Algebraic K-theory and special values of ζ- and L-functions.

R. de Jeu

University of Durham

email: `rob.de-jeu@durham.ac.uk`

website: `http://maths.dur.ac.uk/~dma0rdj`

k: a number field, i.e., for some $f(X)$ irreducible in $\mathbb{Q}[X]$ of degree d, α in $\mathbb{C}$ with $f(\alpha) = 0$,

$$k = \mathbb{Q}(\alpha) = \{b_0 + b_1\alpha + \ldots + b_{d-1}\alpha^{d-1}, \text{ all } b_j \text{ in } \mathbb{Q}\}.$$

$\mathcal{O}_k$: the ring of algebraic integers of k. x in k is in $\mathcal{O}_k$ if it is a zero of some $X^n + a_{n-1}X^{n-1} + \ldots + a_1X + a_0$ with all a_i in $\mathbb{Z}$.

r_1: the number of embeddings $k \to \mathbb{R}$
$2r_2$: the number of nonreal embeddings $k \to \mathbb{C}$
$[k : \mathbb{Q}] = r_1 + 2r_2$
$\mathcal{O}_k^*$ has rank $r = r_1 + r_2 - 1$
Let $\sigma_1, \ldots, \sigma_{r+1}$ be the embeddings of k into $\mathbb{C}$ up to complex conjugaton.
If $u_1, \ldots, u_r$ form a $\mathbb{Z}$-basis of $\mathcal{O}_k^*$/torsion, let

$$R = \frac{2^{r_2}}{[k:\mathbb{Q}]} \left| \begin{vmatrix} 1 & \log|\sigma_1(u_1)| & \ldots & \log|\sigma_1(u_r)| \\ \vdots & \vdots & & \vdots \\ 1 & \log|\sigma_{r+1}(u_1)| & \ldots & \log|\sigma_{r+1}(u_r)| \end{vmatrix} \right|$$

$$\zeta_k(s) = \sum_{\substack{(0)\neq I \subset \mathcal{O}_k \\ I \text{ an ideal of } \mathcal{O}_k}} (\#\mathcal{O}_k/I)^{-s}$$

$$= \prod_{\substack{0\neq \mathcal{P} \subset \mathcal{O}_k \\ \mathcal{P} \text{ prime ideal}}} \frac{1}{1 - (\#\mathcal{O}_k/\mathcal{P})^{-s}}$$

$$\mathrm{Res}_{s=1}\zeta_k(s) = \frac{2^{r_1}(2\pi)^{r_2}R|\mathrm{Cl}(\mathcal{O}_k)|}{w\sqrt{\Delta_k}}$$

Δ_k = the absolute value of the discriminant of k.
$w = |\mathcal{O}^*_{k,\mathrm{tor}}|$ = #roots of unity in k.

$$K_0(\mathcal{O}_k) \cong \mathbb{Z} \oplus \mathrm{Cl}(\mathcal{O}_k)$$

$$K_1(O_k) \cong \mathcal{O}^*_k$$

$$|\mathrm{Cl}(\mathcal{O}_k)| = |K_0(\mathcal{O}_k)_{\mathrm{tor}}|$$

$$w = |K_1(\mathcal{O}_k)_{\mathrm{tor}}|$$

Algebraic K-theory.

R a commutative ring with identity $1 \neq 0$.

$$K_0(R) = \frac{\begin{array}{l}\text{free Abelian group on } [M],\ M \text{ a finitely} \\ \text{generated projective } R\text{-module}\end{array}}{\left\langle\begin{array}{l}[M] - [M'] - [M''] \text{ for each exact} \\ \text{sequence } 0 \to M' \to M \to M'' \to 0\end{array}\right\rangle}$$

View $GL_n(R) \subset GL_{n+1}(R)$ via $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$.

Let $GL(R) = \bigcup_n GL_n(R)$.

$K_1(R) = GL(R)/[GL(R), GL(R)]$

If F is a field, then

$$K_0(F) \cong \mathbb{Z}$$
$$K_1(F) \cong F^* = F \setminus \{0\}$$
$$K_2(F) \cong F^* \otimes_{\mathbb{Z}} F^* / < x \otimes (1-x), x \in F^* \setminus \{1\} > .$$

The class of $a \otimes b$ in $K_2(F)$ is denoted $\{a, b\}$, so $K_2(F)$ is generated by symbols $\{a, b\}$ with a, b in F^*, and rules

$$\{a_1 a_2, b\} = \{a_1, b\} + \{a_2, b\}$$
$$\{a, b_1 b_2\} = \{a, b_1\} + \{a, b_2\}$$
$$\{x, 1-x\} = 0.$$

It follows that $\{a, b\} + \{b, a\} = \{x, -x\} = 0$.

Example

$$K_2(\mathbb{Q}) \cong \{\pm 1\} \oplus \bigoplus_{\substack{p \text{ prime} \\ p>2}} (\mathbb{Z}/p)^* ,$$

p a prime, a in $\mathbb{Q}^*$, $a = \dfrac{u}{v} p^s$ with integers u, v not divisible by p. Set $v_p(a) = s$.
p-component of the isomorphism: the tame symbol

$$T_p : \{a, b\} \mapsto (-1)^{v_p(a) v_p(b)} \frac{a^{v_p(b)}}{b^{v_p(a)}} \bmod p.$$

The map T_2 to $\{\pm 1\}$ is given as follows.
$a = (-1)^i 2^j 5^k \frac{c}{d}$ with $i, k = 0, 1$ and c, $d \equiv 1 \bmod 8$,
$b = (-1)^I 2^J 5^K \frac{c'}{d'}$ similarly. Then

$$T_2(\{a, b\}) = (-1)^{iI + jK + kJ}.$$

There is another map

$$T_\infty : K_2(\mathbb{Q}) \to \{\pm 1\}, \qquad \{a, b\} \mapsto \begin{cases} -1 \text{ if } a, b < 0, \\ 1 \text{ otherwise.} \end{cases}$$

Identify $\{\pm 1\} \subset (\mathbb{Z}/p)^*$ for all primes $p > 2$.

Theorem

$$T_\infty(\{a, b\}) = T_2(\{a, b\}) \prod_{\substack{p>2 \\ p \text{ prime}}} T_p(\{a, b\})^{\frac{p-1}{2}}$$

p and q distinct odd primes,
$\left(\frac{p}{q}\right) = \begin{cases} 1 \text{ if } p \text{ is a square modulo } q \\ -1 \text{ if } p \text{ is not a square modulo } q \end{cases}$
Equivalently,

$$\left(\frac{p}{q}\right) = p^{\frac{q-1}{2}} \bmod q = T_q\left(\{p, q\}\right)^{\frac{q-1}{2}}$$

The theorem says that

$$\begin{aligned} 1 &= T_2(\{p, q\})T_p(\{p, q\})^{\frac{p-1}{2}}T_q(\{p, q\})^{\frac{q-1}{2}} \\ &= (-1)^{\frac{p-1}{2} \cdot \frac{q-1}{2}}\left(\frac{q}{p}\right)\left(\frac{p}{q}\right) \end{aligned}$$

Borel's theorem

k: number field as before
$K_n(\mathcal{O}_k)$ is finitely generated for all $n \geq 0$.
m_n = the rank of $K_n(\mathcal{O}_k)$.

Theorem (Borel) $K_{2n}(\mathcal{O}_k)$ is a finite group if $n \geq 1$.
$K_{2n-1}(\mathcal{O}_k)$ has rank $m_{2n-1} = r_1 + r_2$ if n is odd, and has rank $m_{2n-1} = r_2$ if n is even ($n \geq 2$).
Furthermore, there exists a natural regulator map

$$K_{2n-1}(\mathcal{O}_k) \to \mathbb{R}^{m_{2n-1}}.$$

The image is a lattice with volume V_n of a fundamental domain

$$V_n \sim_{\mathbb{Q}^*} \frac{\zeta_k(n)}{\pi^{n([k:\mathbb{Q}]-m_{2n-1})}\sqrt{\Delta_k}}$$

where Δ_k is the absolute value of the discriminant of k.

[$a \sim_{\mathbb{Q}^*} b$ means $a = qb$ for some q in $\mathbb{Q}^*$.]

Example $\zeta_{\mathbb{Q}}$ is the Riemann zeta function. For $n \geq 2$:
$K_{2n-1}(\mathbb{Z})$ is torsion for n even;
$K_{2n-1}(\mathbb{Z})$ has rank 1 for n odd, and $V_n \sim_{\mathbb{Q}^*} \zeta(n)$.

n	2	3	4	5	6	7	...
m_{2n-1}	0	1	0	1	0	1	...
$\zeta(n)$	$\pi^2/6$	irrat.	$\pi^4/90$	???	$\pi^6/945$	???	...

Generalizations: curves.

E an elliptic curve defined over $\mathbb{Q}$, e.g., defined in $\mathbb{P}^2_{\mathbb{Q}}$ by a Weierstraß equation

$$Y^2 Z = X^3 + AXZ^2 + BZ^3$$

with A, B in $\mathbb{Z}$, and $-4A^3 - 27B^2 \neq 0$.

p a prime number such that E has good reduction E_p at p.
(E.g., in the above example, if $p \neq 2, 3$ and p does not divide $-4A^3 - 27B^2$, then $Y^2 Z = X^3 + \overline{A}XZ^2 + \overline{B}Z^3$ defines an elliptic curve E_p in $\mathbb{P}^2_{\mathbb{F}_p}$).

$$a_p = 1 + p - |E_p(\mathbb{F}_p)|$$

$$L_p(E, s) = \frac{1}{1 - a_p p^{-s} + p^{1-2s}}.$$

There is also $L_p(E, s)$ if E has bad reduction at p.

$$L(E, s) = \prod_{p \text{ prime}} L_p(E, s) \text{ for } \mathrm{Re}(s) > \frac{3}{2}.$$

[Cf. $\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$ for $\mathrm{Re}(s) > 1$.]

Let $E_{\mathbb{C}}$ be the extension of the coefficients to $\mathbb{C}$. (E.g., in the above example, consider the equation in $\mathbb{P}^2_{\mathbb{C}}$.) F the field of meromorphic functions on $E_{\mathbb{C}}$.
Exact localization sequence

$$K_2(E_{\mathbb{C}}) \to K_2(F) \to \coprod_{x \in E_{\mathbb{C}}} \mathbb{C}^*$$

x-component of the last map is the tame symbol again: $\mathrm{ord}_x(f)$ is the order of vanishing of f at x, then

$$\{f, g\} \mapsto (-1)^{\mathrm{ord}_x(f)\,\mathrm{ord}_x(g)} \frac{f^{\mathrm{ord}_x(g)}}{g^{\mathrm{ord}_x(f)}}|_x.$$

If f and g are in F^*, then $\log|f| \mathrm{d} \arg g - \log|g| \mathrm{d} \arg f$ is a closed 1-form on an open part of $E_{\mathbb{C}}$.

$$\log|z| \mathrm{d} \arg(1-z) - \log|1-z| \mathrm{d} \arg z = \mathrm{d} P_2(z),$$

$P_2(z)$ a C^∞-function on $\mathbb{C} \setminus \{0, 1\}$

$$\mathrm{reg} : K_2(F) \to \left\{ \frac{\text{closed 1-forms on open parts}}{\text{exact 1-forms on open parts}} \right\}$$
$$\{f, g\} \mapsto \log|f| \mathrm{d} \arg g - \log|g| \mathrm{d} \arg f$$

This fits into a commutative diagram

$$\begin{array}{ccccccc} & & K_2(E) & & & & \\ & & \downarrow & & & & \\ & & K_2(E_{\mathbb{C}}) & \longrightarrow & K_2(F) & \longrightarrow & \coprod_{x\in E_{\mathbb{C}}} \mathbb{C}^* \\ & & \text{reg}\ \vdots & & \text{reg}\ \downarrow & & \log|\cdot|\ \downarrow \\ 0 & \longrightarrow & H^1_{\mathrm{dR}}(E_{\mathbb{C}};\mathbb{R}) & \longrightarrow & H^1_{\mathrm{dR}}(F;\mathbb{R}) & \xrightarrow[i\cdot\mathrm{res}]{} & \coprod_{x\in E_{\mathbb{C}}} \mathbb{R} \end{array}$$

$H^1_{\mathrm{dR}}(F;\mathbb{R}) = \varinjlim_{U\subset E_{\mathbb{C}}} H^1_{\mathrm{dR}}(U;\mathbb{R})$, U s.t. $E_{\mathbb{C}} \setminus U$ is finite.

Theorem (Bloch) E an elliptic curve over $\mathbb{Q}$ with complex multiplication. For some α in $K_2(E)$,

$$L'(E,0) \sim_{\mathbb{Q}^*} \frac{1}{2\pi} \int_{E_{\mathbb{C}}} \mathrm{reg}(\alpha) \wedge \omega$$

or, using the functional equation for the L-function:

$$\frac{1}{2\pi} L(E,2) \sim_{\mathbb{Q}^*} \int_{E_{\mathbb{C}}} \mathrm{reg}(\alpha) \wedge \omega.$$

[ω a holomorphic form on $E_{\mathbb{C}}$ with $\int_{E(\mathbb{R})} \omega = 1$.]

(Part of) Beilinson's conjectures

$C/\mathbb{Q}$ a complete nonsingular curve such that $C_{\mathbb{C}}$ is a Riemann surface of genus g.
Beilinson defines a regulator map ($n \geq 1$)

$$K_{2n}(C) \to H^1_{\mathrm{dR}}(C_{\mathbb{C}}; \mathbb{R}(n))^+ (\cong \mathbb{R}^g).$$

[$\mathbb{R}(n) = (2\pi i)^n \mathbb{R} \subset \mathbb{C}$; $+$ indicates a subspace invariant with respect to an action of complex conjugation.]
Then for $n \geq 2$ ($n = 1$ needs an additional condition):

(i) $K_{2n}(C)_{\mathbb{Q}} \cong \mathbb{Q}^g$ and the regulator induces an isomorphism $K_{2n}(C)_{\mathbb{R}} \cong H^1_{\mathrm{dR}}(C_{\mathbb{C}}; \mathbb{R}(n))^+$.

(ii) If $\{a_1, \ldots, a_g\}$ is a $\mathbb{Q}$-basis of $\mathrm{reg}(K_{2n}(C)_{\mathbb{Q}})$, A is the transition matrix from $\{a_1, \ldots, a_g\}$ to a $\mathbb{Q}$-basis of $H^1_{\mathrm{sing}}(C_{\mathbb{C}}; \mathbb{Q}(n))^+ \subset H^1_{\mathrm{dR}}(C_{\mathbb{C}}; \mathbb{R}(n))^+$,
then, assuming the L-function can be extended meromorphically,

$$L^*(C, 1-n) \sim_{\mathbb{Q}^*} \det(A),$$

where $L^*(C, b)$ is the first nonvanishing coefficient in the expansion of $L(C, s)$ around $s = b$.

[Using the functional equations for $\zeta_k(s)$ and $L(E, s)$ the previous results come in this form.]

Various results:

(i) Relation between the regulator of a specific element in K-theory and the corresponding L-function for the product of two modular curves (Beilinson).

(ii) Relation of certain parts of the K-theory of cyclotomic fields and Dirichlet L-functions (Beilinson).

(iii) Precise conjecture about the rational numbers involved in the Beilinson conjecture (Bloch-Kato), with various results.

(iv) Generalization of Bloch's result to p-adic regulator and p-adic L-function (Coleman–de Shalit).

(v) Generalization of Bloch's result for CM elliptic curves over $\mathbb{Q}$ to all $K_{2n}(E)$ $(n \geq 2)$ (Deninger).