Algebraic K-theory and special values of ζ- and L-functions.

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Let k be a number field, i.e., for some $f(X)$ an irreducible polynomial of degree d in $\mathbb{Q}[X]$, and α be a root of $f(X)$ in $\mathbb{C}$,

$$k = \mathbb{Q}(\alpha) = \{b_0 + b_1\alpha + \ldots + b_{d-1}\alpha^{d-1},\ \text{all } b_j \text{ in } \mathbb{Q}\}.$$

Let $\mathcal{O}_k$ be the ring of algebraic integers of k: $x \in k$ is an algebraic integer if it is the zero of a polynomial $X^n + a_{n-1}X^{n-1} + \ldots + a_1X + a_0$ with all a_i in $\mathbb{Z}$.

Let r_1 the number of embeddings $k \to \mathbb{R}$, $2r_2$ the number of nonreal embeddings $k \to \mathbb{C}$, so $[k : \mathbb{Q}] = r_1 + 2r_2$.
$\mathcal{O}_k^*$ has rank $r = r_1 + r_2 - 1$. Let $\sigma_1, \ldots, \sigma_{r+1}$ be the embeddings of k into $\mathbb{C}$ up to complex conjugaton.
If $u_1, \ldots, u_r$ form a $\mathbb{Z}$-basis of $\mathcal{O}_k^*/\text{torsion}$, let

$$R = \frac{2^{r_2}}{[k:\mathbb{Q}]} \left| \begin{vmatrix} 1 & \log|\sigma_1(u_1)| & \ldots & \log|\sigma_1(u_r)| \\ \vdots & \vdots & & \vdots \\ 1 & \log|\sigma_{r+1}(u_1)| & \ldots & \log|\sigma_{r+1}(u_r)| \end{vmatrix} \right|$$

The ζ-function of k is defined by

$$\zeta_k(s) = \sum_{\substack{(0)\neq I \subset \mathcal{O}_k \\ I \text{ an ideal of } \mathcal{O}_k}} (\#\mathcal{O}_k/I)^{-s} = \prod_{\substack{0\neq \mathcal{P} \subset \mathcal{O}_k \\ \mathcal{P} \text{ prime ideal}}} \frac{1}{1 - (\#\mathcal{O}_k/\mathcal{P})^{-s}}.$$

The residue of $\zeta_k(s)$ at $s = 1$ is $\dfrac{2^{r_1}(2\pi)^{r_2} R |\mathrm{Cl}(\mathcal{O}_k)|}{w\sqrt{\Delta_k}}$, with $w = |\mathcal{O}_{k,\mathrm{tor}}^*|$ the number of roots of unity in k, and Δ_k the absolute value of the discriminant of k.

This is a statement about algebraic K-theory:
$K_0(\mathcal{O}_k) \cong \mathbb{Z} \oplus \mathrm{Cl}(\mathcal{O}_k)$ and $K_1(O_k) \cong \mathcal{O}_k^*$, so
$|\mathrm{Cl}(\mathcal{O}_k)| = |K_0(\mathcal{O}_k)_{\mathrm{tor}}| =$ and $w = |K_1(\mathcal{O}_k)_{\mathrm{tor}}|$.

Algebraic K-theory

Let R (for simplicity) be a commutative ring with identity $1 \neq 0$.

$$K_0(R) = \frac{\text{free Abelian group on } [M],\ M \text{ a finitely generated projective } R\text{-module}}{\left\langle \begin{matrix} [M] - [M'] - [M''] \text{ for each exact sequence} \\ 0 \to M' \to M \to M'' \to 0 \end{matrix} \right\rangle}.$$

View $GL_n(R) \subset GL_{n+1}(R)$ via $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$.
Let $GL(R) = \bigcup_n GL_n(R)$.

$K_1(R) = GL(R)/[GL(R), GL(R)]$

If F is a field, then we have in general

$$\begin{aligned} &K_0(F) \cong \mathbb{Z} \\ &K_1(F) \cong F^* = F \setminus \{0\} \\ &K_2(F) \cong F^* \otimes_{\mathbb{Z}} F^* / < x \otimes (1-x), x \in F^* \setminus \{1\} > . \end{aligned}$$

The class of $a \otimes b$ in $K_2(F)$ is denoted $\{a, b\}$, so that $K_2(F)$ is generated by symbols $\{a, b\}$ (a, b in F^*) with rules

$$\begin{aligned} &\{a_1 a_2, b\} = \{a_1, b\} + \{a_2, b\} \\ &\{a, b_1 b_2\} = \{a, b_1\} + \{a, b_2\} \\ &\{x, 1-x\} = 0. \end{aligned}$$

It follows that $\{a, b\} + \{b, a\} = \{x, -x\} = 0$.

For a finite field $\mathbb{F}_q$: $\begin{cases} K_{2n-1}(\mathbb{F}_q) \cong \mathbb{Z}/(q^n - 1) \\ K_{2n}(\mathbb{F}_q) = 0 \end{cases} \quad (n \geq 1).$

Example

$$K_2(\mathbb{Q}) \cong \{\pm 1\} \oplus \bigoplus_{\substack{p \text{ prime} \\ p>2}} (\mathbb{Z}/p)^* .$$

The isomorphism is given on the p-component by the tame symbol: for a prime p and a in $\mathbb{Q}^*$ write $a = \dfrac{u}{v} p^s$ with integers u, v not divisible by p. Set $v_p(a) = s$. Then

$$\{a, b\} \mapsto T_p\left(\{a, b\}\right) = (-1)^{v_p(a) v_p(b)} \frac{a^{v_p(b)}}{b^{v_p(a)}} \bmod p.$$

The map T_2 to $\{\pm 1\}$ is given as follows.
For $\{a, b\}$ write $a = (-1)^i 2^j 5^k \frac{c}{d}$ with $i, k = 0, 1$ and c, d integers congruent 1 mod 8, $b = (-1)^I 2^J 5^K \frac{c'}{d'}$ similarly. Then

$$T_2(\{a, b\}) = (-1)^{iI + jK + kJ}.$$

There is another map

$$T_\infty : K_2(\mathbb{Q}) \to \{\pm 1\}, \qquad \{a, b\} \mapsto \begin{cases} -1 \text{ if } a, b < 0, \\ 1 \text{ otherwise.} \end{cases}$$

Identify $\{\pm 1\} \subset (\mathbb{Z}/p)^*$ for all primes $p > 2$.

Theorem $T_\infty(\{a, b\}) = T_2(\{a, b\}) \prod_{\substack{p>2 \\ p \text{ prime}}} T_p(\{a, b\})^{\frac{p-1}{2}}.$

This gives rise to a proof of quadratic reciprocity. Let p and q be distinct odd primes, and put $\left(\frac{p}{q}\right)$ equal to 1 if p is a square modulo q, -1 if not. Equivalently,

$$\left(\frac{p}{q}\right) = p^{\frac{q-1}{2}} \bmod q = T_q\left(\{p,q\}\right)^{\frac{q-1}{2}}$$

The theorem says that

$$\begin{aligned} 1 &= T_2(\{p,q\})T_p(\{p,q\})^{\frac{p-1}{2}}T_q(\{p,q\})^{\frac{q-1}{2}} \\ &= (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}}\left(\frac{q}{p}\right)\left(\frac{p}{q}\right). \end{aligned}$$

Borel's theorem

Let k be a number field as before (with notation as before).
$K_n(\mathcal{O}_k)$ is finitely generated for all $n \geq 0$. Let m_n be the rank of $K_n(\mathcal{O}_k)$.

Theorem (Borel) $K_{2n}(\mathcal{O}_k)$ is a finite group if $n \geq 1$. $K_{2n-1}(\mathcal{O}_k)$ has rank $m_{2n-1} = r_1+r_2$ if n is odd, and has rank $m_{2n-1} = r_2$ if n is even ($n \geq 2$).
Furthermore, there exists a natural regulator map

$$K_{2n-1}(\mathcal{O}_k) \to \mathbb{R}^{m_{2n-1}}$$

such that the image is a lattice with volume of a fundamental domain

$$V_n \sim_{\mathbb{Q}^*} \frac{\zeta_k(n)}{\pi^{n([k:\mathbb{Q}]-m_{2n-1})}\sqrt{\Delta_k}}$$

where Δ_k is the absolute value of the discriminant of k.

[$a \sim_{\mathbb{Q}^*} b$ means $a = qb$ for some q in $\mathbb{Q}^*$.]

Example $\zeta_{\mathbb{Q}}$ is the Riemann zeta function. For $n \geq 2$:
$K_{2n-1}(\mathbb{Z})$ is torsion for n even;
$K_{2n-1}(\mathbb{Z})$ has rank 1 for n odd, and $V_n \sim_{\mathbb{Q}^*} \zeta(n)$.

n	2	3	4	5	6	7	...
m_{2n-1}	0	1	0	1	0	1	...
$\zeta(n)$	$\pi^2/6$	irrational	$\pi^4/90$	???	$\pi^6/945$	???	...

Generalizations: curves.

For simplicity, take E to be an elliptic curve defined over $\mathbb{Q}$, e.g., defined in $\mathbb{P}^2_{\mathbb{Q}}$ by a Weierstraß equation

$$Y^2Z = X^3 + AXZ^2 + BZ^3$$

with A, B in $\mathbb{Z}$, and $-4A^3 - 27B^2 \neq 0$.
Let p be a prime number such that E has good reduction E_p at p. (E.g., in the above example, if $p \neq 2, 3$ and p does not divide $-4A^3 - 27B^2$, then $Y^2Z = X^3 + \overline{A}XZ^2 + \overline{B}Z^3$ defines an elliptic curve E_p in $\mathbb{P}^2_{\mathbb{F}_p}$). Put $a_p = 1 + p - |E_p(\mathbb{F}_p)|$ and

$$L_p(E, s) = \frac{1}{1 - a_p p^{-s} + p^{1-2s}}.$$

There is also a definition if E has bad reduction at p. Then

$$L(E, s) = \prod_{p \text{ prime}} L_p(E, s) \text{ for } \operatorname{Re}(s) > \frac{3}{2}.$$

[Cf. $\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$ for $\operatorname{Re}(s) > 1$.]
Let $E_{\mathbb{C}}$ be the extension of the coefficients to $\mathbb{C}$. (E.g., in the above example, consider the equation in $\mathbb{P}^2_{\mathbb{C}}$.) Let F be the field of meromorphic functions on $E_{\mathbb{C}}$. There is an exact localization sequence

$$K_2(E_{\mathbb{C}}) \to K_2(F) \to \coprod_{x \in E_{\mathbb{C}}} \mathbb{C}^*$$

where the x-component of the last map is the tame symbol again: write $\operatorname{ord}_x(f)$ is the order of vanishing of f at x, then

$$\{f, g\} \mapsto (-1)^{\operatorname{ord}_x(f)\operatorname{ord}_x(g)} \frac{f^{\operatorname{ord}_x(g)}}{g^{\operatorname{ord}_x(f)}}|_x.$$

For two nonzero meromorphic functions f and g on $E_{\mathbb{C}}$, $\log|f| \mathrm{d}\arg g - \log|g| \mathrm{d}\arg f$ is a closed 1-form on an open part of $E_{\mathbb{C}}$. Because

$$\log|z| \mathrm{d}\arg(1-z) - \log|1-z| \mathrm{d}\arg z = \mathrm{d}P_2(z),$$

where $P_2(z)$ is a C^∞-function on $\mathbb{C} \setminus \{0, 1\}$, we get a map

$$\begin{aligned} \operatorname{reg} : K_2(F) &\to \left\{ \frac{\text{closed 1-forms on open parts}}{\text{exact 1-forms on open parts}} \right\} \\ \{f, g\} &\mapsto \log|f| \mathrm{d}\arg g - \log|g| \mathrm{d}\arg f. \end{aligned}$$

This fits into a commutative diagram

$$\begin{array}{ccccccc}
 & & K_2(E) & & & & \\
 & & \downarrow & & & & \\
 & & K_2(E_{\mathbb{C}}) & \longrightarrow & K_2(F) & \longrightarrow & \coprod_{x \in E_{\mathbb{C}}} \mathbb{C}^* \\
 & & \vdots\ \scriptstyle{\operatorname{reg}} & & \downarrow \scriptstyle{\operatorname{reg}} & & \downarrow \scriptstyle{\log|\cdot|} \\
0 & \longrightarrow & H^1_{\mathrm{dR}}(E_{\mathbb{C}}; \mathbb{R}) & \longrightarrow & H^1_{\mathrm{dR}}(F; \mathbb{R}) & \xrightarrow[i \cdot \mathrm{res}]{} & \coprod_{x \in E_{\mathbb{C}}} \mathbb{R}
\end{array}$$

with $H^1_{\mathrm{dR}}(F;\mathbb{R}) = \varinjlim_{U\subset E_{\mathbb{C}}} H^1_{\mathrm{dR}}(U;\mathbb{R})$ where all U are such that $E_{\mathbb{C}}\setminus U$ is finite.

Theorem (Bloch) Let E be an elliptic curve defined over $\mathbb{Q}$, with complex multiplication. Then there exists an element α in $K_2(E)$ with

$$L'(E,0) \sim_{\mathbb{Q}^*} \frac{1}{2\pi}\int_{E_{\mathbb{C}}} \mathrm{reg}(\alpha)\wedge\omega$$

or, using the functional equation for the L-function:

$$\frac{1}{2\pi}L(E,2) \sim_{\mathbb{Q}^*} \int_{E_{\mathbb{C}}} \mathrm{reg}(\alpha)\wedge\omega.$$

[ω is a nonzero holomorphic form on $E_{\mathbb{C}}$ with $\int_{E(\mathbb{R})}\omega = 1$.]

(Part of) Beilinson's conjectures

Let $C/\mathbb{Q}$ be a complete nonsingular curve such that $C_{\mathbb{C}}$ is a Riemann surface of genus g. Beilinson defines a regulator map ($n \geq 1$)

$$K_{2n}(C) \to H^1_{\mathrm{dR}}(C_{\mathbb{C}};\mathbb{R}(n))^+ (\cong \mathbb{R}^g).$$

[$\mathbb{R}(n) = (2\pi i)^n\mathbb{R} \subset \mathbb{C}$; $+$ indicates a subspace invariant with respect to an action of complex conjugation.] Then Beilinson conjectures

(i) $K_{2n}(C)_{\mathbb{Q}} \cong \mathbb{Q}^g$ and the regulator induces an isomorphism $K_{2n}(C)_{\mathbb{R}} \cong H^1_{\mathrm{dR}}(C_{\mathbb{C}};\mathbb{R}(n))^+$.
[$_{\mathbb{Q}}$ (resp. $_{\mathbb{R}}$) means $\otimes_{\mathbb{Q}}$ (resp. $\otimes_{\mathbb{R}}$).]
(ii) Let $\{a_1,\ldots,a_g\}$ be a $\mathbb{Q}$-basis of $K_{2n}(C)_{\mathbb{Q}}$, and let A be the transition matrix from $\{a_1,\ldots,a_g\}$ to a $\mathbb{Q}$-basis of $H^1_{\mathrm{sing}}(C_{\mathbb{C}};\mathbb{Q}(n))^+ \subset H^1_{\mathrm{dR}}(C_{\mathbb{C}};\mathbb{R}(n))^+$. Then, assuming the L-function can be extended meromorphically,

$$L^*(C,1-n) \sim_{\mathbb{Q}^*} \det(A),$$

where $L^*(C,b)$ is the first nonvanishing coefficient in the expansion of $L(C,s)$ around $s=b$.

[Using the functional equations for $\zeta_k(s)$ and $L(E,s)$ the previous results come in this form.]

There are several results in this direction, e.g.:

(i) Relation between the regulator of a specific element in K-theory and the corresponding L-function for the product of two modular curves (Beilinson).
(ii) Relation of certain parts of the K-theory of cyclotomic fields and Dirichlet L-functions (Beilinson).
(iii) Precise conjecture about the rational numbers involved in the Beilinson conjecture (Bloch–Kato), with various results.
(iv) Generalization of Bloch's result to p-adic regulator and p-adic L-function (Coleman–de Shalit).
(v) Generalization of Bloch's result for CM elliptic curves over $\mathbb{Q}$ to all $K_{2n}(E)$ ($n \geq 2$) (Deninger).