1).$$

Next, one looks at k^{n-2} and uses the same reasoning to conclude that, up to factors and terms depending on k, k^{n-1} and their derivatives, k^{n-2} is equivalent to

$$\nu^{n-2} = \det(u_{n+1}, u_n, u_{n-1}, u_{n-3}, \dots, u_1).$$

A recursion argument shows that k^i will be equivalent to

$$\nu^i = \det(u_{n+1}, \dots, u_{i+1}, u_{i-1}, \dots, u_1).$$

It is well known that $\{k^r, \nu^1, \dots, \nu^{i-1}\}$ form a basis for the differential invariants of equi-affine curves, and hence the proof is concluded. □

Comment 2. From these cases one can appreciate certain things. It is in fact the differentiation that generates relative differential invariants - this was of course known -. Still, the transvection process allows us to generate invariants in the kernel of F, the complement to the image of D.

So, despite the fact that F has no geometrical meaning here, it is still very useful to project everything on a complement of (the trivial) im D. In this sense, the basis generated by transvection give natural representatives for first cohomology classes of the invariant variational bicomplex. See [6] for more information. ♠

We will next analyze some non-affine geometries for which, surprisingly, the same algorithm applies, including the role of differentiating and transvecting in the generation of relative and absolute differential invariants.

5. Differential invariants of curves in G/H, $G \subset \mathrm{GL}(n, \mathbb{R})$ semisimple

In this section we will analyze three manifolds, $\mathrm{PSL}(n, \mathbb{R})/H_1 \equiv \mathbb{RP}^n$, $O(n+1, 1)/H_2 \equiv M^n$ the Möbius sphere or local model for flat conformal manifolds, and $\mathrm{Sp}(n)/H_3 \equiv L^n$ the Grasmann Lagrangian. Each H_i is an appropriate isotropy subgroup of a distinguished point. In all cases we will describe how to write a basis of differential invariants for curves in M as a combination of transvectants. Unlike the previous cases, the action of the groups $\mathrm{PSL}(n, \mathbb{R})$, $O(n+1, 1)$ and $\mathrm{Sp}(n)$ on their respective manifolds are not linear as it affects second order frames. Still, we can use an analogous method to the one used in the linear case with proper choices of initial invariant and relative invariant. The main difference is that our relative invariants will not be vectors tangent to the manifold any longer. Several other cases of G/H, $G \subset \mathrm{GL}(n, \mathbb{R})$ semisimple ($G = O(2n, 2n)$ or $O(p+1, q+1)$

for example) follow one of these three models as it is explained in [10]. Their study would be identical to the ones presented here.

5.1. **Differential invariants for parametrized projective curves.** In this section we will show how one can find a generating system of differential invariants for projective $\mathbb{RP}^n$ expressed in terms of transvection.

Given a curve in $\mathbb{RP}^n$, it is known that a complete set of generators of the differential invariants of the curve is given by the so-called *Wilczynski* invariants ([17]). These are given by the formulas

$$k^m = \det(\mu_{n+1}, \mu_n, \dots, \mu_{m-1}, \mu_{m+1}, \dots, \mu)$$

where $\mu = \begin{pmatrix} u \\ 1 \end{pmatrix} W^{-\frac{1}{n+1}}$, $W = \det(u_n, \dots, u_1)$ and where $m = 0, 1, \dots, n-1$.

Proposition 7.

$$\begin{array}{rcl} F(k^m) & = & (m-n)(m+1)k^{m+1} \\ E(k^m) & = & (2(n+1-i)+n)k^m \\ F(W) & = & 0 \end{array}$$

The proof of the propositions is straightforward using the formulas in Lemma 1. It shows how $\{k^m\}$ and their derivatives generate V_λ for $\lambda = 3n+2$ and $v_0 = k^{n-1}$. The functional space generated by V_λ would be the space of differential invariants of the curve.

Wilczynski originally found the invariants k^m as coefficients of the unique $n+1$-st order scalar differential equation with zero n-th order term having each entry of μ as solution. Still, one can look at them as contractions of iterated differentiation of μ. According to previous Theorems we can always find a combination of Wilczynski's invariants generated by transvection. Instead of using the algorithm, we will use a method similar to the one used for equi-affine geometry to generate this basis. Indeed, consider the vector μ above.

Proposition 8. *The vector μ is a relative invariant for the projective action of* $\mathrm{PSL}(n+1, \mathbb{R})$. *Indeed, if $g \in \mathrm{PSL}(n+1, \mathbb{R})$ and we denote by $g^{(r)} \cdot u^{(r)}$ the prolonged projective action of that element on $J^{(r)}(\mathbb{R}, M)$, then*

$$g^{(r)} \cdot \mu = g\mu$$

where $g^{(r)} \cdot \mu$ indicates the application of the prolonged action to each entry of the vector μ and where $g\mu$ is the standard multiplication of matrices.

In fact, this property is an immediate consequence of μ being a column of a left-invariant moving frame for $\mathbb{RP}^n$, see [7]. Once we have a relative invariant with constant weight, we can apply Proposition 6 to generate several independent ones. As before, we will need a differential invariant to start the process. We choose the Wilczynski invariant we know to be in the kernel of F already, namely the one with lowest weight $k = k^{n-1}$. In the case $n = 2$ one cannot find, in general, a differential invariant that behaves as a one form, that is, such that $\phi^* k = \phi_1 k$. Indeed, if $n = 2$ there is only one generating differential invariant for the action of PSL(2) on $\mathbb{RP}^1$, namely the Schwarzian derivative of $u : \mathbb{R} \to \mathbb{RP}^1$. It is well known that

$$\phi^* S(u) = S(u) \circ \phi^{-1} + S(\phi) \circ \phi^{-1}$$

so that only fractional transformations will satisfy $S(\phi) = 0$ and will preserve $S(u)$. If one can find an invariant in the kernel of F behaving like a one form respect to changes of parameter, then the process can be repeated as in the linear case so ensure that the resulting generators are also invariant under reparametrization.

Let us choose $k = k^{n-1}$. Indeed,

$$k^{n-1} = \alpha(\frac{1}{W}, W)^{(2)} + \left(\frac{1}{W}, \det(u_{n+1}, u_n, u_{n-2}, \ldots, u_1)\right)^{(0)},$$

where $\alpha = \frac{1}{2}n\,(n(n+1)+1)!\Gamma\,(-n(n+1))$. Notice that k^{n-1} (and indeed all projective invariants in the kernel of F) *can be written as transvection of equi-affine differential invariants*, although not simply transvection of polynomials.

Define

$$V^i = \tau^1 k \otimes V^{i-1}$$

for $i = 1, \ldots, n+1$, where $V^0 = \mu$. The vectors V^i are all in the kernel of F so we need to use the group preserved contraction to generate invariants.

Theorem 9. *Consider the expressions*

$$\nu^i = \det(V^{n+1}, V^n, \ldots, V^{i+1}, V^{i-1}, \ldots, V^0)$$

for $i = 0, 1, \ldots, n-2$. Then $\{k, \nu^0, \ldots, \nu^{n-2}\}$ form a basis for the space of differential invariants of projective curves. Clearly, they all lie in the kernel of F.

Proof. One has $V^0 = \mu$ and hence, since V^1 is a combination of $\mu_1 = (V^0)_x$ and V^0 with coefficients depending on k and its derivatives, we can safely substitute V^1 by μ_1 in the definition of ν^i for the purposes of this proof, as far as V^0 is also present in the determinant. Likewise, for the purpose of the proof of this theorem, V^2 can be substituted by a multiple of $k\mu_2$ since the other terms appearing in V^2 are combinations of μ and μ_1 with coefficients depending on k and its derivatives. And so on. Therefore, ν^{n-2} is equivalent to

$$\det(V^{n+1}, V^n, V^{n-1}, \mu_{n-3}, \ldots, \mu)$$

in the sense that it is equal to a multiple of it with (nonzero) coefficients depending on k.

We can now write out this determinant to find it is a combination (again with coefficients depending on k and its derivatives) of $k^{n+1} = 1$, $k^n = 0$, $k^{n-1} = k$ and k^{n-2}, the first two Wilczynski invariants. Very clearly the coefficient of k^{n-2} is a nonzero function of k and hence we can conclude that $\{k, \nu^{k-2}\}$ generates the same subspace of differential invariants as $\{k^{n-1}, k^{n-2}\}$. Furthermore they are also functionally independent since $\{k^{n-1}, k^{n-2}\}$ are.

If we now look at ν^{n-3} we see it is equivalent to analyzing the generating and independence properties of k, ν^{n-2} and

$$\det(V^{n+1}, V^n, V^{n-1}, V^{n-2}, \mu_{n-4}, \ldots, \mu).$$

As before, this determinant can be written as a combination of $k^{n+1} = 1$, $k^n = 0$, k^{n-1}, k^{n-2} and k^{n-3}, with coefficients depending on k and its derivatives. The coefficient of k^{n-3} is given by a nonzero function of k. Hence, one can conclude that $\{k, \nu^{n-2}, \nu^{n-3}\}$ generates the same subspace of differential invariants as $\{k^{n-1}, k^{n-2}, k^{n-3}\}$. They are also functionally independent since the Wilczynski invariants are.

Iteration of the argument proves the theorem. □

5.2. **Differential invariants of curves in conformally flat Möbius sphere.** In this subsection we will write the invariants of curves in the Möbius sphere as contraction of successive transvections of u_1. The Möbius sphere is the local model for flat conformal manifolds and can be identified with $O(n+1,1)/H$ where H is given by matrices of the form

$$\begin{pmatrix} \alpha & 0 & 0 \\ v & A & 0 \\ \beta & w^T & \gamma \end{pmatrix} \in O(n+1,1)$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ and $v, w \in \mathbb{R}^n$. The quotient $O(n+1,1)/H$ can be locally identified with the section defined by matrices of the form

$$\begin{pmatrix} 1 & u^T & \frac{1}{2}u \cdot u \\ 0 & I & u \\ 0 & 0 & 1 \end{pmatrix}$$

which can itself be identified with $u \in M^n$, the n-dimensional flat conformal sphere. A complete set of differential invariants for conformal curves was originally found in [3] and later related to the group based picture in [8]. Here we will follow the procedure followed by Wilczynski in the projective case to write a basis of invariants as contraction of successive transvections of u_1.

As in the projective case we will first find a relative invariant for the quadratic action of $O(n+1,1)$ and a first differential invariant. We will then produce a family of relative invariants using transvection and will use a group preserved contraction to generate invariants.

Consider

$$\mu = \frac{1}{\ell}\begin{pmatrix} \frac{1}{2}u \cdot u \\ u \\ 1 \end{pmatrix}$$

where $\ell = (u_1 \cdot u_1)^{1/2}$. As it was the case with $\mathbb{RP}^n$, one can check that μ is a relative invariant for the quadratic action of $O(n+1,1)$ induced on $O(n+1,n)/H$ via left multiplication. That is

$$g^{(n)} \cdot \mu = g\mu$$

where by $g^{(n)}$ we denote the prolonged action of the element $g \in G$ on the n-jet space, and where $g\mu$ denotes multiplication of matrices. As before, μ is a column in a group based moving frame on the Möbius sphere and that guarantees the property. See [8].

With this setting we will be able to apply Proposition 6 to generate relative invariants by recursion. They will all be in the kernel of the operators F and hence any invariant obtained by contracting the relative invariants will also be in the kernel of F. Besides, if we choose the first invariant to be the conformal arc length (that is, so that $\phi^* k = \phi_1 k$ for some change of variable ϕ) then we can generate the relative invariants and their associated invariants so that they are invariant under reparametrizations also. We merely need to choose the proper power of k at each transvection, as it was done in the linear case.

One of the conformal differential invariants is known to be given by

$$\hat{I} = p_{33} - 6p_{12}p_{23} - p_{13}^2 + 6p_{12}^2 p_{13} + 9p_{12}^2 p_{22} - 9p_{12}^4$$

where, as before, $p_{ij} = \frac{u_i \cdot u_j}{u_1 \cdot u_1}$. This is not the third order invariant with lowest weight ($\omega_{\hat{I}} = 8$). Indeed,

$$I = p_{13} + \frac{3}{2}p_{22} - 3p_{12}^2$$

is also a conformal differential invariant with lower weight ($\omega_I = 4$). Both these invariants lie in the kernel of F and hence they both can be chosen as initial invariant, depending on the purpose of the computation. Notice also that $\phi^*\hat{I} = \phi_1^4\hat{I}$, so we can use $\hat{I}^{1/4}$ as element of arc-length.

Consider the contraction

$$\hat{C} : \mathbb{R}^n \otimes \mathbb{R}^n \to \mathbb{R}$$

given by the dot product and use it to define the transvectant as in Definition 9. Then, one can see that

$$I = \frac{1}{\ell^2}(u_1, u_1)^{(2)} + 3(\tau^1\frac{1}{\ell} \otimes u_1, \tau^1\frac{1}{\ell} \otimes u_1)^{(0)}$$

and

$$\frac{1}{9}\hat{I} + \frac{3}{2}I^2 = (\tau^2\frac{1}{\ell} \otimes u_1, \tau^2\frac{1}{\ell} \otimes u_1)^{(0)}.$$

Notice that, unlike the previous affine cases, the second transvection is necessary to generate differential invariants. Both I and $\hat{I}$ are also generated following the systematic process used in the previous section, which uses a different contraction. Consider the $O(n+1,1)$-invariant contraction

$$C : \mathbb{R}^{n+2} \otimes \mathbb{R}^{n+2} \to \mathbb{R}$$

defined by $C(v,w) = v^T Jw$, where J is given by

$$J = \begin{pmatrix} 0 & 0 & -1 \\ 0 & I & 0 \\ -1 & 0 & 0 \end{pmatrix} \in O(n+1,1),$$

and use it to define the transvectant as in Definition 9.

Theorem 10. *Let μ be given as above and let $k = I$ or $\hat{I}$. Define*

$$V^i = \tau^1 k \otimes V^{i-1}$$

for $i = 1, \ldots, n+1$, where $V^0 = \mu$. Define $k_i = (V^i)^T JV^i = (V^i, V^i)^{(0)}$. Then either $I, k^3, k^4, \ldots, k^{n+1}$ or $\hat{I}, k^2, k^4, \ldots, k^{n+1}$, depending on the initial choice, are basis for the space of differential invariants of conformal curves. They all lie in the kernel of the operator F.

Notice that the contraction used in this theorem is different from the one used above to define I and $\hat{I}$ as transvection. In fact, as we already saw and we will see also in the next subsection, there is not a unique way to express invariants as contractions of transvections. Along the proof of this theorem we will show how I and $\hat{I}$ can be also generated by transvection using the group-invariant contraction.

Proof. Let us look at the first few values of k^i. It is known that $\mu^T J\mu = 0$. In fact, the vector used to define μ is determined by that property and having the last entry equals 1. Hence $\mu_1^T J\mu = 0$ and so $(V^1)^T JV^0 = (V^0)^T JV^0 = 0$. On the other hand, it is straightforward to show that $\mu_1^T J\mu_1 = 1$ and so $(V^1)^T JV^1$ does not add new generators to the original one we start with. Neither does $\mu_2^T J\mu_1$ since it is the derivative of the previous one.

Hence, let us look at $(V^2)^T J V^2$ and, among its terms, at $k^2 \mu_2^T J \mu_2$, the only term that, in principle, might not depend on k and its derivatives. We have

$$\mu_2^T J \mu_2 = \left(\frac{u}{\ell}\right)_2 \cdot \left(\frac{u}{\ell}\right)_2 - \left(\frac{1}{\ell}\right)_2 \left(\frac{u \cdot u}{\ell}\right)_2 = 2I.$$

Hence, if we start with the conformal arc length, k^2 generates I. If we start with I, k^2 does not add to the set of independent generators. As before $\mu_3^T J \mu_i$, $i = 0, 1, 2$, can be obtained from previous ones by differentiation and so will be $(V^3)^T J V^i$, $i = 0, 1, 2$. Hence, we next look at k^3 and, in particular, at $k^3 \mu_3^T J \mu_3$ as the only term that might not depend on I (or $I, \hat{I}$ if that was the choice). One can check that

$$\mu_3^T J \mu_3 = \left(\frac{u}{\ell}\right)_3 \cdot \left(\frac{u}{\ell}\right)_3 - \left(\frac{1}{\ell}\right)_3 \left(\frac{u \cdot u}{\ell}\right)_3 = \hat{I} + 4I^2.$$

Hence, if we start with I, k^3 will add $\hat{I}$ to the list. If we start with $\hat{I}$, k^2 adds I and k^3 does not add anything.

We now look at the other products. The relevant term in $(V^r)^T J V^r$ is given by $k^2 \mu_r^T J \mu_r$ where

$$\mu_r^T J \mu_r = \left(\frac{u}{\ell}\right)_r \cdot \left(\frac{u}{\ell}\right)_r - \left(\frac{1}{\ell}\right)_r \left(\frac{u \cdot u}{\ell}\right)_r.$$

One can see that this expression contains no terms on u (that is with zero derivatives) and hence it is a homogeneous polynomial on p_{ij}. Observe that the linear part of the polynomial is given by the terms with no derivative of $\frac{1}{\ell}$ involved. That is, the linear part is p_{rr}. Also, k^r has E-eigenvalue $4(r-1)$. In [8] the author found a set of generators $\{I_r\}$, for the conformal differential invariants of curves, homogeneous polynomials on p_{ij}. The linear part of I_r was given by $p_{r+1,r+1}$ for $r = 3, \dots n$, while I and $\hat{I}$ where as given in this paper. The eigenvalue of I_r coincided with that of k^{r-1}. Given that $\{k^i\}$ are all differential invariants and hence generated by $\{I_r\}$, one sees that the set of differential invariants $\{k^r\}$, $r = 4, \dots, n+1$, together with I and $\hat{I}$, generates the same space as the set $\{I_r\}$, $r = 1, \dots, n$ in [8]. The theorem follows. $\square$

5.3. **Differential invariants of curves of Lagrangian planes.** Consider the manifold $\mathrm{Sp}(n)/H \equiv L^n$ identified with the space of Lagrangian planes in $\mathbb{R}^{2n}$. The set of differential invariants of curves in this manifold under the action of $\mathrm{Sp}(n)$ was classified in [9]. Some of these invariants are projectively invariant and had been previously found in [13]. A basis for the differential invariants can be described as follows. For more details see [9].

A Lagrangian plane in $\mathbb{R}^{2n}$ can be identified with a symmetric matrix and, in fact, the quotient $\mathrm{Sp}(n)/H$ can be locally identified with matrices of the form

$$\begin{pmatrix} I & u \\ 0 & I \end{pmatrix}$$

where u is $n \times n$ and symmetric and where I represents the unit $n \times n$ matrix. The subgroup H can thus be represented by matrices of the form

$$\begin{pmatrix} I & 0 \\ S & I \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g^{-T} \end{pmatrix}$$

with $g \in \mathrm{GL}(n, \mathbb{R})$ and S symmetric. With this description, it is known that a basis for the space of differential invariants of Lagrangian curves is given by the eigenvalues of the *Lagrangian Schwarzian* derivative of u

$$S(u) = u_1^{-1/2}\left(u_3 - \frac{3}{2}u_2u_1^{-1}u_2\right)u_1^{-1/2}$$

together with the off-diagonal entries of the matrix of differential invariants

$$I = u_1^{-1/2}\left(u_4 - 2u_3u_1^{-1}u_2 - 2u_2u_1^{-1}u_3 + 3u_2u_1^{-1}u_2u_1^{-1}u_2\right)u_1^{-1/2}$$

It was shown in [9] that I contains in its diagonal the derivative of the eigenvalues of $S(u)$. The Lagrangian Schwarzian was first defined in [13] as generating projective differential invariants for Lagrangian planes. Recall that $u_1^{1/2}$ is defined as a certain canonical form of matrices B such that $BB^T = u_1$ whenever u_1 is symmetric and positive definite. Such matrix B can be found in terms of the matrix used to diagonalize u_1 and its eigenvalues. The entries of such a matrix are functions of the entries of u_1 and, therefore, they belong to the kernel of F. We can then conclude that $F_v(u_1^{1/2}) = 0$.

We now proceed as before. One can show that, if $g = \Theta u_1^{-1/2}$, where $\Theta \in O(n)$ is such that $\Theta S(u)\Theta^T$ is diagonal (uniquely determined by Gram-Schmidt), then

$$\mu = \begin{pmatrix} u \\ I \end{pmatrix} g^T$$

is a relative invariant for the action of $\mathrm{Sp}(n)$ on the space of Lagrangian planes. In fact, $F_v(g) = 0$ so that μ is in the kernel of F_v. This fact will become obvious later on. We will denote by $\mathbf{D} = \Theta S(u)\Theta^T$ the diagonalization of $S(u)$. These will be our initial differential invariants. It is shown below that they, indeed, are generated by a certain contraction of a transvectant of u_1.

Let us consider the group-invariant contraction

$$C : M_{2n\times n} \otimes M_{2n\times n} \to M_{n\times n}$$

given by $C(v, w) = v^T Jw$, where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

and use it to define the transvectant as in Definition 9.

Theorem 11. *Let us call $V^0 = \mu$ and define $V^{i+1} = (\mathbf{D}, V^i)^{(1)}$, $i = 1, 2, \dots$. Then, the entries of $\mathbf{D}$ and $(V^2, V^0)^{(0)} = (V^2)^T JV^0$ generate all differential invariants for curves of Lagrangian planes under the action of the symplectic group.*

Proof. Let's choose $\mathbf{D}$ as our initial invariant. We could choose a power of $\mathbf{D}$ instead of $\mathbf{D}$ but, as it was in the projective case, $\mathbf{D}$ does not behave as a one form with respect to changes of variables, so there is no advantage on doing that.

It is very clear that $(V^i)^T JV^i = 0$ and also that

$$\begin{pmatrix} u_r & 0 \end{pmatrix} J \begin{pmatrix} u_s \\ 0 \end{pmatrix} = 0.$$

Using this when calculating $(V^j)^T JV^i$ will simplify our calculations enormously. Indeed, the first nonzero contraction will be, for some appropriate non zero constant β

$$(V^1)^T JV^0 = \beta\mathbf{D}\mu_1^T JV^0 = \beta\mathbf{D}g\begin{pmatrix} u_1 & 0 \end{pmatrix} J \begin{pmatrix} u \\ I \end{pmatrix} g^T = \beta\mathbf{D}gu_1g^T = \beta\mathbf{D}.$$

We then calculate the next nonzero differential invariants, namely the contraction of V^2 and V^0. As before V^2 will involve μ, μ_1 and μ_2. The first two will not generate anything new when contracted with V^0 since they will result in combinations of functions depending on $\mathbf{D}$ and its derivatives with previous results, which also depend on $\mathbf{D}$ and its derivatives. The new terms will come from a multiple of

$$\mathbf{D}\mu_2^T JV^0 = \mathbf{D}\begin{pmatrix} gu_2 + 2g_x u_1 & 0 \end{pmatrix} J \begin{pmatrix} u \\ I \end{pmatrix} g^T = \mathbf{D}\left(gu_2 g^T + 2g_x g^{-1}\right).$$

It was shown in [9] that the off-diagonal entries of $gu_2g^T + 2g_xg^{-1}$, together with the entries in $\mathbf{D}$, form a basis for the differential invariants for curve of Lagrangian planes under the action of the symplectic group. $\square$

There is not a unique way of writing these differential invariants as transvection of u_1. Although the method presented in this paper is systematic and common for all cases, other ways also produce the generating invariants using different transvections and contractions. The following theorem illustrates this fact.

Theorem 12. *The eigenvalues of $S(u)$ can be written as functions of the second transvection of entries of u_1. Furthermore, the eigenvalues of $S(u)$ together with the off-diagonal entries of $\left(12\tilde{C}\tau^1(\tau^2u_1 \otimes u_1) \otimes u_1 + \tau^2(\tau^1 u_1 \otimes u_1) \otimes u_1\right)$ for a certain contraction map $\tilde{C}$, form a basis of differential invariants for Lagragian curves under the action of the symplectic group.*

Proof. First of all, we will define the following contraction map

$$\tilde{C} : V_0^{\otimes m} \to V$$

$$\tilde{C}(u_{i_1} \otimes \cdots \otimes u_{i_m}) = u_1^{-1/2} u_{i_1} u_1^{-1/2} \ldots u_1^{-1/2} u_{i_m} u_1^{-1/2}.$$

Clearly $F\tilde{C} = \tilde{C}F_v$ as before and so we can generate differential invariants by transvecting at the Lagrangian planes level and contracting the result. With that in mind we can write

$$S(u) = 3\tilde{C}\left(\tau^2 u_1 \otimes u_1\right)$$

which explains why $S(u)$ is in the kernel of the operator F. The eigenvalues of this matrix can be written as

$$\mathbf{D} = \Theta S(u)\Theta^T$$

where $\Theta \in O(n)$ is the diagonalizing matrix. Since this is obtained using the Gram-Schmidt process, its entries are functions of the entries of $S(u)$, themselves combinations of second transvections of the entries of u_1. That is, $\mathbf{D}$ is written as a function of second transvections of different entries of u_1, and so is Θ.

Finally, it is a direct computation to show that I as above can be written as

$$I = 2(7!)\tilde{C}\left(12\tau^1(\tau^2u_1 \otimes u_1) \otimes u_1 + \tau^2(\tau^1u_1 \otimes u_1) \otimes u_1\right) + 3D\tilde{C}\left(\tau^2u_1 \otimes u_1\right)$$

and so we can ignore the last differentiation to conclude that the off diagonal entries of

$$\tilde{C}\left(12\tau^1(\tau^2u_1 \otimes u_1) \otimes u_1 + \tau^2(\tau^1u_1 \otimes u_1) \otimes u_1\right)$$

complete our basis of differential invariants. $\square$

6. Concluding remarks

We have shown that transvectants provide us with a natural language in which to express differential invariants. The method is surprisingly simple and its use is particularly convenient when we seek to identify generators that do not include the differential of lower order differential invariants. Although the splitting given in Theorem 1 does not imply that the orders of v^j's are lower than that of p, it is so when proper procedures are adopted. In fact, it would be very interesting to know if we can expect this to be always the case for differential invariants of parametrized curves. Although we have not covered the general homogeneous manifold case, all best-known geometries are included in the examples developed in this paper.

Obvious generalizations of this study of differential invariants of curves would involve differential invariants of surfaces, and it would seem that this could lead to a systematic treatment of higher order submanifolds. It would be very interesting to see how transvectants can be used to identify syzygies, or algebraic relations among generating differential invariants of surfaces (for example, Codazzi-Mainardi equations in Euclidean geometry). The F-operator theory can also be easily adapted to higher dimensional situations involving differential forms, see [16].

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