$K$-theory, regulators and $L$-functions for curves over number fields

R. de Jeu

University of Durham

e-mail: rob.de-jeu@durham.ac.uk

website: http://maths.dur.ac.uk/~dma0rdj
$k$: a number field.
$\mathcal{O}_k$: the ring of algebraic integers of $k$.

$r_1$: the number of embeddings $k \to \mathbb{R}$
$2r_2$: the number of nonreal embeddings $k \to \mathbb{C}$

$[k : \mathbb{Q}] = r_1 + 2r_2$
$\mathcal{O}_k^*$ has rank $r = r_1 + r_2 - 1$

Let $\sigma_1, \ldots, \sigma_{r+1}$ be the embeddings of $k$ into $\mathbb{C}$ up to complex conjugation.
If $u_1, \ldots, u_r$ form a $\mathbb{Z}$-basis of $\mathcal{O}_k^*/$torsion, let

$$R = \frac{2^{r_2}}{[k : \mathbb{Q}]} \begin{vmatrix}
1 & \log |\sigma_1(u_1)| & \ldots & \log |\sigma_1(u_r)| \\
\vdots & \vdots & \ddots & \vdots \\
1 & \log |\sigma_{r+1}(u_1)| & \ldots & \log |\sigma_{r+1}(u_r)|
\end{vmatrix}$$

$$\zeta_k(s) = \sum_{I \text{ an ideal of } \mathcal{O}_k} (\#\mathcal{O}_k/I)^{-s} \prod_{\mathcal{O}_k \text{ prime ideal}} \frac{1}{1 - (\#\mathcal{O}_k/\mathcal{P})^{-s}}$$

$$\text{Res}_{s=1} \zeta_k(s) = \frac{2^{r_1}(2\pi)^{r_2} R |\text{Cl}(\mathcal{O}_k)|}{w\sqrt{\Delta_k}}$$

$\Delta_k =$ the absolute value of the discriminant of $k$.
$w = |\mathcal{O}_k^{*,\text{tor}}| = \#\text{roots of unity in } k$. 

2
\[ K_0(\mathcal{O}_k) \cong \mathbb{Z} \oplus \text{Cl}(\mathcal{O}_k) \]
\[ K_1(\mathcal{O}_k) \cong \mathcal{O}^*_k \]
\[ |\text{Cl}(\mathcal{O}_k)| = |K_0(\mathcal{O}_k)_{\text{tor}}| \]
\[ w = |K_1(\mathcal{O}_k)_{\text{tor}}| \]

If \( F \) is a field, then
\[ K_0(F) \cong \mathbb{Z} \]
\[ K_1(F) \cong F^* = F \setminus \{0\} \]
\[ K_2(F) \cong F^* \otimes_{\mathbb{Z}} F^*/ < x \otimes (1 - x), x \in F^* \setminus \{1\} > . \]

The class of \( a \otimes b \) in \( K_2(F) \) is denoted \( \{a, b\} \), so \( K_2(F) \) is generated by symbols \( \{a, b\} \) with \( a, b \) in \( F^* \), and rules
\[ \{a_1a_2, b\} = \{a_1, b\} + \{a_2, b\} \]
\[ \{a, b_1b_2\} = \{a, b_1\} + \{a, b_2\} \]
\[ \{x, 1 - x\} = 0. \]

It follows that \( \{a, b\} + \{b, a\} = \{x, -x\} = 0. \)

\( F \) a field. Then for \( n \geq 1 \)
\[ K_n(F)_\mathbb{Q} = K_n^{(1)}(F) \oplus K_n^{(2)}(F) \oplus \ldots \oplus K_n^{(n)}(F) \]
and a similar decomposition for \( K_n(X)_\mathbb{Q} \) for a reasonable scheme \( X \).
[Here and elsewhere, \( A_\mathbb{Q} = A \otimes \mathbb{Z} \mathbb{Q} \).]
Borel’s theorem

$k$: number field
$K_n(\mathcal{O}_k)$ is finitely generated for all $n \geq 0$.
$m_n$ = the rank of $K_n(\mathcal{O}_k)$.

**Theorem (Borel)** $K_{2n}(\mathcal{O}_k)$ is a finite group if $n \geq 1$.
$K_{2n-1}(\mathcal{O}_k)$ has rank $m_{2n-1} = r_1 + r_2$ if $n$ is odd, and has
rank $m_{2n-1} = r_2$ if $n$ is even ($n \geq 2$).
Furthermore, there exists a natural regulator map

$$K_{2n-1}(\mathcal{O}_k) \to \mathbb{R}^{m_{2n-1}}.$$ 

The image is a lattice with volume $V_n$ of a fundamental domain

$$V_n \sim_{\mathbb{Q}^*} \frac{\zeta_k(n)}{\pi^n([k:\mathbb{Q}] - m_{2n-1}) \sqrt{\Delta_k}}$$

where $\Delta_k$ is the absolute value of the discriminant of $k$.

[ $a \sim_{\mathbb{Q}^*} b$ means $a = qb$ for some $q$ in $\mathbb{Q}^*$ .]

**Example** $\zeta_\mathbb{Q}$ is the Riemann zeta function. For $n \geq 2$:
$K_{2n-1}(\mathbb{Z})$ is finite for $n$ even;
$K_{2n-1}(\mathbb{Z})$ has rank 1 for $n$ odd, and $V_n \sim_{\mathbb{Q}^*} \zeta(n)$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_{2n-1}$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>...</td>
</tr>
<tr>
<td>$\zeta(n)$</td>
<td>$\pi^2/6$</td>
<td>irrat.</td>
<td>$\pi^4/90$</td>
<td>???</td>
<td>$\pi^6/945$</td>
<td>???</td>
<td>...</td>
</tr>
</tbody>
</table>
Curves.

$E/\mathbb{Q}$ an elliptic curve.
$E_{\mathbb{C}}$ the extension of the coefficients to $\mathbb{C}$.
$F$ the field of meromorphic functions on $E_{\mathbb{C}}$.

Exact localization sequence

\[ K_2(E_{\mathbb{C}}) \xrightarrow{T} K_2(F) \xrightarrow{T} \prod_{x \in E_{\mathbb{C}}} \mathbb{C}^* \]

$T$ is the tame symbol. With $\text{ord}_x(f)$ the order of vanishing of $f$ at $x$, $T_x$ is given by:

\[ \{f, g\} \mapsto (-1)^{\text{ord}_x(f) \text{ord}_x(g)} \frac{f^{\text{ord}_x(g)}}{g^{\text{ord}_x(f)}}|_x. \]

For $f$ and $g$ in $F^*$, put $\eta(f, g) = \log |f| \text{d} \arg g - \log |g| \text{d} \arg f$, a closed 1-form on an open part of $E_{\mathbb{C}}$.

\[ \log |z| \text{d} \arg (1 - z) - \log |1 - z| \text{d} \arg z = \text{d}P_2(z), \]

$P_2(z)$ a $C^\infty$-function on $\mathbb{C} \setminus \{0, 1\}$

\[ \text{reg}: K_2(F) \rightarrow \left\{ \begin{array}{l} \text{closed 1-forms on open parts} \\ \text{exact 1-forms on open parts} \end{array} \right\} \]

\[ \{f, g\} \mapsto \eta(f, g) \]
This fits into a commutative diagram

\[
\begin{array}{cccccc}
K_2(E) & \longrightarrow & K_2(E_\mathbb{C}) & \longrightarrow & K_2(F) & \longrightarrow & \prod_{x \in E_\mathbb{C}} \mathbb{C}^* \\
\downarrow^{reg} & & \downarrow^{reg} & & \downarrow^{log|\cdot|} & & \\
0 & \longrightarrow & H^1_{dR}(E_\mathbb{C}; \mathbb{R}) & \longrightarrow & H^1_{dR}(F; \mathbb{R}) & \longrightarrow & \prod_{x \in E_\mathbb{C}} \mathbb{R}
\end{array}
\]

\[H^1_{dR}(F; \mathbb{R}) = \lim_{\to} H^1_{dR}(U; \mathbb{R}), \text{ U s.t. } E_\mathbb{C} \setminus U \text{ is finite.}\]

**Theorem (Bloch)** \(E\) an elliptic curve over \(\mathbb{Q}\) with complex multiplication. For some \(\alpha\) in \(K_2(E)\),

\[L'(E, 0) \sim_{\mathbb{Q}^*} \frac{1}{2\pi} \int_{E(\mathbb{C})} \text{reg}(\alpha) \wedge \omega\]

or, using the functional equation for the \(L\)-function:

\[\frac{1}{2\pi} L(E, 2) \sim_{\mathbb{Q}^*} \int_{E(\mathbb{C})} \text{reg}(\alpha) \wedge \omega.\]

[\(\omega\) a holomorphic form on \(E_\mathbb{C}\) with \(\int_{E(\mathbb{R})} \omega = 1\).]
Getting a hold on higher $K$-groups.

"Algebraic $K$-theory is a functor that associates to your favourite exact category Abelian groups $K_n \ (n \geq 0)$, about which you know nothing."

Let $F$ be an infinite field, and write $F_Q^*$ for $F^* \otimes \mathbb{Q}$. $B_n(F)$: a free $\mathbb{Q}$-vector space on $[x]_n$, $x$ in $F$, $x \neq 0, 1$, modulo some inductively defined relations.

Complex $\Gamma(F, n)$ in degrees $1, \ldots, n$ for $n \geq 2$:

$$B_n(F) \rightarrow B_{n-1}(F) \otimes F_Q^* \rightarrow \ldots \rightarrow B_2(F) \otimes \bigwedge^{n-2} F_Q^* \rightarrow \bigwedge^n F_Q^*$$

$$d[x]_l \otimes y_1 \wedge \ldots \wedge y_{n-l} = [x]_{l-1} \otimes x \wedge y_1 \wedge \ldots \wedge y_{n-l} (l \geq 3)$$

$$d[x]_2 \otimes y_1 \wedge \ldots \wedge y_{n-2} = (1-x) \wedge x \wedge y_1 \wedge \ldots \wedge y_{n-2}$$

$C$ is a complete nonsingular curve over an infinite field $k$, $F = k(C)$: the field of rational functions on $C$.

$\Gamma(n, C)$: total complex associated to double complex

$$\begin{array}{ccc}
B_n(F) & \rightarrow & B_{n-1}(F) \otimes F_Q^* \\
\downarrow & & \downarrow \\
0 & \rightarrow & \coprod B_{n-1}(k(x)) \\
\end{array}$$

with the coproduct over the closed points $x$ in $C$. The vertical maps are "based on" $[f]_m \otimes g \mapsto \text{ord}_x(g) \cdot [f(x)]_m$ with $[0]_m = [\infty]_m = 0$.  

7
Conjecture (Zagier): \( k \) a number field. Then for \( n \geq 2 \),

\[
K_{2n-1}(k)_\mathbb{Q} \cong H^1(\Gamma(k, n)) = \text{Ker}(d_n),
\]

with

\[
d_n : B_n(k) \to B_{n-1}(k) \otimes k^*_\mathbb{Q} \\
[x]_n \mapsto [x]_{n-1} \otimes x
\]

for \( n \geq 3 \), and

\[
d_2 : B_2(k) \to \bigwedge^2 k^*_\mathbb{Q} \\
[x]_2 \mapsto (1 - x) \wedge x
\]

together with a formula for the regulator in terms of polylogarithms on \( \text{Ker}(d_n) \).

Conjecture (Goncharov) \((n \geq 2)\)

1. \( H^p(\Gamma(n, F)) \cong K^{(n)}_{2n-p}(F) \) if \( F \) is an infinite field.
2. \( H^p(\Gamma(n, C)) \cong K^{(n)}_{2n-p}(C) \) if \( C \) is a complete smooth curve over an infinite field \( k \).

Theorem

(i) (Deligne-Beilinson, RdJ) There is an injection

\[
H^1(\Gamma(n, k)) \to K_{2n-1}(k)_\mathbb{Q}
\]

with the expected formula for the regulator.

(ii) (Suslin/Goncharov) For \( n = 2 \) or 3 it is also surjective.

(iii) (Zagier) It is also surjective if \( k \) is a cyclotomic field.
**Theorem (RdJ)** Let $C$ be a complete, smooth, geometrically irreducible curve over a number field $k$, $F = k(C)$. Then there exist complexes $\widetilde{\mathcal{M}}_{(n)}(F)$ and $\widetilde{\mathcal{M}}_{(n)}(C)$ similar to Goncharov’s (with $B_n$ replaced by $\widetilde{M}_n$, also generated by $[x]_n$’s), with maps to the $K$-theory as follows.

1. $n = 3$, $p \geq 2$; in particular,
   \[
   H^2(\widetilde{\mathcal{M}}_{(3)}(F')) \to K_4^{(3)}(F)
   \]
   and
   \[
   H^2(\widetilde{\mathcal{M}}_{(3)}(C')) \to K_4^{(3)}(C') + K_3^{(2)}(k) \cup F^*/K_3^{(2)}(k) \cup F^*.
   \]
   [Note: if $k$ is totally real, then $K_3^{(2)}(k) = 0$.]

2. $n = 4$, $p \geq 3$, and for $p = 2$ more or less:
   \[
   H^2(\widetilde{\mathcal{M}}_{(4)}(F')) \leftrightarrow H^2(\mathcal{M}_{(4)}(F')) \to \frac{K_6^{(4)}(F)}{K_4^{(2)}(F) \cup K_2^{(2)}(F)}
   \]
   and
   \[
   H^2(\widetilde{\mathcal{M}}_{(4)}(C')) \leftrightarrow H^2(\mathcal{M}_{(4)}(F)) \to \frac{K_6^{(4)}(C') + K_4^{(2)}(F) \cup K_2^{(2)}(F) + K_5^{(3)}(k) \cup F^*}{K_4^{(2)}(F) \cup K_2^{(2)}(F) + K_5^{(3)}(k) \cup F^*}.
   \]
   $[\widetilde{\mathcal{M}}_{(n)}(F')$ is a quotient complex of a rather similar complex $\mathcal{M}_{(n)}(F)$, and similarly for $\widetilde{\mathcal{M}}_{(n)}(C)$.]

9
Theorem (RdJ, also using results by Goncharov)  
Let $C$ be as above. Then

$$H^2(\Gamma(3, C)), \ H^2(\widetilde{\mathcal{M}}(3)(C)) \text{ and } K_4^{(3)}(C)$$

all have the same image in $H^1_{dR}(C \otimes \mathbb{Q} \mathbb{C}; \mathbb{R}(2))^+$ under the regulator. Similarly,

$$H^2(\Gamma(4, C)), \ H^2(\mathcal{M}(4)(C)), \ H^2(\widetilde{\mathcal{M}}(4)(C)) \text{ and } K_6^{(4)}(C)$$

all have the same image in $H^1_{dR}(C \otimes \mathbb{Q} \mathbb{C}; \mathbb{R}(3))^+$ under the regulator.

[Here and elsewhere, $\mathbb{R}(n) = (2\pi i)^n \mathbb{R} \subset \mathbb{C}$.]

Polylogarithms

$L_i n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$ for $n \geq 1$ on $|z| < 1$.  
$L_i 1(z) = -\log(1 - z)$

Using $dL_i n+1(z) = L_i n(z) d \log z$ for $n \geq 1$, $L_i n(z)$ extends to a multivalued function on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

Single valued versions

$(\pi_{n-1} : \mathbb{C} = \mathbb{R}(n) \oplus \mathbb{R}(n-1) \rightarrow \mathbb{R}(n-1) \text{ projection})$:

$$P_n(z) = \pi_{n-1} \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \log^k |z| L_i n-k(z)$$

$$P_n,\text{Zag}(z) = \pi_{n-1} \sum_{k=0}^{n-1} \frac{2^k B_k}{k!} \log^k |z| L_i n-k(z)$$

[$B_k$: Bernoulli number.]

Then $P_n,\text{Zag}(z) + (-1)^n P_n,\text{Zag}(z^{-1}) = 0$.  

10
Formulae for the regulator

(1) $k$ a number field. [As $K_n(O_k)_\mathbb{Q} = K_n(k)_\mathbb{Q}$ for $n \geq 2$, this provides information about Borel’s theorem.]

$H^1(\tilde{\mathcal{M}}_n(k)) \hookrightarrow K_{2n-1}^+(k) \to (\mathbb{R}(n-1)\sigma)^+_\sigma: k \to \mathbb{C}$

maps $[x]_n$ to $\pm((n-1)! \cdot P_{n,Zag}(\sigma(x)))_\sigma$ ($n \geq 2$).

(2) $C$ complete, smooth, geometrically irreducible curve over a number field $k$, $F = k(C)$. Fix $\omega$ in $H^0(C \otimes \mathbb{Q} \mathbb{C}, \Omega^1_C)^+$ (+: certain invariance w.r.t. complex conjugation).

\[
H^2(\tilde{\mathcal{M}}_3(F)) \cong H^1_{dR}(F \otimes \mathbb{Q} \mathbb{C}; \mathbb{R}(2)^+) \xrightarrow{\int \cdot \wedge \omega} \mathbb{R}(1)
\]

is

\[
[f]_2 \otimes g \mapsto \pm \frac{8}{3} \int_{C \otimes \mathbb{Q} \mathbb{C}} \log |g| \eta(f, 1 - f) \wedge \overline{\omega}
\]

or \( \pm \frac{8}{3} \int_{C \otimes \mathbb{Q} \mathbb{C}} P_2 \circ f \, d \log |g| \wedge \overline{\omega} \)

with $\eta(h_1, h_2) = \log |h_1| di \arg h_2 - \log |h_2| di \arg h_1$.

For $H^2(\tilde{\mathcal{M}}_4(F'))$ we get

\[
[f]_3 \otimes g \mapsto \pm 6 \int_{C \otimes \mathbb{Q} \mathbb{C}} \log |g| \log |f| \eta(f, 1 - f) \wedge \overline{\omega}.
\]
Coleman integration

$$\mathbb{C}_p = \hat{\mathbb{Q}}_p$$

$|\cdot|$: $p$-adic valuation with $|p| = p^{-1}$

$\mathcal{O}$: ring of integers of $\mathbb{C}_p$

$\mathbb{F}_p$: residue field

$X/\mathcal{O}$: smooth curve over $\mathcal{O}$ (=smooth projective surjective scheme of relative dimension 1)

For $x$ in $X(\mathbb{F}_p)$, put

$U_x = \text{residue disc of } x = \{\text{all pts in } X(\mathbb{C}_p) \text{ reducing to } x\}$, a copy of the maximal ideal of $\mathcal{O}$.

$Y \subseteq X_{\mathbb{F}_p}$ nonempty open affine subscheme, smooth over $\mathbb{F}_p$, so $X(\mathbb{F}_p) = Y(\mathbb{F}_p) \bigsqcup \{e_1, \ldots, e_n\}$.

$U_r = \text{rigid space obtained by removing discs of radius } r < 1 \text{ from } X(\mathbb{C}_p) \text{ for all } e_i: e_i \text{ locally given by } \overline{h} = 0$, so leave out $|h| \leq r$.

$U = \lim_{r \to 1} U_r$ is independent of the choices

Make a choice of logarithm $\log : \mathbb{C}_p^* \to \mathbb{C}_p$ such that

1. $\log ab = \log a + \log b$
2. $\log(1 + z) = \text{usual powerseries expansion for } |z| \text{ small}$.
   (I.e., fix a choice of $\log p$.)
For \( x \in Y(\overline{F}_p) \), put
\[
A(U_x) = \{ \sum_{n=0}^{\infty} a_n z^n \ \text{conv. for } |z| < 1 \}
\]
\[
A_{\log}(U_x) = A(U_x)
\]
\[
\Omega_{\log}(U_x) = A_{\log}(U_x) dz_x
\]
\([z = z_x \text{ is a local parameter on } U_x.]
\]
For \( x \notin Y(\overline{F}_p) \) (i.e., \( x = e_1, \ldots, e_n \)), put
\[
A(U_x) = \{ \sum_{n=-\infty}^{\infty} a_n z^n \ \text{conv. for some } r < |z| < 1 \}
\]
\[
A_{\log}(U_x) = A(U_x)[\log z]
\]
\[
\Omega_{\log}(U_x) = A_{\log}(U_x) dz_x
\]

Put
\[
A_{loc}(U) = \prod_{x \in X(\overline{F}_p)} A_{\log}(U_x)
\]
(locally analytic functions, with choice of logs around the \( e_i \))
\[
\Omega_{loc}(U) = \prod_{x \in X(\overline{F}_p)} \Omega_{\log}(U_x)
\]
(locally analytic forms, with choice of log around the \( e_i \))

\[
0 \to \prod_{x \in X(\overline{F}_p)} \mathbb{C}_p \to A_{loc}(U) \to \Omega_{loc}(U) \to 0
\]
is exact as \( d \log z = \frac{dz}{z} \).

**Coleman:** there exists a subspace \( A_{Col}(U) \) of \( A_{loc}(U) \), containing the rigid analytic functions \( A(U) \) on \( U \), such that with \( \Omega_{col}(U) = A_{Col}(U) \otimes \Omega^1(U/\mathbb{C}_p) \)

\[
0 \to \mathbb{C}_p \to A_{Col}(U) \to \Omega_{col}(U) \to 0
\]
is exact.

Let \( P \) and \( Q \) be in \( U \), \( \omega \) in \( \Omega_{col}(U) \), \( F_\omega \) in \( A_{Col}(U) \) with \( dF_\omega = \omega \). Put \( \int_P^Q \omega = F_\omega(Q) - F_\omega(P) \).
Example

\[ \begin{align*}
X &= \mathbb{P}^1_{\mathbb{C}_p} \\
Y &= \mathbb{P}^1_{\mathbb{F}_p} \setminus \{1, \infty\} \\
U &= \mathbb{C}_p \setminus U_1 \bigcup U_\infty.
\end{align*} \]

Put \( Li_{n+1}(z) = \int_0^z Li_n(z) \, d\log z \) starting with \( Li_0(z) = \frac{z}{1-z} \).

\[ Li_n(z) = \sum_{k=1}^\infty \frac{z^k}{kn} \quad \text{for } |z| < 1. \]

[In fact, \( Li_n(z) \) extends naturally to \( \mathbb{C}_p \setminus \{1\} \).]

\[ L_{\text{mod}n}(z) = \sum_{k=0}^{n-1} \alpha_k Li_{n-k}(z) \log^k z \]

satisfies \( L_{\text{mod}n}(z) + (-1)^n L_{\text{mod}n}(z^{-1}) = 0 \) for suitable \( \alpha_m \).

**Theorem (Besser and RdJ)** Let \( k \subset \mathbb{C}_p \) be a number field, and let \( n \geq 2 \). Then

\[ H^1(\widetilde{\mathcal{M}}(n)(k)) \to K_{2n-1}^{(n)}(k) = K_{2n-1}^{(n)}(\mathcal{O}_k) \xrightarrow{r_{\text{syn}}} \mathbb{C}_p \]

is

\[ [x]_n \mapsto (-1)^n (n-1)! \ L_{\text{mod}n}(x) \]

if \( x \) is a root of unity, or \( x \) is a special unit in \( \mathcal{O} \), i.e., both \( x \) and \( 1-x \) are units in \( \mathcal{O} = \mathcal{O}_{\mathbb{C}_p} \).
$C/\mathbb{C}_p$ smooth complete irreducible curve with good reduction. Coleman and de Shalit define

$$r_p : K_2(\mathbb{C}_p(C)) \to H^0(C, \Omega_{C/\mathbb{C}_p}^1)^\vee$$

$$\{f, g\} \mapsto \left[ \omega \mapsto \int_{(f)} \log(g) \cdot \omega \right]$$

(Here, if $(f) = \sum_j a_P(P)$, $\int_{(f)} \rho = \sum_j a_P F_p(P).$)

**Theorem (Coleman and de Shalit)** If $E/\mathbb{Q}$ is an elliptic curve with complex multiplication, $p$ a prime that splits in the CM field of $E$, then for the same $\alpha$ in $K_2(E)$ as in Bloch’s theorem, and the same $\omega$, $r_p(\alpha)(\omega) = a_\alpha \Omega_p L_p(E, 0)$ for the same $a_\alpha$ as for Bloch. [$\Omega_p$ is a $p$-adic period.]

[Cf. Bloch: $\int_{E(\mathbb{C})} r(\alpha) \cup \omega = a_\alpha \Omega L^*(E, 0)$, where $L^*(E, s)$ is the usual $L$-function multiplied by the $\Gamma$ factor.]

**Theorem (Besser)** $C/\mathbb{C}_p$ as above. $C/\mathcal{O}$ model of $C/\mathbb{C}_p$. Then $r_p$ equals $[r_{\text{syn}}$ the syntomic regulator]

$$K_2(C') = K_2(C) \xrightarrow{r_{\text{syn}}} H^1_{dR}(C/\mathbb{C}_p)^{\text{Tr}(f \cup \omega)} \to \mathbb{C}_p.$$ 

**Theorem (Besser and RdJ)** $C$ as above, defined over a number field $k \subset \mathbb{C}_p$. $\mathcal{C}$: a model of $C$ over $k \cap \mathcal{O}$. Then

$$H^2(\tilde{\mathcal{M}}_{(3)}(C)) \to K_4^{(3)}(C) \xrightarrow{r_{\text{syn}}} K_4^{(3)}(\mathcal{C}) \xrightarrow{\text{Tr}(\cdot \cup \omega)} H^1_{dR}(C/\mathbb{C}_p) \to \mathbb{C}_p$$

is

$$[f]_2 \otimes g \mapsto 2 \int_{(g)} L_2(f) \cdot \omega,$$

provided that $f$, $1 - f$ and $g$ do not have a zero or pole along the special fibre of $C$.

$[L_2(z) = Li_2(z) + \log(z) \cdot \log(1 - z).]$
For $K^{(3)}_4(F)$ and $K^{(3)}_4(C)$ we are looking at

\[
\begin{array}{c}
\tilde{M}_3(F) \xrightarrow{\delta} \tilde{M}_2(F) \otimes \mathbb{Q} F^\ast_P \xrightarrow{d} \bigwedge^3 F^\ast_P \\
0 \xrightarrow{\delta} \coprod \tilde{M}_2(k(x)) \xrightarrow{\delta} \coprod \bigwedge^2 k(x)^\ast_P
\end{array}
\]

with
\[d[f]_3 = [f]_2 \otimes f\]
\[d[f]_2 \otimes g = (1 - f) \wedge f \wedge g\]
\[\delta_x[f]_2 \otimes g = \text{ord}_x(g) \cdot [f(x)]_2.\]

(Coproduct over all (closed) points $x$ in $C$.)

16