K-theory, regulators and L-functions for curves over number fields

R. de Jeu

University of Durham

email: rob.de-jeu@durham.ac.uk

website: http://maths.dur.ac.uk/~dma0rdj

k: a number field.
$\mathcal{O}_k$: the ring of algebraic integers of k.

r_1: the number of embeddings $k \to \mathbb{R}$
$2r_2$: the number of nonreal embeddings $k \to \mathbb{C}$
$[k : \mathbb{Q}] = r_1 + 2r_2$
$\mathcal{O}_k^*$ has rank $r = r_1 + r_2 - 1$
Let $\sigma_1, \ldots, \sigma_{r+1}$ be the embeddings of k into $\mathbb{C}$ up to complex conjugation.
If $u_1, \ldots, u_r$ form a $\mathbb{Z}$-basis of $\mathcal{O}_k^*/\text{torsion}$, let

$$R = \frac{2^{r_2}}{[k : \mathbb{Q}]} \left| \begin{vmatrix} 1 & \log|\sigma_1(u_1)| & \ldots & \log|\sigma_1(u_r)| \\ \vdots & \vdots & & \vdots \\ 1 & \log|\sigma_{r+1}(u_1)| & \ldots & \log|\sigma_{r+1}(u_r)| \end{vmatrix} \right|$$

$$\begin{aligned} \zeta_k(s) &= \sum_{\substack{(0)\neq I \subset \mathcal{O}_k \\ I \text{ an ideal of } \mathcal{O}_k}} (\#\mathcal{O}_k/I)^{-s} \\ &= \prod_{\substack{0 \neq \mathcal{P} \subset \mathcal{O}_k \\ \mathcal{P} \text{ prime ideal}}} \frac{1}{1 - (\#\mathcal{O}_k/\mathcal{P})^{-s}} \end{aligned}$$

$$\text{Res}_{s=1}\zeta_k(s) = \frac{2^{r_1}(2\pi)^{r_2} R |\text{Cl}(\mathcal{O}_k)|}{w\sqrt{\Delta_k}}$$

$\Delta_k =$ the absolute value of the discriminant of k.
$w = |\mathcal{O}_{k,\text{tor}}^*| = \#$roots of unity in k.

$$K_0(\mathcal{O}_k) \cong \mathbb{Z} \oplus \mathrm{Cl}(\mathcal{O}_k)$$

$$K_1(\mathcal{O}_k) \cong \mathcal{O}_k^*$$

$$|\mathrm{Cl}(\mathcal{O}_k)| = |K_0(\mathcal{O}_k)_{\mathrm{tor}}|$$

$$w = |K_1(\mathcal{O}_k)_{\mathrm{tor}}|$$

If F is a field, then

$$\begin{aligned}
&K_0(F) \cong \mathbb{Z} \\
&K_1(F) \cong F^* = F \setminus \{0\} \\
&K_2(F) \cong F^* \otimes_{\mathbb{Z}} F^* / < x \otimes (1-x), x \in F^* \setminus \{1\} > .
\end{aligned}$$

The class of $a \otimes b$ in $K_2(F)$ is denoted $\{a, b\}$, so $K_2(F)$ is generated by symbols $\{a, b\}$ with a, b in F^*, and rules

$$\begin{aligned}
&\{a_1 a_2, b\} = \{a_1, b\} + \{a_2, b\} \\
&\{a, b_1 b_2\} = \{a, b_1\} + \{a, b_2\} \\
&\{x, 1-x\} = 0.
\end{aligned}$$

It follows that $\{a, b\} + \{b, a\} = \{x, -x\} = 0$.

F a field. Then for $n \geq 1$

$$K_n(F)_{\mathbb{Q}} = K_n^{(1)}(F) \oplus K_n^{(2)}(F) \oplus \ldots \oplus K_n^{(n)}(F)$$

and a similar decomposition for $K_n(X)_{\mathbb{Q}}$ for a reasonable scheme X.
[Here and elsewhere, $A_{\mathbb{Q}} = A \otimes_{\mathbb{Z}} \mathbb{Q}$.]

Borel's theorem

k: number field
$K_n(\mathcal{O}_k)$ is finitely generated for all $n \geq 0$.
$m_n =$ the rank of $K_n(\mathcal{O}_k)$.

Theorem (Borel) $K_{2n}(\mathcal{O}_k)$ is a finite group if $n \geq 1$. $K_{2n-1}(\mathcal{O}_k)$ has rank $m_{2n-1} = r_1 + r_2$ if n is odd, and has rank $m_{2n-1} = r_2$ if n is even ($n \geq 2$).
Furthermore, there exists a natural regulator map

$$K_{2n-1}(\mathcal{O}_k) \to \mathbb{R}^{m_{2n-1}}.$$

The image is a lattice with volume V_n of a fundamental domain

$$V_n \sim_{\mathbb{Q}^*} \frac{\zeta_k(n)}{\pi^{n([k:\mathbb{Q}]-m_{2n-1})}\sqrt{\Delta_k}}$$

where Δ_k is the absolute value of the discriminant of k.

[$a \sim_{\mathbb{Q}^*} b$ means $a = qb$ for some q in $\mathbb{Q}^*$.]

Example $\zeta_{\mathbb{Q}}$ is the Riemann zeta function. For $n \geq 2$:
$K_{2n-1}(\mathbb{Z})$ is finite for n even;
$K_{2n-1}(\mathbb{Z})$ has rank 1 for n odd, and $V_n \sim_{\mathbb{Q}^*} \zeta(n)$.

n	2	3	4	5	6	7	...
m_{2n-1}	0	1	0	1	0	1	...
$\zeta(n)$	$\pi^2/6$	irrat.	$\pi^4/90$	???	$\pi^6/945$	???	...

Curves.

$E/\mathbb{Q}$ an elliptic curve.
$E_{\mathbb{C}}$ the extension of the coefficients to $\mathbb{C}$.
F the field of meromorphic functions on $E_{\mathbb{C}}$.

Exact localization sequence

$$K_2(E_{\mathbb{C}}) \longrightarrow K_2(F) \xrightarrow{T} \coprod_{x \in E_{\mathbb{C}}} \mathbb{C}^*$$

T is the *tame symbol*. With $\mathrm{ord}_x(f)$ the order of vanishing of f at x, T_x is given by:

$$\{f, g\} \mapsto (-1)^{\mathrm{ord}_x(f)\,\mathrm{ord}_x(g)} \frac{f^{\mathrm{ord}_x(g)}}{g^{\mathrm{ord}_x(f)}}|_x.$$

For f and g in F^*, put $\eta(f, g) = \log|f| \mathrm{d}\arg g - \log|g| \mathrm{d}\arg f$, a closed 1-form on an open part of $E_{\mathbb{C}}$.

$$\log|z| \mathrm{d}\arg(1-z) - \log|1-z| \mathrm{d}\arg z = \mathrm{d}P_2(z),$$

$P_2(z)$ a C^∞-function on $\mathbb{C} \setminus \{0, 1\}$

$$\mathrm{reg} : K_2(F) \to \left\{ \frac{\text{closed 1-forms on open parts}}{\text{exact 1-forms on open parts}} \right\}$$
$$\{f, g\} \mapsto \eta(f, g)$$

This fits into a commutative diagram

$$
\begin{array}{ccccccc}
 & & K_2(E) & & & & \\
 & & \downarrow & & & & \\
 & & K_2(E_{\mathbb{C}}) & \longrightarrow & K_2(F) & \xrightarrow{T} & \coprod_{x \in E_{\mathbb{C}}} \mathbb{C}^* \\
 & & \vdots\,\text{reg} & & \downarrow\,\text{reg} & & \downarrow\,\log|\cdot| \\
0 & \longrightarrow & H^1_{\mathrm{dR}}(E_{\mathbb{C}};\mathbb{R})) & \longrightarrow & H^1_{\mathrm{dR}}(F;\mathbb{R})) & \xrightarrow[i\cdot\mathrm{res}]{} & \coprod_{x \in E_{\mathbb{C}}} \mathbb{R}
\end{array}
$$

$H^1_{\mathrm{dR}}(F;\mathbb{R}) = \varinjlim_{U \subset E_{\mathbb{C}}} H^1_{\mathrm{dR}}(U;\mathbb{R})$, U s.t. $E_{\mathbb{C}} \setminus U$ is finite.

Theorem (Bloch) E an elliptic curve over $\mathbb{Q}$ with complex multiplication. For some α in $K_2(E)$,

$$L'(E,0) \sim_{\mathbb{Q}^*} \frac{1}{2\pi} \int_{E(\mathbb{C})} \mathrm{reg}(\alpha) \wedge \omega$$

or, using the functional equation for the L-function:

$$\frac{1}{2\pi} L(E,2) \sim_{\mathbb{Q}^*} \int_{E(\mathbb{C})} \mathrm{reg}(\alpha) \wedge \omega.$$

[ω a holomorphic form on $E_{\mathbb{C}}$ with $\int_{E(\mathbb{R})} \omega = 1$.]

Getting a hold on higher K-groups.

> "Algebraic K-theory is a functor that associates to your favourite exact category Abelian groups K_n $(n \geq 0)$, about which you know nothing."

Let F be an infinite field, and write $F^*_{\mathbb{Q}}$ for $F^* \otimes_{\mathbb{Z}} \mathbb{Q}$.
$B_n(F)$: a free $\mathbb{Q}$-vector space on $[x]_n$, x in F, $x \neq 0, 1$, modulo some inductively defined relations.

Complex $\Gamma(F, n)$ in degrees $1, \ldots, n$ for $n \geq 2$:

$$B_n(F) \to B_{n-1}(F) \otimes F^*_{\mathbb{Q}} \to \ldots \to B_2(F) \otimes \bigwedge^{n-2} F^*_{\mathbb{Q}} \to \bigwedge^{n} F^*_{\mathbb{Q}}$$

$$\mathrm{d}[x]_l \otimes y_1 \wedge \ldots \wedge y_{n-l} = [x]_{l-1} \otimes x \wedge y_1 \wedge \ldots \wedge y_{n-l} (l \geq 3)$$
$$\mathrm{d}[x]_2 \otimes y_1 \wedge \ldots \wedge y_{n-2} = (1-x) \wedge x \wedge y_1 \wedge \ldots \wedge y_{n-2}$$

C is a complete nonsingular curve over an infinite field k,
$F = k(C)$: the field of rational functions on C
$\Gamma(n, C)$: total complex associated to double complex

$$\begin{array}{ccccc} B_n(F) & \longrightarrow & B_{n-1}(F) \otimes F^*_{\mathbb{Q}} & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \coprod B_{n-1}(k(x)) & \longrightarrow & \cdots \end{array}$$

with the coproduct over the closed points x in C. The vertical maps are "based on" $[f]_m \otimes g \mapsto \mathrm{ord}_x(g) \cdot [f(x)]_m$ with $[0]_m = [\infty]_m = 0$.

Conjecture (Zagier): k a number field. Then for $n \geq 2$,

$$K_{2n-1}(k)_{\mathbb{Q}} \cong H^1(\Gamma(k,n)) = \mathrm{Ker}(\mathrm{d}_n),$$

with

$$\begin{aligned} \mathrm{d}_n : B_n(k) &\to B_{n-1}(k) \otimes k^*_{\mathbb{Q}} \\ [x]_n &\mapsto [x]_{n-1} \otimes x \end{aligned}$$

for $n \geq 3$, and

$$\begin{aligned} \mathrm{d}_2 : B_2(k) &\to \bigwedge^2 k^*_{\mathbb{Q}} \\ [x]_2 &\mapsto (1-x) \wedge x \end{aligned}$$

together with a formula for the regulator in terms of polylogarithms on $\mathrm{Ker}(\mathrm{d}_n)$.

Conjecture (Goncharov)$(n \geq 2)$

(1) $H^p(\Gamma(n,F)) \cong K^{(n)}_{2n-p}(F)$ if F is an infinite field.

(2) $H^p(\Gamma(n,C)) \cong K^{(n)}_{2n-p}(C)$ if C is a complete smooth curve over an infinite field k.

Theorem

(i) **(Deligne-Beilinson, RdJ)** There is an injection

$$H^1(\Gamma(n,k)) \to K_{2n-1}(k)_{\mathbb{Q}}$$

with the expected formula for the regulator.

(ii) **(Suslin/Goncharov)** For $n = 2$ or 3 it is also surjective.

(iii) **(Zagier)** It is also surjective if k is a cyclotomic field.

Theorem (RdJ) Let C be a complete, smooth, geometrically irreducible curve over a number field k, $F = k(C)$. Then there exist complexes $\widetilde{\mathcal{M}}_{(n)}(F)$ and $\widetilde{\mathcal{M}}_{(n)}(C)$ similar to Goncharov's (with B_n replaced by $\widetilde{M}_n$, also generated by $[x]_n$'s), with maps to the K-theory as follows.

(1) $n = 3$, $p \geq 2$; in particular,

$$H^2(\widetilde{\mathcal{M}}_{(3)}(F)) \to K_4^{(3)}(F)$$

and

$$H^2(\widetilde{\mathcal{M}}_{(3)}(C)) \to K_4^{(3)}(C) + K_3^{(2)}(k) \cup F^*/K_3^{(2)}(k) \cup F^*.$$

[Note: if k is totally real, then $K_3^{(2)}(k) = 0$.]

(2) $n = 4$, $p \geq 3$, and for $p = 2$ more or less:

$$H^2(\widetilde{\mathcal{M}}_{(4)}(F)) \twoheadleftarrow H^2(\mathcal{M}_{(4)}(F)) \to \frac{K_6^{(4)}(F)}{K_4^{(2)}(F) \cup K_2^{(2)}(F)}$$

and

$$H^2(\widetilde{\mathcal{M}}_{(4)}(C)) \twoheadleftarrow H^2(\mathcal{M}_{(4)}(F)) \to \frac{K_6^{(4)}(C) + K_4^{(2)}(F) \cup K_2^{(2)}(F) + K_5^{(3)}(k) \cup F^*}{K_4^{(2)}(F) \cup K_2^{(2)}(F) + K_5^{(3)}(k) \cup F^*}.$$

[$\widetilde{\mathcal{M}}_{(n)}(F)$ is a quotient complex of a rather similar complex $\mathcal{M}_{(n)}(F)$, and similarly for $\widetilde{\mathcal{M}}_{(n)}(C)$.]

Theorem (RdJ, also using results by Goncharov)
Let C be as above. Then

$$H^2(\Gamma(3,C)),\ \ H^2(\widetilde{\mathcal{M}}_{(3)}(C)) \text{ and } K_4^{(3)}(C)$$

all have the same image in $H^1_{\mathrm{dR}}(C \otimes_{\mathbb{Q}} \mathbb{C}; \mathbb{R}(2))^+$ under the regulator. Similarly,

$$H^2(\Gamma(4,C)),\ \ H^2(\mathcal{M}_{(4)}(C)),\ \ H^2(\widetilde{\mathcal{M}}_{(4)}(C)) \text{ and } K_6^{(4)}(C)$$

all have the same image in $H^1_{\mathrm{dR}}(C \otimes_{\mathbb{Q}} \mathbb{C}; \mathbb{R}(3))^+$ under the regulator.
[Here and elsewhere, $\mathbb{R}(n) = (2\pi i)^n \mathbb{R} \subset \mathbb{C}$.]

Polylogarithms
$Li_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$ for $n \geq 1$ on $|z| < 1$.
$Li_1(z) = -\mathrm{Log}(1-z)$
Using $\mathrm{d}Li_{n+1}(z) = Li_n(z) \mathrm{d}\log z$ for $n \geq 1$, $Li_n(z)$ extends to a multivalued function on $\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$.
Single valued versions
($\pi_{n-1} : \mathbb{C} = \mathbb{R}(n) \oplus \mathbb{R}(n-1) \to \mathbb{R}(n-1)$ projection):

$$P_n(z) = \pi_{n-1} \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \log^k |z| Li_{n-k}(z)$$

$$P_{n,\mathrm{Zag}}(z) = \pi_{n-1} \sum_{k=0}^{n-1} \frac{2^k B_k}{k!} \log^k |z| Li_{n-k}(z)$$

[B_k: Bernoulli number.]
Then $P_{n,\mathrm{Zag}}(z) + (-1)^n P_{n,\mathrm{Zag}}(z^{-1}) = 0$.

Formulae for the regulator

(1) k a number field. [As $K_n(\mathcal{O}_k)_{\mathbb{Q}} = K_n(k)_{\mathbb{Q}}$ for $n \geq 2$, this provides information about Borel's theorem.]
$H^1(\widetilde{\mathcal{M}}_{(n)}(k)) \hookrightarrow K_{2n-1}^{(n)}(k) \to (\mathbb{R}(n-1)_\sigma)^+_{\sigma:k\hookrightarrow\mathbb{C}}$
maps $[x]_n$ to $\pm((n-1)!\,P_{n,\mathrm{Zag}}(\sigma(x)))_\sigma$ $(n \geq 2)$.

(2) C complete, smooth, geometrically irreducible curve over a number field k, $F = k(C)$. Fix ω in $H^0(C \otimes_{\mathbb{Q}} \mathbb{C}, \Omega^1_{/\mathbb{C}})^+$
(+: certain invariance w.r.t. complex conjugation).

$$H^2(\widetilde{\mathcal{M}}_{(3)}(F)) \longrightarrow H^1_{\mathrm{dR}}(F \otimes_{\mathbb{Q}} \mathbb{C}; \mathbb{R}(2))^+ \xrightarrow{\int \cdot \wedge \overline{\omega}} \mathbb{R}(1)$$

is

$$[f]_2 \otimes g \mapsto \pm \frac{8}{3} \int_{C \otimes_{\mathbb{Q}} \mathbb{C}} \log|g|\, \eta(f, 1-f) \wedge \overline{\omega}$$

$$\text{or } \pm \frac{8}{3} \int_{C \otimes_{\mathbb{Q}} \mathbb{C}} P_2 \circ f \;\mathrm{d}\log|g| \wedge \overline{\omega}$$

with $\eta(h_1, h_2) = \log|h_1|\mathrm{d}i \arg h_2 - \log|h_2|\mathrm{d}i \arg h_1$.
For $H^2(\widetilde{\mathcal{M}}_{(4)}(F))$ we get

$$[f]_3 \otimes g \mapsto \pm 6 \int_{C \otimes_{\mathbb{Q}} \mathbb{C}} \log|g| \log|f|\, \eta(f, 1-f) \wedge \overline{\omega}.$$

Coleman integration

$\mathbb{C}_p = \hat{\overline{\mathbb{Q}}}_p$
$|\cdot|$: p-adic valution with $|p| = p^{-1}$
$\mathcal{O}$: ring of integers of $\mathbb{C}_p$
$\overline{\mathbb{F}}_p$: residue field

$X/\mathcal{O}$: smooth curve over $\mathcal{O}$ (=smooth projective surjective scheme of relative dimension 1)

For x in $X(\overline{\mathbb{F}}_p)$, put
U_x = residue disc of x = {all pts in $X(\mathbb{C}_p)$ reducing to x}, a copy of the maximal ideal of $\mathcal{O}$.

$Y \subseteq X_{\overline{\mathbb{F}}_p}$ nonempty open affine subscheme, smooth over $\overline{\mathbb{F}}_p$, so $X(\overline{\mathbb{F}}_p) = Y(\overline{\mathbb{F}}_p) \coprod \{e_1, \ldots, e_n\}$.

U_r = rigid space obtained by removing discs of radius $r < 1$ from $X(\mathbb{C}_p)$ for all e_i: e_i locally given by $\overline{h} = 0$, so leave out $|h| \leq r$.

$U = \lim_{r \to 1} U_r$ is independent of the choices

Make a choice of logarithm $\log : \mathbb{C}_p^* \to \mathbb{C}_p$ such that
(1) $\log ab = \log a + \log b$
(2) $\log(1 + z)$ = usual powerseries expansion for $|z|$ small.

(I.e., fix a choice of $\log p$.)

For $x \in Y(\overline{\mathbb{F}}_p)$, put
$A(U_x) = \{\sum_{n=0}^{\infty} a_n z^n \text{ conv. for } |z| < 1\}$
$A_{\log}(U_x) = A(U_x)$
$\Omega_{\log}(U_x) = A_{\log}(U_x)\mathrm{d}z_x$
[$z = z_x$ is a local parameter on U_x.]

For $x \notin Y(\overline{\mathbb{F}}_p)$ (i.e., $x = e_1, \ldots, e_n$), put
$A(U_x) = \{\sum_{n=-\infty}^{\infty} a_n z^n \text{ conv. for some } r < |z| < 1\}$
$A_{\log}(U_x) = A(U_x)[\log z]$
$\Omega_{\log}(U_x) = A_{\log}(U_x)\mathrm{d}z_x$

Put
$A_{\mathrm{loc}}(U) = \prod_{x \in X(\overline{\mathbb{F}}_p)} A_{\log}(U_x)$
(locally analytic functions, with choice of logs around the e_i)
$\Omega_{\mathrm{loc}}(U) = \prod_{x \in X(\overline{\mathbb{F}}_p)} \Omega_{\log}(U_x)$
(locally analytic forms, with choice of log around the e_i)

$$0 \to \prod_{x \in X(\overline{\mathbb{F}}_p)} \mathbb{C}_p \to A_{\mathrm{loc}}(U) \to \Omega_{\mathrm{loc}}(U) \to 0$$

is exact as $\mathrm{d}\log z = \frac{\mathrm{d}z}{z}$.

Coleman: there exists a subspace $A_{\mathrm{Col}}(U)$ of $A_{\mathrm{loc}}(U)$, containing the rigid analytic functions $A(U)$ on U, such that with $\Omega_{\mathrm{col}}(U) = A_{\mathrm{Col}}(U) \otimes \Omega^1(U/\mathbb{C}_p)$

$$0 \to \mathbb{C}_p \to A_{\mathrm{Col}}(U) \to \Omega_{\mathrm{col}}(U) \to 0$$

is exact.

Let P and Q be in U, ω in $\Omega_{\mathrm{col}}(U)$, F_ω in $A_{\mathrm{Col}}(U)$ with $\mathrm{d}F_\omega = \omega$. Put $\int_P^Q \omega = F_\omega(Q) - F_\omega(P)$.

Example

$X = \mathbb{P}^1_{\mathbb{C}_p}$
$Y = \mathbb{P}^1_{\overline{\mathbb{F}}_p} \setminus \{1, \infty\}$
$U = \mathbb{C}_p \setminus U_1 \coprod U_\infty$.
Put $Li_{n+1}(z) = \int_0^z Li_n(z) \mathrm{d}\log z$ starting with $Li_0(z) = \frac{z}{1-z}$.

$Li_n(z) = \sum_{k=1}^\infty \frac{z^k}{k^n}$ for $|z| < 1$.

[In fact, $Li_n(z)$ extends naturally to $\mathbb{C}_p \setminus \{1\}$.]

$$L_{\mathrm{mod}\,n}(z) = \sum_{k=0}^{n-1} \alpha_k Li_{n-k}(z) \log^k z$$

satisfies $L_{\mathrm{mod}\,n}(z) + (-1)^n L_{\mathrm{mod}\,n}(z^{-1}) = 0$ for suitable α_m.

Theorem (Besser and RdJ) Let $k \subset \mathbb{C}_p$ be a number field, and let $n \geq 2$. Then

$$H^1(\widetilde{\mathcal{M}}_{(n)}(k)) \to K_{2n-1}^{(n)}(k) = K_{2n-1}^{(n)}(\mathcal{O}_k) \overset{r_{\mathrm{syn}}}{\to} \mathbb{C}_p$$

is

$$[x]_n \mapsto (-1)^n (n-1)!\; L_{\mathrm{mod}\,n}(x)$$

if x is a root of unity, or x is a *special unit* in $\mathcal{O}$, i.e., both x and $1-x$ are units in $\mathcal{O} = \mathcal{O}_{\mathbb{C}_p}$.

$C/\mathbb{C}_p$ smooth complete irreducible curve with good reduction. Coleman and de Shalit define

$$r_p : K_2(\mathbb{C}_p(C)) \to H^0(C, \Omega^1_{C/\mathbb{C}_p})^\vee$$
$$\{f, g\} \mapsto \left[\omega \mapsto \int_{(f)} \log(g) \cdot \omega\right]$$

(Here, if $(f) = \sum_j a_P(P)$, $\int_{(f)} \rho = \sum_j a_P F_\rho(P)$.)

Theorem (Coleman and de Shalit) If $E/\mathbb{Q}$ is an elliptic curve with complex multiplication, p a prime that splits in the CM field of E, then for the same α in $K_2(E)$ as in Bloch's theorem, and the same ω, $r_p(\alpha)(\omega) = a_\alpha \Omega_p L_p(E, 0)$ for the same a_α as for Bloch. [Ω_p is a p-adic period.]

[Cf. Bloch: $\int_{E(\mathbb{C})} r(\alpha) \cup \omega = a_\alpha \Omega L^*(E, 0)$, where $L^*(E, s)$ is the usual L-function multiplied by the Γ factor.]

Theorem (Besser) $C/\mathbb{C}_p$ as above. $\mathcal{C}/\mathcal{O}$ model of $C/\mathbb{C}_p$. Then r_p equals [r_{syn} the syntomic regulator]

$$K_2(C) = K_2(\mathcal{C}) \overset{r_{\text{syn}}}{\to} H^1_{\text{dR}}(C/\mathbb{C}_p) \overset{\text{Tr}(\cdot \cup \omega)}{\to} \mathbb{C}_p.$$

Theorem (Besser and RdJ) C as above, defined over a number field $k \subset \mathbb{C}_p$. $\mathcal{C}$: a model of C over $k \bigcap \mathcal{O}$. Then

$$H^2(\widetilde{\mathcal{M}}_{(3)}(C)) \to K_4^{(3)}(C) = K_4^{(3)}(\mathcal{C}) \overset{r_{\text{syn}}}{\to} H^1_{\text{dR}}(C/\mathbb{C}_p) \overset{\text{Tr}(\cdot \cup \omega)}{\to} \mathbb{C}_p$$

is

$$[f]_2 \otimes g \mapsto 2 \int_{(g)} L_2(f) \cdot \omega,$$

provided that f, $1 - f$ and g do not have a zero or pole along the special fibre of $\mathcal{C}$.

[$L_2(z) = Li_2(z) + \log(z) \cdot \log(1 - z)$.]

For $K_4^{(3)}(F)$ and $K_4^{(3)}(C)$ we are looking at

$$\begin{array}{ccccc}
\widetilde{M}_3(F) & \longrightarrow & \widetilde{M}_2(F) \otimes_{\mathbb{Q}} F_{\mathbb{Q}}^* & \xrightarrow{\ \mathrm{d}\ } & \bigwedge^3 F_{\mathbb{Q}}^* \\
\downarrow & & \downarrow{\scriptstyle \delta} & & \downarrow \\
0 & \longrightarrow & \coprod \widetilde{M}_2(k(x)) & \longrightarrow & \coprod \bigwedge^2 k(x)_{\mathbb{Q}}^*
\end{array}$$

with
$\mathrm{d}[f]_3 = [f]_2 \otimes f$
$\mathrm{d}[f]_2 \otimes g = (1-f) \wedge f \wedge g$
$\delta_x [f]_2 \otimes g = \mathrm{ord}_x(g) \cdot [f(x)]_2.$

(Coproduct over all (closed) points x in C.)