HAMILTONIAN STRUCTURE OF THE REVERSIBLE NONSEMISIMPLE 1:1 RESONANCE

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ABSTRACT. We show that a reversible non-Hamiltonian vector field at nonsemisimple 1:1 resonance can be split into a Hamiltonian and a non-Hamiltonian part in such a way that after reduction to the orbit space for the S^1-action coming from the semisimple part of the linearized vector field the non-Hamiltonian part vanishes. As a consequence the reduced reversible vector field is Hamiltonian. We furthermore show that for vector fields in normal form on the orbit space being Hamiltonian is equivalent to being reversible.

1. Introduction

We consider a reversible - but non-Hamiltonian - vector field in the neighborhood of a 1:1 resonance. There has been a recent interest in the reversible nonsemisimple 1:1 resonance (See [5], [10], [6], [3] and other references given in this last paper) which occurs when two pairs of purely imaginary eigenvalues of the linearized system collide on the imaginary axis. From the analysis of the bifurcation in [6] it becomes clear that there is an apparent resemblance with the Hamiltonian Hopf bifurcation [12]. Part of the resemblance has been clarified in [3] where it was shown that, on the orbit space with respect to the S^1-action coming from the semisimple part of the linearized vector field, the reduced vector field was Hamiltonian with respect to a "hidden" Poissonstructure.

In [5, eqn. (46)] the equations of interest are given in complex form by

$$\begin{aligned} \frac{dA}{dt} &= i(\omega_0 + \hat{P}(u, \Omega, \mu))A + B, \\ \frac{dB}{dt} &= i(\omega_0 + \hat{P}(u, \Omega, \mu))B + \hat{Q}(u, \Omega, \mu))A, \end{aligned} \tag{1}$$

where $(A, B) \in C^2$, $u = |A|^2$, $\Omega =\mathrm{Im}(\bar{A}B)$. $\hat{P}$ and $\hat{Q}$ are real polynomials in u and Ω with μ-dependent coefficients, such that $\hat{P}(0, 0, 0) = \hat{Q}(0, 0, 0) = 0$.

Note that at $\mu = 0$ the linearized system has a pair of purely imaginary eigenvalues $\pm i\omega_0$ with two dimensional Jordan blocks. The reversibility is present in the symmetry $\tilde{R}$ defined by $\tilde{R}(A, B) = (\bar{A}, -\bar{B})$ which anticommutes with the vector field. The vector field is equivariant with respect to the S^1-action $\theta \cdot (A, B) = (e^{i\theta} A, e^{i\theta} B)$, which is the flow of the semisimple part of the linearized vector field. Furthermore the normalized vector field has two absolute invariants:

$$\Omega, H = -\frac{1}{2}|B|^2 + \frac{1}{2}\int_0^u \hat{Q}(s, \Omega, \mu)ds \tag{2}$$

In order to compare these equations to other results we will first change to real coordinates. Set $A = y_1 + iy_2$ and $B = x_1 + ix_2$. Then we obtain the equations

$$\begin{aligned} \frac{dx}{dt} &= \omega_0 Jx + \hat{P}(u, \Omega, \mu)Jx + \hat{Q}(u, \Omega, \mu)y, \\ \frac{dy}{dt} &= \omega_0 Jy + \hat{P}(u, \Omega, \mu)Jy + x, \end{aligned} \tag{3}$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, u = y_1^2 + y_2^2, \Omega = x_1 y_2 - x_2 y_1. \tag{4}$$

Note that the *reversor* we consider is now given by

$$R = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{5}$$

In the following sections we will first show that equations (3) are actually the general normal form for reversible systems on $\mathbb{R}^4$ at nonsemisimple 1:1 resonance. After that we will show that the Poissonstructure on the orbit space is natural and that for vector fields in normal form on the orbit space being Hamiltonian is equivalent to being reversible, if the appropriate choices are made for the *reversor* and the Poisson structure. We furthermore show that the Hamiltonian structure can be identified explicitly in the original non-Hamiltonian reversible vector field. For the basics on Hamiltonian systems resp. Poisson structures we refer the reader to the textbooks [1] and [7].

2. Linear normal form

In this section we consider the linear normal form of a reversible system at non-semisimple 1:1 resonance. We show that the linear system and the *reversor* can be simultaneously normalized. A more general version of this can be found in [9].

Consider a system

$$\dot{z} = f(z, \lambda),\ z \in \mathbb{R}^4,\ \lambda \in \mathbb{R}^p, \tag{6}$$

where $f(0, \lambda) = 0$, $\forall \lambda$. Define $A(\lambda) = D_z f(0, \lambda) \in \mathcal{L}(\mathbb{R}^4)$. We call system (6) reversible if

$$f(Rz, \lambda) = -Rf(z, \lambda),\ R \in \mathcal{L}(\mathbb{R}^4),\ R^2 = I.$$

Let $A_0 = A(0)$ then obviously $A_0 R = -RA_0$.

Now suppose that $\pm i$ are non-semisimple eigenvalues of A_0, i.e. dim $\ker(A_0^2 + I) = 2$ and $\ker(A_0^2 + I)^2 = \mathbb{R}^4$.

LEMMA 1. *Let $A_0 = A_S + A_N$ be the Jordan decomposition of A_0. Then A_S and A_N are reversible, i.e. $A_S R = -RA_S$ and $A_N R = -RA_N$.*

Proof. From $A_0 = A_S + A_N$ and $A_0 = -RA_0R$ we find $A_0 = -RA_SR - RA_NR$. Now $-RA_SR$ is semisimple, $-RA_NR$ is nilpotent, and $-RA_SR$ and $-RA_NR$ commute. Because of the uniqueness of the Jordan decomposition it follows that $A_S = -RA_SR$ and $A_N = -RA_NR$. □

Let U =ker$(A_0^2 + I)$, thus dim$U = 2$. Then $K = A_0|_U \in \mathcal{L}(U)$ is semisimple, thus $K = A_S|_U$ and $A_N|_U = 0$. Furthermore $A_S^2 = -I$, that is, A_S generates an

S^1-action on $\mathbb{R}^4$ given by $\{e^{A_S\phi} \mid \phi \in S^1 \cong \mathbb{R}/2\pi\mathbb{Z}\}$. This S^1-action together with R generates an $O(2)$-action on $\mathbb{R}^4$. U is invariant under this action, actually it is irreducible under this action. Next set $R_0 = R|_U \in \mathcal{L}(U)$, then

$$R_0^2 = I_U \text{ and } KR_0 = -R_0K.$$

Now let $\mathbb{R}^4 = U \oplus V$, where V is chosen to be the $O(2)$-invariant complement to U with respect to the given $O(2)$-action. Consequently V is invariant under A_S and R. Let $\hat{K} = A_S|_V \in \mathcal{L}(V)$ and $\hat{R}_0 = R|_V \in \mathcal{L}(V)$. Then

$$\hat{K}\hat{R}_0 = -\hat{R}_0\hat{K},\ \hat{R}_0^{\ 2} = I_V,\ \text{and } \hat{K}^2 = -I_V.$$

Clearly $\dim V = 2$, and V is irreducible under the S^1- and $O(2)$-actions.

LEMMA 2. $\hat{A}_N = A_N|_V \in \mathcal{L}(V, U)$ *is an isomorphism.*

Proof. Consider $W = A_N(V)$. W is invariant under the $O(2)$-action since $A_NA_S = A_SA_N$ and $A_NR = -RA_N$. Because A_N commutes with the S^1-action it follows from Schur's lemma that either $A_N(V) = 0$ or that $A_N|_V \in \mathcal{L}(V, W)$ is an isomorphism. If $A_N(V) = 0$ then $A_N = 0$, which is impossible since this implies $A_0 = A_S$ is semisimple. Consequently $W = \mathrm{Im}A_N$ and $U = \ker A_N$. If $U \cap W = \{0\}$ then A_N is invertible on its image W, which is impossible since A_N is nilpotent. Thus $U \cap W$ is a nontrivial subspace of U, which is invariant under the $O(2)$-action. From this it follows that $U \cap W = U$ and consequently $U = W$. Thus we may conclude that $\hat{A}_N = A_N|_V \in \mathcal{L}(V, U)$ is an isomorphism. □

From $A_SA_N = A_NA_S$ we get that $K\hat{A}_N = \hat{A}_NK$, while from $A_NR = -RA_N$ we obtain that $\hat{A}_NR_0 = -R_0\hat{A}_N$.

LEMMA 3. *There exists a linear isomorphism*

$$\Phi : U \times U \to \mathbb{R}^4$$

such that

$$\Phi^{-1}A_0\Phi(u_1, u_2) = (Ku_1 + u_2, Ku_2)$$

and

$$\Phi^{-1}R\Phi(u_1, u_2) = (R_0u_1, -R_0u_2).$$

Proof. Define $\Phi \in \mathcal{L}(U \times U; I\!R^4)$ by

$$\Phi(u_1, u_2) = u_1 + \hat{A}_N^{-1} u_2,$$

obviously, by lemma 2, ϕ is an isomorphism. We have

$$\begin{aligned} \Phi^{-1} A_0 \Phi(u_1, u_2) &= \Phi^{-1}(A_S + A_N)(u_1 + \hat{A}_N^{-1} u_2) \\ &= \Phi^{-1}(K u_1 + \hat{K} \hat{A}_N^{-1} u_2 + u_2) \\ &= \Phi^{-1}(K u_1 + u_2 + \hat{A}_N^{-1} \hat{K} u_2) \\ &= (K u_1 + u_2, K u_2), \end{aligned}$$

and

$$\begin{aligned} \Phi^{-1} R \Phi(u_1, u_2) &= \Phi^{-1} R(u_1 + \hat{A}_N^{-1} u_2) \\ &= \Phi^{-1}(R_0 u_1 + \hat{R}_0 \hat{A}_N^{-1} u_2) \\ &= \Phi^{-1}(R_0 u_1 + \hat{A}_N^{-1} R_0 u_2) \\ &= (R_0 u_1, -R_0 u_2). \end{aligned}$$

□

Identifying $U \times U$ with$I\!R^4$ we can assume that

$$A_0 = \begin{pmatrix} K & I_U \\ 0 & K \end{pmatrix}, \text{ and } R = \begin{pmatrix} R_0 & 0 \\ 0 & -R_0 \end{pmatrix}. \tag{7}$$

LEMMA 4. *U has a basis$\{e_1, e_2\}$ such that*

$$K e_1 = e_2,\ K e_2 = -e_1,\ R_0 e_1 = e_1,\ R_0 e_2 = -e_2,$$

i.e. K and R_0 are represented by the matrices

$$K \to J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } R_0 \to \begin{pmatrix} 1 & \\ 0 & -1 \end{pmatrix}.$$

Proof. Since $R_0^2 = I_U$, R_0 has only eigenvalues $\epsilon = \pm 1$. If ϵ is an eigenvalue of R_0 with eigenvector u_0, then $R_0 u_0 = \epsilon u_0$ and $R_0(K u_0) = -K R_0 u_0 = -\epsilon K u_0$, i.e. $-\epsilon$ is an eigenvalue with eigenvector $K u_0$. Since dim$U$=2 we have that ± 1 are both simple eigenvalues of R_0. Let e_1 be an eigenvector of R_0 with eigenvalue $+1$, and let

$e_2 = Ke_1$. Then the lemma follows. $\square$

Consequently, on $I\!R^4$ with coordinates (x,y), we may, without loss of generality, suppose the linear system (at $\lambda = 0$) to be

$$\begin{aligned} \frac{dx}{dt} &= \omega_0 J x, \\ \frac{dy}{dt} &= \omega_0 J y + x, \end{aligned} \tag{8}$$

with R given by (5) and J given by (4).

The following theorem shows us how to obtain a normal form for the parameter dependent case.

THEOREM 1. *There exists a mapping*

$$\Psi : I\!R^p \to \mathcal{L}(U \times U; I\!R^4),$$

defined and smooth in a neighborhood of the origin, such that

(i) $\Psi(0) = \Phi$*, with* Φ *as in lemma 3.*

(ii) $\Psi(\lambda)^{-1} R \Psi(\lambda)(u_1, u_2) = (R_0 u_1, -R_0 u_2)$.

(iii) $\Psi(\lambda)^{-1} A(\lambda) \Psi(\lambda)(u_1, u_2) = ((1+\beta(\lambda))K u_1 + u_2, \alpha(\lambda) u_1 + (1+\beta(\lambda))K u_2)$*, for some smooth functions* $\beta(\lambda)$ *and* $\alpha(\lambda)$*, with* $\alpha(0) = \beta(0) = 0$.

That is, identifying $U \times U$ *with* $I\!R^4$ *using* $\Psi(\lambda)$ *we can bring the original system in a form for which*

$$A(\lambda) = \begin{pmatrix} (1+\beta(\lambda))K & I_U \\ \alpha(\lambda) I_U & (1+\beta(\lambda))K \end{pmatrix}, \text{ and } R = \begin{pmatrix} R_0 & 0 \\ 0 & -R_0 \end{pmatrix}. \tag{9}$$

Proof. On U we have an $O(2)$-action generated by $\{e^{K\phi} \mid \phi \in S^1 \cong I\!R/2\pi Z\!\!Z\}$ and R_0. Let $<\,,\,>$ denote an inner product on U for which this action is orthogonal. Then $< e^{K\phi}u_1, e^{K\phi}u_2 > = < u_1, u_2 >$, for all ϕ and thus $K^T = -K$, $R_0^T R_0 = I_U$, and $R_0^2 = I_U$. Consequently $R_0^T = R_0$. We extend this scalar product to $U \times U$ by $< (u_1, u_2), (\tilde{u}_1, \tilde{u}_2) > = < u_1, \tilde{u}_1 > + < u_2, \tilde{u}_2 >$. We may assume $A(\lambda)$ to be such that A_0 and R are given by (7). Then $R^T = R$ and

$$A_0^T = \begin{pmatrix} -K & 0 \\ I_U & -K \end{pmatrix}. \tag{10}$$

Before continuing the proof of the theorem we will first prove a few lemma's.

LEMMA 5. *There exists a mapping*

$$\hat{\Psi} : I\!\!R^p \to \mathcal{L}(U \times U),$$

defined and smooth in a neighborhood of the origin, such that

(i) $\hat{\Psi}(0) = I_{U \times U}$.

(ii) $R\hat{\Psi}(\lambda) = \hat{\Psi}(\lambda)R$.

(iii) $A_0^T(\hat{\Psi}(\lambda)^{-1}A(\lambda)\hat{\Psi}(\lambda) - A_0) = (\hat{\Psi}(\lambda)^{-1}A(\lambda)\hat{\Psi}(\lambda) - A_0)A_0^T$.

Proof. (Lemma 5) Let

$$T_+ = \{\psi \in \mathcal{L}(U \times U) \mid \psi R = R\psi\},$$

$$T_- = \{\psi \in \mathcal{L}(U \times U) \mid \psi R = -R\psi\}$$

(thus $A(\lambda) \in T_-$),

$$\tilde{T}_+ = \{\psi \in T_+ \mid \psi \text{ is invertible }\}.$$

Define $F : \tilde{T}_+ \times I\!\!R^p \to T_-$ by $F(\psi, \lambda) = \psi^{-1}A(\lambda)\psi$. Then $F(I, 0) = A_0$ and

$$D_\psi F(I, 0)\bar{\psi} = A_0\bar{\psi} - \bar{\psi}A_0 = (\mathrm{Ad}A_0)\bar{\psi}, \text{ for all } \bar{\psi} \in T_+,$$

with $\mathrm{Ad}A_0 \in \mathcal{L}(T_+, T_-)$.

On $\mathcal{L}(U \times U)$ we introduce the scalar product by $< B, C >= \mathrm{trace}(B^TC)$, where adjoint and trace are defined with respect to the inner product $<,>$ on $U \times U$. Then

$$\begin{aligned} < (\mathrm{Ad}A_0)B, C > &= < A_0B - BA_0, C > \\ &= \mathrm{trace}((A_0B - BA_0)^TC) \\ &= \mathrm{trace}((B^TA_0^T - A_0^TB^T)C) \\ &= \mathrm{trace}(B^T(A_0^TC - CA_0^T)) \\ &= < B, (\mathrm{Ad}A_0^T)C > . \end{aligned} \qquad (11)$$

Moreover, from $A_0R = -RA_0$ we get $A_0^TR = -RA_0^T$, i.e. $A_0^T \in T_-$, and $\mathrm{Ad}A_0^T \in \mathcal{L}(T_-, T_+)$. From (11) we see that $(\mathrm{Ad}A_0)^T = \mathrm{Ad}A_0^T$, hence

$$T_- = \mathrm{Im}(\mathrm{Ad}A_0) \oplus \ker(\mathrm{Ad}A_0^T).$$

Let $Q \in \mathcal{L}(T_-)$ be the projection in T_- such that $\mathrm{Im} Q = \mathrm{Im}(\mathrm{Ad} A_0)$, and $\ker Q = \ker(\mathrm{Ad} A_0^T)$. Define $\tilde{F} : \tilde{T}_+ \times I\!R^p \to \mathrm{Im} Q$ by $\tilde{F}(\psi, \lambda) = Q(F(\psi, \lambda) - A_0)$. Then $\tilde{F}(I, 0) = 0$ and

$$D_\psi \tilde{F}(I, 0) = Q(\mathrm{Ad} A_0) = \mathrm{Ad} A_0$$

is surjective on $\mathrm{Im} Q$. By the implicit function theorem there exists a smooth function $\hat{\psi} : I\!R^p \to \tilde{T}_+$ with $\hat{\psi}(0) = I$, such that $\tilde{F}(\hat{\psi}(\lambda), \lambda) = 0$, for all λ near zero. Thus $F(\hat{\psi}(\lambda), \lambda) - A_0 \in \ker(\mathrm{Ad} A_0^T)$. □

LEMMA 6.

$$\{B \in \mathcal{L}(U) \mid BK = KB \text{ and } BR_0 = R_0 B\} = \{\alpha I_U \mid \alpha \in I\!R\}$$

and

$$\{C \in \mathcal{L}(U) \mid CK = KC \text{ and } CR_0 = -R_0 C\} = \{\beta K \mid \beta \in I\!R\}$$

Proof. (Lemma 6) We first observe that U is irreducible for the action generated by K. Let $B \in \mathcal{L}(U)$ be such that $BK = KB$ and let $\alpha + i\beta$ be an eigenvalue of A.

If $\beta = 0$, let $\tilde{U} = \ker(A - \alpha I_U)$. Then $\tilde{U}$ is K-invariant and non-trivial. Consequently, by irreducibility, $\tilde{U} = U$ and $B = \alpha I_U$.

If $\beta \neq 0$, let $\tilde{U} = \ker((A - \alpha I_U)^2 + \beta^2 I_U)$. By the same argument we have $\tilde{U} = U$.

Let $\tilde{K} = -\beta^{-1}(A - \alpha I_U) \in \mathcal{L}(U)$, then $\tilde{K}^2 = -I_U$ and $(\tilde{K} - K)(\tilde{K} + K) = \tilde{K}^2 - K^2 = 0$. Thus not both $\tilde{K} - K$ and $\tilde{K} + K$ are isomorphisms, but both commute with K, consequently, by Schur's lemma, either $\tilde{K} = K$ or $\tilde{K} = -K$. Therefore $B = \alpha I_U \pm \beta K$. So

$$\{B \in \mathcal{L}(U) \mid BK = KB\} = \{\alpha I_U + \beta K \mid \alpha, \beta \in I\!R\}$$

The result of the lemma now follows using $R_0 K = -K R_0$. □

We will now continue the proof of theorem 1. Let

$$\Psi(\lambda) = \Phi \hat{\Psi}(\lambda).$$

Set $D = \hat{\Psi}(\lambda)^{-1} A(\lambda) \hat{\Psi}(\lambda) - A_0$. Recall that we assume $A(\lambda)$ to be such that A_0 and R are given by (7). Write

$$D = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}, \; D_i \in \mathcal{L}(U), \; 1 \leq i \leq 4.$$

Then by lemma 5 D has to fulfill the equations $A_0^T D = DA_0^T$ and $RD = -DR$ with R given by (7) and A_0^T given by (10). From these equations and lemma 6 we get

$$D_2 = 0,\ D_1 = D_4 = \beta K,\ D_3 = \alpha I_U,$$

which proves the theorem. □

A further transformation

$$(u_1(t), u_2(t)) = (\tilde{u}_1(1+\beta(\lambda))t, (1+\beta(\lambda))\tilde{u}_2((1+\beta(\lambda))t))$$

and a choice of basis as in lemma 4 give

$$A(\lambda) = \begin{pmatrix} J & I \\ \tilde{\alpha} I & J \end{pmatrix},$$

with $\tilde{\alpha} = \alpha(\lambda)/(1+\beta(\lambda))^2$, and J as in (4). When $\lambda \in I\!R$ and $\tilde{\alpha}'(0) \neq 0$ then we can redefine the parameter such that $\tilde{\alpha}(\lambda) = \mu$.

Remark 1. Note that with respect to the standard symplectic form ω

$$\omega = \sum_{i=1}^{2} dx_i \wedge dy_i, \tag{12}$$

(8) is Hamiltonian and that $A(\mu)$ is in normal form as a infinitesimal symplectic matrix. Furthermore R is anti-symplectic.

3. Nonlinear normal form

In [4, eqn. (8.2)] the normal form for a vector field on $I\!R^4$ in nonsemisimple 1:1 resonance, i.e. the linearized system has a pair of eigenvalues $\pm i$ and two dimensional Jordan blocks, is given as

$$\begin{aligned} \frac{dx}{dt} &= (1+F_1(u,\Omega,\mu))Jx + F_2(u,\Omega,\mu)y + F_3(u,\Omega,\mu)x + F_4(u,\Omega,\mu)Jy, \\ \frac{dy}{dt} &= (1+F_1(u,\Omega,\mu))Jy + x + F_3(u,\Omega,\mu)y, \end{aligned} \tag{13}$$

where J, u, and Ω are given by (4).

This normal form is obtained by the common Lie algebraic methods. For completeness we will formulate the reversible normal form theorem which shows that formally a reversible system can be put in normal form in such a way that the *reversor* remains the same.

We will first prove the following lemma. Let X_{A_0} denote the reversible linear vector field corresponding to A_0, and let $[\ ,\ ]$ be the Lie bracket of vector fields.

LEMMA 7. *$R[X_{A_0}, V](z) = \epsilon[X_{A_0}, V](Rz)$, where $\epsilon = +1$ if V is reversible, and $\epsilon = -1$ if V is R-equivariant, i.e. $RV(z) = V(Rz)$.*

Proof.

$$\begin{aligned} R[X_{A_0}, V](z) &= R[X_{A_0}(z), V(z)] \\ &= RD_zV(z) \cdot X_{A_0}(z) - RD_zX_{A_0}(z) \cdot V(z) \\ &= RD_zV(z)R \cdot RX_{A_0}(z) - RD_zX_{A_0}(z)R \cdot RV(z) \\ &= \epsilon RD_zV(Rz) \cdot X_{A_0}(Rz) - \epsilon RD_zX_{A_0}(Rz) \cdot V(Rz) \\ &= \epsilon(D_uV(u) \cdot X_{A_0}(u) - D_uX_{A_0}(u) \cdot V(u)) \text{ with } u = Rz \\ &= \epsilon[X_{A_0}(u), V(u)] \\ &= \epsilon[X_{A_0}, V](Rz), \end{aligned}$$

where $\epsilon = +1$ if V is reversible, and $\epsilon = -1$ if V is R-equivariant. □

We now come to the normal form theorem where as usual we show that given a vector field in normal form up to order $k-1$ one can find a transformation normalizing the vector field up to order k. By $\mathrm{ad}(X_{A_0})$ we denote the mapping defined by $\mathrm{ad}(X_{A_0})(V) = [V, X_{A_0}]$, for V some arbitrary vector field.

THEOREM 2 (Reversible normal form theorem). *Let*

$$f(z,\mu) = f_1(z,\mu) + f_2(z,\mu) + ... + f_k(z,\mu) + h.o.t.,\ z \in I\!R^4,\ \mu \in I\!R^p,$$

with $f_k(z,\mu)$ homogeneous of order k in z, be a formal power series vector field, reversible with respect to R, which is in normal form up to order $k-1$ with respect to $f_1(z,0) = A_0z$. Then there exists a transformation exp(ad(P)), with $P(z,\mu)$ a homogeneous vector field of order k in z and R-equivariant, such that exp(ad(P))$f(z,\mu)$ is in normal form up to order k and reversible with respect to R.

Proof. We will start with considering the parameter independent case. Therefore let $\mu = 0$. In order to define the normal form we need to embed A_N in a subalgebra of the Lie algebra of vector fields isomorphic to sl(2,$I\!R$). We denote the generators by A_N, A_M, and A_T, A_M being the nilpotent element dual to A_N (cf. [4]). We then have a splitting of the space of vector fields into

$$[\ker(\mathrm{ad}(X_{A_S})) \cap \ker(\mathrm{ad}(X_{A_M}))] \oplus \mathrm{im}(\mathrm{ad}(X_{A_0})). \tag{14}$$

The k-th order term in $\exp(\mathrm{ad}(P))f(z)$, $P(z)$ homogeneous of order k, is given by

$$f_k(z) + [X_{A_0}, P](z).$$

Now the reversible vector fields form a subspace, thus we may split $f_k(z)$ according to (14) as

$$f_k(z) = \bar{f}_k + \hat{f}_k,\ \bar{f}_k \in \ker(\mathrm{ad}(X_{A_S})) \cap \ker(\mathrm{ad}(X_{A_M})),\ \hat{f}_k \in \mathrm{im}(\mathrm{ad}(X_{A_0})),$$

and choose P such that

$$[X_{A_0}, P](z) = -\hat{f}_k, P \in im(ad(X_{A_S} + X_{A_M})).$$

This way, P is determined uniquely by f_k. It follows that the part of f_k in $\mathrm{im}(\mathrm{ad}(X_{A_0}))$ vanishes after applying the transformation $\exp(\mathrm{ad}(P))$. According to lemma 7 P has to be chosen R-equivariant. Consequently the *reversor* is not changed, and the remaining part $\bar{f}_k$ of f_k is in normal form and reversible.

The transformation $\exp(\mathrm{ad}(P))$ determined so far only normalizes the part of $f_k(z, \mu)$ which does not depend on μ. That is,

$$(\exp(\mathrm{ad}(P))f_k(z, \mu))_k = \bar{f}_k(z) + \tilde{f}_k(z, \mu),$$

with $\tilde{f}_k(z, 0) = 0$. Let $\mathcal{P}_k$ denote the homogeneous polynomials of order k. We will now show that we can find an additional transformation which normalizes the μ depending part $\tilde{f}_k$. We will use an implicit function theorem argument as in lemma 5. I.e. we will show that there exists a smooth mapping

$$\hat{P} : I\!R^p \to \mathcal{P}_k,$$

with $\hat{P}(0) = 0$, $\hat{P}(\mu)(Rz) = R\hat{P}(\mu)(z)$, and such that

$$\exp(\mathrm{ad}(\hat{P}))f_k(z, \mu) \in \ker(\mathrm{ad}(X_{A_S})) \cap \ker(\mathrm{ad}(X_{A_M})).$$

Let

$$\mathcal{P}_{k,+} = \{P \in \mathcal{P}_k \mid P(Rz) = RP(z)\},$$

$$\mathcal{P}_{k,-} = \{P \in \mathcal{P}_k \mid P(Rz) = -RP(z)\}.$$

Define $F : \mathcal{P}_{k,+} \times I\!\!R^p \to \mathcal{P}_{k,-}$ by $F(P,\mu) = f_k(z,\mu) + [X_{A_0}, P](z,\mu)$. Then $F(0,0) = \bar{f}_k$ and $D_P F(0,0) = \mathrm{ad}X_{A_0}$. Let Q be the projection in $\mathcal{P}_{k,-}$ such that $\mathrm{Im}Q = \mathrm{im}(\mathrm{ad}(X_{A_0}))$ and $\ker Q = \ker(\mathrm{ad}(X_{A_S})) \cap \ker(\mathrm{ad}(X_{A_M}))$. Define $\tilde{F} : \mathcal{P}_{k,+} \times I\!\!R^p \to \mathrm{Im}Q$ by $\tilde{F}(P,\mu) = Q(F(P,\mu) - \bar{f}_k)$. Then $D_P\tilde{F}(0,0) = \mathrm{ad}X_{A_0}$ is surjective on $\mathrm{Im}Q$. By the implicit function theorem there exists a smooth function $\hat{P} : I\!\!R^p \to \mathcal{P}_{k,+}$ with $\hat{P}(0,0) = 0$, such that $\tilde{F}(\hat{P}(\mu),\mu) = 0$, for all μ near zero. Thus $(\exp(\mathrm{ad}(\hat{P}))f_k(z,\mu))_k = \bar{f}_k + \tilde{f}_k(z,\mu) + [X_{A_0}, P](z,\mu) \in \ker(\mathrm{ad}(X_{A_S})) \cap \ker(\mathrm{ad}(X_{A_M}))$. □

Remains to give a general expression for the normal form. Such a general expression can be obtained from (13) and is given in lemma 8.

LEMMA 8. *The equations (13) are reversible if and only if $F_3 = F_4 = 0$.*

Proof. The condition for reversibility is $f(Rz) = -Rf(z)$. This gives for (13) the following equations. The F_i are R-invariant because u and Ω are R-invariant. We will write the F_i without there arguments.

$$\begin{aligned}
-(1+F_1)x_2 + F_2y_1 - F_3x_1 + F_4y_2 &= -(1+F_1)x_2 + F_2y_1 + F_3x_1 - F_4y_2, \\
-(1+F_1)x_1 - F_2y_2 + F_3x_2 + F_4y_1 &= -(1+F_1)x_1 - F_2y_2 - F_3x_2 - F_4y_1, \\
(1+F_1)y_2 - x_1 + F_3y_1 &= (1+F_1)y_2 - x_1 - F_3y_1, \\
(1+F_1)y_1 + x_2 - F_3y_2 &= (1+F_1)y_1 + x_2 + F_3y_2 \qquad (15)
\end{aligned}$$

Which gives

$$\begin{aligned}
-F_3x_1 + F_4y_2 &= F_3x_1 - F_4y_2, \\
F_3x_2 + F_4y_1 &= -F_3x_2 - F_4y_1, \\
F_3y_1 &= -F_3y_1, \\
-F_3y_2 &= F_3y_2 \qquad (16)
\end{aligned}$$

The conclusion of the lemma is now straightforward. □

Thus by putting $F_3 = F_4 = 0$ in (13) we obtain a general normal form for reversible vector fields at nonsemisimple 1:1 resonance as follows.

$$\begin{aligned}
\frac{dx}{dt} &= (1 + F_1(u,\Omega,\mu))Jx + F_2(u,\Omega,\mu)y, \\
\frac{dy}{dt} &= (1 + F_1(u,\Omega,\mu))Jy + x. \qquad (17)
\end{aligned}$$

This is precisely the system (3) if we put $\omega_0 = 1$ (this can be done without loss of generality), $\hat{P} = F_1$, and $\hat{Q} = F_2$.

In the Hamiltonian context the normalization proceeds along the same lines. In this case we have the standard symplectic form on $I\!R^4$ and the normalizing transformations need to be symplectic, i.e. the generating vector field P must be a Hamiltonian vector field. In the Hamiltonian case the normalization procedure is performed with the Hamiltonian functions, which, with the Poisson bracket induced by the symplectic form, also form a Lie algebra, rather then with vector fields (see [12]). The normal form for the Hamiltonian function at nonsemisimple 1:1 resonance is given by

$$H(x, y) = \Omega + n + F(u, \Omega, \mu), \tag{18}$$

with $F(\Omega, 0) = F(0, 0) = 0$, and $n = \frac{1}{2}(x_1^2 + x_2^2)$. We obtain the vector field

$$\begin{aligned} \frac{dx}{dt} &= (1 + \frac{\partial F}{\partial \Omega}((u, \Omega, \mu))Jx + 2\frac{\partial F}{\partial u}((u, \Omega, \mu)y, \\ \frac{dx}{dt} &= (1 + \frac{\partial F}{\partial \Omega}((u, \Omega, \mu))Jy + x. \end{aligned} \tag{19}$$

Thus we obtain

LEMMA 9. *The equations (13) are Hamiltonian with a Hamiltonian of the form (18) if and only if* $F_3 = F_4 = 0$ *and* $(F_1, F_2) = (\frac{\partial F}{\partial \Omega}, 2\frac{\partial F}{\partial u})$ *for some* $F(u, \Omega, \mu)$.

Consequently the reversible system (17) is Hamiltonian if and only if $(F_1, F_2) = (\frac{\partial F}{\partial \Omega}, 2\frac{\partial F}{\partial u})$, a remark which is also made in [5, p 243].

4. Reduction to the orbit space

Consider a system

$$\dot{z} = f(z, \mu),\ z \in I\!R^n,\ \mu \in I\!R^p, \tag{20}$$

which is equivariant with respect to a compact symmetry group G acting linearly on $I\!R^n$, i.e. $gf(z, \mu) = f(gz, \mu)$ for $g \in G$. according to a theorem of Hilbert the polynomials invariant under a compact group action are generated by finitely many invariants (a Hilbert basis), which can be chosen to be homogeneous polynomials, say $\sigma_1, \sigma_2, ..., \sigma_p$. We may now define the map

$$\sigma : I\!R^n \to I\!R^p; z \mapsto (\sigma_1, \sigma_2, ..., \sigma_p).$$

This map is called the orbit map, its image can be identified with $\mathbb{R}^n/G$ and is called the orbit space. Each point in the orbit space corresponds to precisely one G-orbit. (see [8]). A G-equivariant system on $\mathbb{R}^n$ can now be lifted to the orbit space, i.e. be written as a system of equations in the invariants. It seems that the orbit space is the natural setting to study symmetric systems. However, it has the disadvantage that in general the orbit space is a semi-algebraic variety.

In the context of Hamiltonian systems the concept of reduction is well known. In this case the group action is symplectic and contains in general the flow of some integral, i.e. there exists a momentum mapping. In the case of, for instance, one integral I the reduced phase spaces are given by $\sigma(I^{-1}(c))$, c some value of the integral. (For more details see [2]).

In the case of the nonsemisimple 1:1 resonance we consider a system (13) which is in normal form, that is , the system is symmetric with respect to the S^1-action given by

$$\{e^{A_S\phi} \mid \phi \in S^1 \cong \mathbb{R}/2\pi\mathbb{Z}\},$$

which is the flow of the semisimple part of the linearized vector field. By remark 1 we may consider the linearized system to be a Hamiltonian system. Consequently the S^1-action can be seen as a symplectic action. The Hamiltonian function corresponding to this symplectic S^1-action is Ω.

LEMMA 10. *Ω is an integral for the system (13) if and only if*

$$2F_3(u,\Omega,\mu)\Omega - F_4(u,\Omega,\mu)u = 0. \tag{21}$$

Proof. A straightforward calculation shows that

$$\frac{d\Omega}{dt} = 2F_3(u,\Omega,\mu)\Omega - F_4(u,\Omega,\mu)u.$$

□

From this it is clear that the reversible normal form has Ω as an absolute invariant.

A Hilbert basis generating the S^1-invariant polynomials is given by $\Omega, \frac{1}{2}u, n, T$, with $T = x_1y_1 + x_2y_2$. Note that these are the Hamiltonian functions corresponding to A_S, A_M, A_N, and A_T respectively, i.e. $A_S z = X_\Omega(z)$, $A_M z = X_u(z)$, $A_N z = X_n(z)$, and $A_T z = X_T(z)$, where X_f denotes the Hamiltonian vector field with Hamiltonian function f.

On the C^∞ functions the symplectic form induces a Poisson bracket $\{\ ,\ \}$ by

$$\omega(X_f, X_g) = \{f, g\},$$

where, writing $z = (z_1, z_2, z_3, z_4) = (x, y) = (x_1, x_2, y_1, y_2) \in I\!R^4$,

$$\begin{aligned} \{f, g\}(z) &= \sum_{i=1,j=1}^{4} W_{ij} \frac{\partial f}{\partial z_i} \frac{\partial g}{\partial z_j} \\ &= \sum_{i=1}^{2} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} \right). \end{aligned}$$

The bracket $\{\ ,\ \}$ defines a Poissonstructure on the C^∞ functions. The C^∞ functions together with this bracket form a Poisson algebra. The matrix

$$W_{ij} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

is called the structure matrix for the Poisson structure. u, n,and T generate a Lie subalgebra of $(C^\infty, \{\ ,\ \})$ isomorphic to $\mathrm{sl}(2, I\!R)$.

The orbit map for the S^1 action is given by

$$\rho : I\!R^4 \to I\!R^4; (x, y) \mapsto (\Omega, \frac{1}{2}u, n, T). \tag{22}$$

The image is determined by the relation

$$2un - \Omega^2 - T^2 = 0. \tag{23}$$

The standard Poisson structure on $I\!R^4$ induces a natural Poisson structure on the image of the orbit map given by the $\mathrm{sl}(2, I\!R)$ bracket relations.

We may now lift the equation (13) to the orbit space. First we introduce on the orbit space new coordinates by choosing a somewhat different set of generating invariants:

$$y_1 = -2\Omega,\ y_2 = 4n,\ y_3 = 2T,\ y_4 = u. \tag{24}$$

In these new coordinates the equations on the orbit space are (see [4])

$$\begin{pmatrix} \dot{y_1} \\ \dot{y_2} \\ \dot{y_3} \\ \dot{y_4} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ y_2 \\ y_3 \end{pmatrix} + 2F_3(y_1, y_4, \mu) \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} + 2F_2(y_1, y_4, \mu) \begin{pmatrix} 0 \\ y_3 \\ y_4 \\ 0 \end{pmatrix} +$$

$$+2F_4(y_1, y_4, \mu) \begin{pmatrix} y_4 \\ y_1 \\ 0 \\ 0 \end{pmatrix}. \qquad (25)$$

We call this the reduced equations. Again these equations are in normal form. On the invariants the *reversor* (5) becomes

$$\bar{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \qquad (26)$$

Obviously the vector field (25) is reversible with respect to $\bar{R}$ if and only if $F_3 = F_4 = 0$. Thus a reduced reversible vector field in normal form has the general form

$$\begin{pmatrix} \dot{y_1} \\ \dot{y_2} \\ \dot{y_3} \\ \dot{y_4} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ y_2 \\ y_3 \end{pmatrix} + 2F_2(y_1, y_4, \mu) \begin{pmatrix} 0 \\ y_3 \\ y_4 \\ 0 \end{pmatrix}. \qquad (27)$$

The bracket relations among the y_i are $\{y_1, y_2\} = \{y_1, y_3\} = \{y_1, y_4\} = 0$, $\{y_2, y_3\} = 4y_2$, $\{y_2, y_4\} = 4y_3$,and $\{y_3, y_4\} = 4y_4$. Thus the Poisson structure on the orbit space is given by the structure matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 4y_2 & 4y_3 \\ 0 & -4y_2 & 0 & 4y_4 \\ 0 & -4y_3 & -4y_4 & 0 \end{pmatrix} \qquad (28)$$

Let $[,]$ denote the Poisson bracket for this Poisson structure. Then the vector field (27) is Hamiltonian with respect to this Poisson structure, i.e. can be written as $\dot{y} = [y, \bar{H}]$, with

$$\bar{H} = -\frac{1}{4}y_2 + \frac{1}{2}\int_0^{y_4} F_2(y_1, s, \mu)ds. \qquad (29)$$

We obtain $\bar{H}$ as an absolute invariant for the reversible system. So far we have shown that a reduced reversible vector field in normal form is Hamiltonian. By lemma 9 a reduced Hamiltonian vector field in normal form is reversible. Thus we have shown

THEOREM 3. *Consider a vector field on $I\!R^4$ at nonsemisimple 1:1 resonance which is in normal form. Then the corresponding reduced vector field (25) is reversible with respect to (26) if and only if it is Hamiltonian with respect to (28).*

Because Ω is an invariant, a reduced reversible vector field has a natural restriction to surfaces $\Omega = constant$. On the images of the surfaces $\Omega = constant$ the Poisson structure, in a natural way, restricts to a symplectic structure. I.e. on the orbit space the surfaces $\rho(\Omega^{-1}(c))$ are the symplectic leaves for the Poisson structure. The surfaces $\rho(\Omega^{-1}(c))$ are called the reduced phase spaces. A system which on the orbit space is Hamiltonian with respect to the Poisson structure on the reduced phase spaces restricts to a genuine Hamiltonian system which is Hamiltonian with respect to the symplectic structure.

5. Hamiltonian structure in the non-reduced reversible vector field

The reversible system (17) is Hamiltonian if and only if there exists an $F(u, \Omega, \mu)$ such that $\frac{\partial F}{\partial \Omega} = F_1$ and $2\frac{\partial F}{\partial u} = \frac{1}{2}F_2$. The Hamiltonian function with respect to the standard symplectic form is then given by

$$G = \Omega - n + F, \tag{30}$$

where $n = \frac{1}{2}(x_1^2 + x_2^2)$. Now let $\hat{F}$ be such that $\hat{F}_u = \frac{1}{2}F_2$, and write $\hat{V} = F_1 - \hat{F}_\Omega$. Then it is clear that we can split our reversible vector field in a Hamiltonian (and reversible) part corresponding to $H = -n + \hat{F}$, which is precisely one of the absolute invariants, and a reversible part $(\omega_0 - \hat{V})(Jx, Jy)$, i.e. the vector field can be written as

$$\begin{aligned} \frac{dx}{dt} &= (\omega_0 - \hat{V})Jx + \frac{\partial H}{\partial y}, \\ \frac{dy}{dt} &= (\omega_0 - \hat{V})Jy - \frac{\partial H}{\partial x}. \end{aligned} \tag{31}$$

Let X_H denote the Hamiltonian vector field corresponding to H. Then the vector field (31) can be written as $(\omega_0 - \hat{V})X_\Omega + X_H$. The "non-Hamiltonian" part of the vector field $(\omega_0 - \hat{V})X_\Omega$ is thus equivalent to the Hamiltonian vector field X_Ω in the sense that the trajectories coincide in a neighborhood of the origin. However, the flow is non-Hamiltonian, which is obvious because its period, which is determined by $\omega_0 - \hat{V}$, does not depend on Ω alone.

The vector field $(\omega_0 - \hat{V})X_\Omega$ might still contain a Hamiltonian part because each vector field of the form $f(\Omega)X_\Omega$ is obviously Hamiltonian. Let $W(\mu, \Omega)$ be such that $W_\Omega = -\hat{V}(\mu, 0, \Omega)$. Furthermore let $u\tilde{V}(\mu, u, \Omega) = -\hat{V}(\mu, u, \Omega) + \hat{V}(\mu, 0, \Omega)$, and let $\hat{H} = W + H$. Then we can write the vector field as

$$u\tilde{V}X_\Omega + X_{\hat{H}}. \tag{32}$$

The vector field for the reversible 1:1 resonance can thus be seen as a non-Hamiltonian perturbation, of at least order two, of a Hamiltonian vector field. The perturbation generates a shift along the X_Ω trajectories. Consequently after reduction to the orbit space the non-Hamiltonian effect vanishes.

6. Bifurcations of periodic solutions

The non-Hamiltonian part of the reversible vector field, which is in F_1 vanishes after reduction. This is obvious from the geometry. The trajectories of the "non-Hamiltonian" part, $(\omega_0 - \hat{V})X_\Omega$, of the reversible vector field (31) coincide with the X_Ω orbits and thus will vanish on the orbit space. Thus on the reduced phase spaces we are left with the reduced Hamiltonian systems corresponding to H, i.e. which have reduced Hamiltonian $\bar{H}$. The periodic solutions are precisely the stationary points of the reduced Hamiltonian vector fields. These points correspond to the critical values of the map

$$H \times \Omega : I\!R^4 \to I\!R^2; (x, y) \mapsto (H, \Omega), \tag{33}$$

which is obvious if we factorize this map through the orbit map. The linear stability type of the periodic solution corresponds to the stability type of the stationary point of the reduced vector field (see [3]). Consequently the geometry of the bifurcation of periodic solutions at nonsemisimple 1:1 resonance of the reversible case is exactly the same as for the Hamiltonian Hopf bifurcation [12].

Because of the presence of two absolute invariants the phase space of the system (1) is fibered into invariant surfaces given by $\Omega = constant$ and $H = constant$, i.e. the fibers of the map (33). Thus we obtain exactly the same fibration as in the Hamiltonian case (see [12]). The difference lies in the fact that on these fibers the flow of the reversible system (1) is non-Hamiltonian by the fact that a small non-Hamiltonian shift is added in the direction of the X_Ω trajectories.

Like in the Hamiltonian case one can use a reversible version of Weinstein-Moser reduction ([12], [11]) to show persistence of the periodic solutions under higher order perturbations which destroy the normal form.

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