K_2 of elliptic curves over non-Abelian cubic and quartic fields

Joint work with
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Borel, Bloch, Beilinson

Borel (1977) (+Quillen+Soulé) For a number field F with ring of algebraic integers $\mathcal{O}_F$, and $n \geq 2$, $K_{2n-1}(F) = K_{2n-1}(\mathcal{O}_F)$ is finitely generated. Using a regulator based on continuous group cohomology of $\mathrm{GL}(\mathbb{C})$, as well as the embeddings of $F \to \mathbb{C}$, he defined a regulator $R_{n,F}$ for $K_{2n-1}(\mathcal{O}_F)$ and showed a relation between it and $\zeta_F(n)$.

Bloch (1978). For CM elliptic curves over $\mathbb{Q}$ he defined an element in $K_2(E) \otimes_{\mathbb{Z}} \mathbb{Q}$ and related an ad hoc regulator of it with $L(E, 2)$.

Beilinson (∼1985) Defined a theory of regulators for the K-groups of regular projective varieties over $\mathbb{Q}$, and conjectured relations with the L-functions at certain points. Using the image of the K-group of a regular proper model of the variety if it exists; nowadays use alterations (Scholl).

Some evidence for Beilinson's conjectures for curves

Constructing as many elements as Beilinson predicts, and relating their regulator to the L-value is done by, for example:

- Beilinson (K_{2n} of modular curves, $n \geq 1$; 1986),
- Deninger (K_{2n} of certain CM elliptic curves over number fields, $n \geq 1$; 1989),
- dJ (K_4 of $y^2 = x^3 - 2x^2 + 1$ over $\mathbb{Q}$, numerical relation with $L^*(E, -1)$; 1996)
- Dokchitser-dJ-Zagier (K_2 of hyperelliptic curves over $\mathbb{Q}$, numerically; 2006)
- Ito (K_2 of three elliptic curves over $\mathbb{Q}$; 2018)
- Asakura (K_2 of some elliptic curves over $\mathbb{Q}$, either theoretically or numerically; 2018)
- Brunault (for K_2 of strongly modular curves over Abelian number fields; 2018)
- Brunault (for K_4 of all elliptic curves over $\mathbb{Q}$ of conductor at most 50, numerically; preprint 2020/2022)

'Integrality' for curves; choice of model

L any field: $K_2(L) = L^\times \otimes_{\mathbb{Z}} L^\times / \langle l \otimes (1-l) \rangle$; $\{l_1, l_2\}$ = class of $l_1 \otimes l_2$

For a regular, proper, irreducible curve C over the number field F, fix a regular, flat, proper model $\mathcal{C}$ of C over the ring of algebraic integers $\mathcal{O}_F$ of F. Then we define

$$K_2^T(C) = \ker\left(K_2(F(C)) \xrightarrow{\prod T_P} \oplus_{P \in C^{(1)}} F(P)^\times\right)$$

$$K_2^T(C)_{\text{int}} = \ker\left(K_2(F(C)) \xrightarrow{\prod T_{\mathcal{D}}} \oplus_{\mathcal{D}} \mathbb{F}(\mathcal{D})^\times\right)$$

$\mathcal{D}$: an irreducible curve on $\mathcal{C}$; $\mathbb{F}(\mathcal{D})$: the residue field at $\mathcal{D}$.

$$T_{\mathcal{D}} : \{a, b\} \mapsto (-1)^{v_{\mathcal{D}}(a) v_{\mathcal{D}}(b)} \frac{a^{v_{\mathcal{D}}(b)}}{b^{v_{\mathcal{D}}(a)}}(\mathcal{D}) \quad \text{tame symbol for } \mathcal{D}$$

where $v_{\mathcal{D}}$ is the valuation on $F(C)$ corresponding to $\mathcal{D}$.

$K_2^T(C)_{\text{int}}$ is the subgroup of $K_2^T(C)$ consisting of integral elements.

Theorem (Liu-dJ, 2015)

The subgroup $K_2^T(C)_{\text{int}}$ is the image of $K_2(\mathcal{C})$ in $K_2(F(C))$ under localisation, and does not depend on the choice of $\mathcal{C}$.

The Beilinson regulator for K_2 of curves

As a starter,

- $C/\mathbb{C}$ be a regular, proper curve
- $\alpha = \sum_j \{f_j, g_j\}$ in $K_2^T(C)$
- γ in $H_1(C(\mathbb{C}), \mathbb{Z})$
- their regulator pairing is (well-)defined by

$$\langle \gamma, \alpha \rangle = \frac{1}{2\pi} \int_\gamma \sum_j \eta(f_j, g_j)$$

with $\eta(f, g) = \log|f| \mathrm{d}\arg(g) - \log|g| \mathrm{d}\arg(f)$ for non-zero functions f and g on C; we use a representative of γ that avoids all zeroes and poles of the functions involved.

Beilinson regulator for K_2 of curves (continued)

As main course,

• C a regular, proper, geometrically irreducible curve over a number field F of degree m, of genus g; let $n = mg$

• X the Riemann surface consisting of all $\mathbb{C}$-valued points of C, a disjoint union of the complex points of m curves C^σ over $\mathbb{C}$, indexed by the embeddings σ of F into $\mathbb{C}$. Complex conjugation acts through its action on $\mathbb{C}$, and $H_1(X,\mathbb{Z})^- \simeq \mathbb{Z}^n$

• Define a pairing

$$\begin{aligned} H_1(X,\mathbb{Z}) \times K_2^T(C) &\to \mathbb{R} \\ (\gamma,\alpha) &\mapsto \langle \gamma,\alpha\rangle_X = \sum_\sigma \langle \gamma_\sigma, \alpha^\sigma\rangle \end{aligned}$$

if $\gamma = (\gamma_\sigma)_\sigma$ in $H_1(X,\mathbb{Z}) = \oplus_\sigma H_1(C^\sigma(\mathbb{C}),\mathbb{Z})$, α^σ the pullback of α to C^σ.

Beilinson's conjecture for K_2 of curves (continued)

Assume $L(C,s)$ can be analytically continued to the complex plane and satisfies a functional equation for s versus $2-s$ as in the Hasse-Weil conjecture.
Then $L(C,s)$ should have a zero of order n at $s=0$, and we let $L^*(C,0)=(n!)^{-1}L^{(n)}(C,0)$ be the first non-vanishing coefficient in its Taylor expansion in s at 0.

Conjecture

- *Let $\gamma_1,\ldots,\gamma_n$ and $\alpha_1,\ldots,\alpha_n$ form $\mathbb{Z}$-bases of $H_1(X,\mathbb{Z})^-$ and $K_2^T(C)_{\text{int}}$ modulo torsion respectively borrowing finite generation of $K_2^T(C)_{\text{int}}$ from Bass's conjecture*

Let the Beilinson regulator of the α_j be $R=|\det(\langle\gamma_i,\alpha_j\rangle_X)_{i,j}|$. Then

$$L^*(C,0)=Q\cdot R$$

for some Q in $\mathbb{Q}^\times$

They considered (hyper)elliptic curves C of genus $g \geq 1$ defined by, e.g.,

$$y^2 + f(x)y + x^{2g+2} = 0$$

where $f(x) = 2x^{g+1} + b_g x^g + \cdots + b_1 x + b_0$; b_i in $\mathbb{Q}$, $b_g \neq 0$, and $-x^{2g+2} + f(x)^2/4$ has no multiple roots

For $f(x) = 2x^{g+1} \pm (v_1 x + 1) \ldots (v_g x + 1)$ with the v_i integers, all $\{\frac{y^2}{x^{2g+2}}, v_i x + 1\}$ are in $K_2^T(C)_{\text{int}}$

The Beilinson conjecture was verified numerically for many curves in the above (and other) families.

(dJ) Limit results for the Beilinson regulator for fixed $v_1, \ldots, v_{g-1}$ and $|v_g| \to \infty$, which imply linear independence for $|v_g| \gg 0$.

More general higher genus curves over $\mathbb{Q}$ (Liu-dJ, 2015)

Define C as the normalisation of the projective closure of

$$\prod_{i=1}^{N} \prod_{j=1}^{N_j} L_{i,j} = t$$

with $L_{i,j} = a_i x + b_i y + c_{i,j}$ distinct, non-parallel for distinct i.
If C has regular affine part then C has genus
$g = 1 - \sum_{1 \le i \le N} N_i + \sum_{1 \le i < j \le N} N_i N_j$
Then $K_2^T(C)$ contains 'rectangular' and 'triangular' elements

- $\left\{ \frac{L_{i,j}}{L_{i,k}}, \frac{L_{l,m}}{L_{l,n}} \right\}$ $(i \neq l)$
- $\left\{ \frac{[i,m] L_{k,l}}{[k,m] L_{i,j}}, \frac{[i,k] L_{l,m}}{[m,k] L_{i,j}} \right\}$ $(i, k, m$ distinct; $[i,k] = a_i b_k - a_k b_i)$ with

some relations, giving at most g independent elements.

More general higher genus curves over $\mathbb{Q}$ (Liu-dJ, 2015)

Theorem

If no three of the $L_{i,j}$ pass through an affine point and the a_i, b_i and $c_{i,j}$ are real, then there are $\alpha_1, \dots, \alpha_g$ among the rectangular and/or linear elements with $\lim_{t \to 0} \frac{R(\alpha_1, \dots, \alpha_g)}{|\log^g |t||} = 1$

Theorem

If the defining equation is

$$\lambda \prod_{i=1}^{N_1} (x + a_i) \prod_{j=1}^{N_2} (y + b_j) \prod_{k=1}^{N_3} (y - x + c_k) = 1$$

($N_1 \geq N_2 \geq N_3 \geq 0$, $N_2 \geq 1$) with λ, a_i, b_j and c_k algebraic integers, then all rectangular and triangular elements are integral.

Example

For fixed integers a_i, b_j, c_k this gives g linearly independent elements in $K_2^T(C)_{\text{int}}$ if λ is an integer with $|\lambda| \gg 0$. C is not hyperelliptic when $N_2 + N_3 > 2$.

Some special cubic number fields

Now on to the joint work with Brunault, Liu, and Fernando Villegas

We need exceptional units in (hence special) cubic number fields.

Lemma

For every integer a, and all $\varepsilon, \varepsilon'$ in $\{\pm 1\}$

$$f_a(X) = X^3 + aX^2 - (a + \varepsilon + \varepsilon' + 1)X + \varepsilon$$

is irreducible in $\mathbb{Q}[X]$.

A cubic field F has an element u such that $F = \mathbb{Q}(u)$ and both u and $1-u$ are in $\mathcal{O}_F^\times$ precisely when u is a root of some $f_a(X)$.

$F/\mathbb{Q}$ is cyclic if and only if $\varepsilon = \varepsilon' = 1$ or $|2a - \varepsilon + \varepsilon' + 3| = 7$.

- For $\varepsilon = \varepsilon' = 1$ we get the simplest cubic fields (Shanks).
- The fields are totally real for $|a| \gg 0$
- $u \mapsto 1-u$ and $u \mapsto u^{-1}$ generate some identifications; we end up with two 'half' families.

First construction

Theorem

Let $F = \mathbb{Q}(u)$ with u a root of $f_a(X)$ as for the special cubic families. Let $p = u - 1$, $h(x) = (p^2 + p + 1)x + p^2(p+1)^2$. Then for $\lambda = 1$, 2, 3 or 4 the normalisation of the curve defined by

$$y^2 + \left(2x^3 + \lambda h(x)^2\right)y + x^6 = 0$$

with a few exceptions is an elliptic curve E. The elements

$$M = \left\{-\frac{y}{x^3}, \frac{h(x)}{h(0)}\right\} \qquad M_q = C_\lambda \left\{-\frac{y}{x^3}, q^{-2}x + 1\right\}$$

$(q = p, p+1, p(p+1);\ C_1 = 6,\ C_2 = 4,\ C_3 = 3,\ C_4 = 2)$ are in $K_2^T(E)_{\mathrm{int}}$ and satisfy $2C_\lambda M = \sum_q M_q$.

The Beilinson regulator $R = R(a)$ of the M_q satisfies

$$\lim_{|a|\to\infty} \frac{R(a)}{\log^3|a|} = 16C_\lambda^3$$

For $\lambda \neq 4$ the support of the divisor $(q^{-2} + 1)$ is in general not contained in the torsion of E.

First construction (numerical results)

$f_a(X) = X^3 + aX^2 - (a+1)X + 1$, $a \geq 0$, one of the special cubic families F is non-Abelian for $a \neq 3$

- $\widetilde{Q}$: rational number in the Beilinson conjecture for M_p, M_{p+1}, M
- d: discriminant of F • c: conductor norm of E

Data for $\lambda = 1$ red: F not totally real

a	d	c	$L^*(E,0)$	$\widetilde{Q}$
0	-23	$2^3 \cdot 17 \cdot 107$	132.724179260406391	$2^{-4} \cdot 3$
1	-31	$2^3 \cdot 3^4 \cdot 17$	168.814511547175067	$2^{-4} \cdot 3$
2	-23	$2^3 \cdot 19 \cdot 37$	53.401946956784239	2^{-4}
3	7^2	$2^3 \cdot 127$	37.1776384769406512	$2^{-4} \cdot 3 \cdot 7^{-1}$
4	257	$2^3 \cdot 3^4$	-721.242054102691853	$-2^{-3} \cdot 3$
5	$17 \cdot 41$	$2^3 \cdot 19$	1414.02549043158906	$2^{-2} \cdot 3$
6	1489	$2^3 \cdot 17 \cdot 19$	83163.7726064265207	$2 \cdot 3^3$
7	2777	$2^3 \cdot 3^4 \cdot 37$	2915249.85675393311	$2^2 \cdot 3^3 \cdot 13$
8	4729	$2^3 \cdot 71 \cdot 163$	33679082.6389894579	$2 \cdot 3^3 \cdot 241$
9	7537	$2^3 \cdot 37 \cdot 863$	260954243.280987485	$2 \cdot 3^3 \cdot 1567$

Data for $\lambda = 2$ and $\lambda = 3$ red: F not totally real

a	d	c	$L^*(E,0)$	$\widetilde{Q}$
0	-23	$2^6 \cdot 11 \cdot 23 \cdot 37$	4486.8160562777558	$2^{-1} \cdot 3 \cdot 5$
1	-31	$2^6 \cdot 3^3 \cdot 11 \cdot 13$	3599.55769844723823	$2^{-1} \cdot 3^2$
2	-23	$2^6 \cdot 5^2 \cdot 59$	837.55573566513198	2
3	7^2	$2^6 \cdot 13 \cdot 83$	−2498.99534192761051	-3
4	257	$2^6 \cdot 3^3 \cdot 37$	−64543.3050825583931	$-2^2 \cdot 3^3$
5	$17 \cdot 41$	$2^6 \cdot 11^2 \cdot 13 \cdot 23$	−16392164.6852019715	$-2^2 \cdot 3^4 \cdot 53$
6	1489	$2^6 \cdot 23 \cdot 47 \cdot 179$	437520185.347094640	$2^5 \cdot 3^2 \cdot 1187$

a	d	c	$L^*(E,0)$	$\widetilde{Q}$
0	-23	$2^3 \cdot 3^9 \cdot 19$	25300.9847248343307	$3 \cdot 17$
1	-31	$2^3 \cdot 3^{11}$	−21806.9954627600874	$-2^2 \cdot 3 \cdot 5$
2	-23	$2^3 \cdot 3^9 \cdot 17$	−21113.3123276958079	$-2^2 \cdot 3 \cdot 5$
3	7^2	$2^3 \cdot 3^9$	5601.39536780219401	$2^2 \cdot 3$
4	257	$2^3 \cdot 3^{11} \cdot 19$	−26042785.9143510709	$-2^3 \cdot 3^3 \cdot 233$

Data for $\lambda = 4$ red: F not totally real

a	d	c	$L^*(E,0)$	$\widetilde{Q}$
0	-23	$2^6 \cdot 5 \cdot 7$	19.1718016489393019	2^{-3}
1	-31	$2^6 \cdot 3^2$	8.95758063575193728	$2^{-3} \cdot 3^{-1}$
2	-23	$2^6 \cdot 5 \cdot 11$	-25.4138019939166741	-2^{-3}
3	7^2	$2^6 \cdot 7 \cdot 13$	241.273298483854998	$2^{-1} \cdot 3$
4	257	$2^6 \cdot 3^2 \cdot 5$	-2647.23969149488937	-3^2
5	$17 \cdot 41$	$2^6 \cdot 5 \cdot 11 \cdot 17$	441097.703795075666	$2^3 \cdot 3^3 \cdot 5$
6	1489	$2^6 \cdot 7 \cdot 13 \cdot 19$	-4149007.28165801473	$-2^7 \cdot 3^2 \cdot 7$
7	2777	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2423760.93043136419	$2 \cdot 3^3 \cdot 73$
8	4729	$2^6 \cdot 11 \cdot 17 \cdot 23$	99044008.9977606699	$2^7 \cdot 3^2 \cdot 11^2$
9	7537	$2^6 \cdot 5 \cdot 13 \cdot 19$	66308672.9214609161	$2^5 \cdot 3^2 \cdot 7 \cdot 41$
10	$7^2 \cdot 233$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	41156246.2610705047	$2^4 \cdot 3^3 \cdot 107$

A new integrality criterion

Proposition

Let C be a regular, projective, geometrically irreducible curve over a number field F, with regular, flat, proper model $\mathcal{C}$ over the ring of algebraic integers $\mathcal{O}_F$. Suppose f, g in $F(C)^\times$ are such that $(f) = N(P) - N(O)$, $(g) = N(Q) - N(O)$ for some $N \geq 1$ and distinct F-rational points O, P and Q on C, and $f(Q) = g(P) = 1$. Then $\alpha = \{f, g\}$ is in $K_2^T(C)$.

Let $\mathfrak{P}$ be a maximal ideal of $\mathcal{O}_F$, with residue field k, fibre $\mathcal{F} = \mathcal{C}_{\mathfrak{P}}$.

- *If O, P and Q all hit the same irreducible component of $\mathcal{F}$, then $T_{\mathcal{D}}(\alpha) = 1$ for all $\mathcal{D}$ in $\mathcal{F}$.*
- *If O, P and Q hit two irreducible components of $\mathcal{F}$, then $T_{\mathcal{D}}(\alpha)$ is a constant function on $\mathcal{D}$ for every irreducible component $\mathcal{D}$ of $\mathcal{F}$. If M is the order of the image of $\varepsilon = (-g/f)(O)$ in $k^\times$, $M|N$ as $\varepsilon^N = 1$, then M is an exponent of $T_{\mathcal{D}}(\alpha)$ for $\mathcal{D}$ in a certain part of $\mathcal{F}$ determined by how the points hit the two components.*

A new integrality criterion (continued)

Corollary

If C is of genus 1 then the 'certain part' is always $\mathcal{F}$, so $M'\alpha$ is in $K_2^T(E)_{\mathrm{int}}$ with M' the order of ε in $\mathcal{O}_F^\times$.

Example

On the elliptic curve over $\mathbb{Q}$ defined by $y^2 = x^3 + 1$, with $P = (2,3)$, $Q = -P = (2,-3)$, $N = 6$,

$$f = \frac{1}{108}\frac{(y-2x+1)^3}{y+1} \qquad g = \frac{1}{108}\frac{(-y-2x+1)^3}{-y+1},$$

$\varepsilon = -1$. The reduction at $p = 3$ is of type III. It has two irreducible components (meeting tangentially in one point), $\mathcal{A}$, hit by O, and $\mathcal{B}$, hit by P and Q. Then $T_{\mathcal{A}}(\{f,g\}) = -1$ and $T_{\mathcal{B}}(\{f,g\}) = 1$.

Second construction

Let E be an elliptic curve over a field F, and P an F-rational point on E of order N. For $1 \leqslant s \leqslant N-1$, let $f_{P,s}$ in $F(E)^\times$ be a function with divisor $(f_{P,s}) = N(sP) - N(O)$.

In $K_2^T(E)$ define

$$T_{P,s,t} = \left\{ \frac{f_{P,s}}{f_{P,s}(tP)}, \frac{f_{P,t}}{f_{P,t}(sP)} \right\} \quad (s \neq t)$$

$$S_{P,s} = \{f_{P,s}, -f_{P,s}\} + \sum_{t=1, t\neq s}^{N-1} T_{P,s,t} \quad (1 \leq s \leq N-1)$$

Remark

$S_{P,s} + S_{-P,s}$ is in $K_2(F)$

Second construction

Let F be a field and $N \geq 4$. A pair (E, P) with E an elliptic curve over F and an F-rational point P on E of order N has a unique Weierstraß model Tate normal form

$$E : y^2 + (1-g)xy - fy = x^3 - fx^2$$

with f in $F^\times$, g in F, and $P = (0,0)$. Parametrised by $X_1(N)$:

N	f	g	Δ
7	$t^3 - t^2$	$t^2 - t$	$t^7(t-1)^7(t^3-8t^2+5t+1)$
8	$2t^2 - 3t + 1$	$\frac{2t^2-3t+1}{t}$	$\frac{(t-1)^8(2t-1)^4(8t^2-8t+1)}{t^4}$
10	$\frac{2t^5-3t^4+t^3}{(t^2-3t+1)^2}$	$\frac{-2t^3+3t^2-t}{t^2-3t+1}$	$\frac{t^{10}(t-1)^{10}(2t-1)^5(4t^2-2t-1)}{(t^2-3t+1)^{10}}$

Remark

We want $X_1(N)$ to have genus 0, and $N > 6$ to get enough elements in $K_2^T(E)_{\text{int}}$ for F non-Abelian. For $N = 9$, 12 the situation is a bit different from that for $N = 7, 8, 10$.

Second construction (integrality)

Theorem

If $E = E_t$ is an elliptic curve over a number field F, in Tate normal form for $N = 7, 8$ or 10, and $P = (0,0)$, then $2P$ hits the O-component in each fibre of the minimal regular model over $\mathcal{O}_F$ if

- $t, 1-t$ *are in* $\mathcal{O}_F^\times$, *for* $N = 7$
- $\frac{1}{t} - 1, \frac{1}{t} - 2$ *are in* $\mathcal{O}_F^\times$, *for* $N = 8$
- $\frac{1}{t} - 1, 1 - 2t$ *are in* $\mathcal{O}_F^\times$, *for* $N = 10$

In that case

- *each* $S_{P,s}$ *is in* $K_2^T(E)_{\mathrm{int}}$ *for* $N = 7$;
- *each* $N' \cdot S_{P,s}$ *is in* $K_2^T(E)_{\mathrm{int}}$ *for* $N = 8, 10$; $N' = \gcd(N, \# F_{\mathrm{tor}}^\times)$.

Let $F = \mathbb{Q}(t)$, t satisfying the condition. $N = 7, 8$: F is cubic if and only if it is one of the special cubic fields. $N = 10$: 40 families (identifications under a dihedral group of order 8: for $u = 1 - 2t$, u and $\frac{1-u}{1+u}$ are in $\mathcal{O}_F^\times$); the Galois closure in a family almost always has group S_4 (28×), D_4 (10×), C_4 (1×) simplest quartic fields (Gras), $C_2 \times C_2$ (1×).

Table for our special quartic fields

We list b, c, ε of the polynomials $X^4 + aX^3 + bX^2 + cX + \varepsilon$, with a in $\mathbb{Z}$ (sometimes with congruence condition), defining such fields (with 28 reducible exceptions), as well as the Galois groups Gal of the splitting field for general a

b	c	ε	Gal	b	c	ε	Gal
-2	$-a \pm 1$	1	S_4	0	$-a \pm 1$	-1	S_4
-2	$-a \pm 2$	1	S_4	0	$-a \pm 2$	-1	S_4
-2	$-a \pm 4$	1	S_4	0	$-a \pm 4$	-1	D_4
-2	$-a \pm 8, 2 \mid a$	1	S_4	0	$-a \pm 8,\ 2 \mid a$	-1	S_4
-2	$-a \pm 16, 4 \mid a$	1	S_4	0	$-a \pm 16,\ a \equiv 2 \pmod 4$	-1	S_4
-2 ± 1	$-a$	1	D_4	± 1	$-a$	-1	S_4
-2 ± 2	$-a$	1	D_4	± 2	$-a$	-1	S_4
2	$-a$	1	$C_2 \times C_2$	± 4	$-a$	-1	S_4
-6	$-a$	1	C_4				
-2 ± 8	$-a, 2 \mid a$	1	D_4	± 8	$-a,\ 2 \mid a$	-1	S_4
-2 ± 16	$-a, 4 \mid a$	1	D_4	± 16	$-a,\ a \equiv 2 \pmod 4$	-1	S_4

Second construction (integrality and regulator)

Theorem

Define fields F, with element t, parametrised by an integer a.

- *Let u be a root of an $f_a(X)$ defining a special cubic field $F = \mathbb{Q}(u)$, and put $t = u$ $(N = 7)$ or $t = 1/(u+1)$ $(N = 8)$.*
- *Let u be a root of an $f_a(X)$ defining a special quartic field $F = \mathbb{Q}(u)$, and put $t = \frac{1-u}{2}$ for $N = 10$.*

If the Tate normal form for (N, t) defines an elliptic curve E/F, then, with $P = (0, 0)$:

- *the $\gcd(N, 2) \cdot S_{P,s}$ for $s = 1, \ldots, N-1$ are in $K_2^T(E)_{\text{int}}$;*
- *for the Beilinson regulator $R(a)$ of the first $\lfloor \frac{N-1}{2} \rfloor$ we have*

$$\lim_{|a|\to\infty} \frac{R(a)}{\log^{\lfloor \frac{N-1}{2} \rfloor} |a|} = C_N \cdot \left| \det \left(\frac{N^4}{3} B_3\left(\left\{ \frac{ij}{N} \right\} \right) \right)_{1 \le i,j \le \lfloor \frac{N-1}{2} \rfloor} \right| \neq 0$$

($B_3(X) = X^3 - \frac{3}{2}X^2 + \frac{1}{2}X$: third Bernoulli polynomial; $\{x\}$: the fractional part of x; $C_7 = 1$, $C_8 = C_{10} = 4$)

Second construction (idea of proof of limit result)

The Bloch-Wigner dilogarithm is the unique continuous function $D\colon \mathbb{P}^1(\mathbb{C}) \to \mathbb{R}$ with $D(z) = \mathrm{im}(\sum_{n=1}^{\infty} z^n/n^2) + \mathrm{Arg}(1-z)\log|z|$ for $|z| \leqslant 1$, $z \neq 0, 1$, and $D(1/z) = -D(z)$ for every such z.
For q in $\mathbb{C}^\times$ with $|q| < 1$ Bloch's elliptic dilogarithm D_q is

$$D_q\colon \mathbb{C}^\times/q^{\mathbb{Z}} \to \mathbb{R}, \qquad z \mapsto \sum_{n\in\mathbb{Z}} D(zq^n).$$

Also define $J(z) = \log|z|\log|1-z|$ and $J_q\colon \mathbb{C}^\times/q^{\mathbb{Z}} \to \mathbb{R}$ by

$$J_q(z) = \sum_{n=0}^{\infty} J(zq^n) - \sum_{n=1}^{\infty} J(z^{-1}q^n) + \frac{1}{3}\log^2|q|\, B_3\left(\frac{\log|z|}{\log|q|}\right),$$

and $R_q\colon \mathbb{C}^\times/q^{\mathbb{Z}} \to \mathbb{C}$ as $R_q = D_q - iJ_q$.

If $q = exp(2\pi i\tau) = \mathrm{e}(\tau)$ then we get R_τ, etc., on $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ by composing R_q, etc., with $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau) \simeq \mathbb{C}^\times/q^{\mathbb{Z}}$.

Second construction (idea of proof of limit result)

Let γ_0 be the path from 0 to 1 in $E(\mathbb{C}) \cong \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$. Then for γ in $H_1(E(\mathbb{C}),\mathbb{Z})$ and $\alpha = \sum_j \{f_j, g_j\}$ in $K_2^T(E)$,

$$\langle \gamma, \alpha \rangle = -\frac{1}{2\pi} \mathrm{im}\Big(\frac{\Omega_\gamma}{\mathrm{im}(\tau)\Omega_{\gamma_0}} \sum_j R_\tau((f_j) \diamond (g_j)) \Big).$$

with, for any holomorphic 1-form $\omega \neq 0$ on E, $\Omega_\delta = \int_\delta \omega$, and $(f) \diamond (g) = \sum m_i n_j (a_i - b_j)$ if $(f) = \sum m_i (a_i)$, $(g) = \sum n_j (b_j)$.

Then

$$\langle \gamma_0, S_{P,s} \rangle = -\frac{N^3}{2\pi \mathrm{im}(\tau)} J_\tau(sP).$$

Fourier expansion: If $u = a + b\tau$ with $0 \leqslant a, b < 1$, and $\tau = x + iy$ with x real and y positive, then

$$J_\tau(u) = \frac{4\pi^2 y^2}{3} B_3(b) - \pi y \sum_{\substack{m, n+b \in \mathbb{Z} \\ m,n \neq 0}} \mathrm{e}(-ma)\mathrm{e}(mnx) \frac{n}{|m|} e^{-2\pi |mn| y}.$$

Second construction (idea of proof of limit result)

Then the limit result follows by

• knowing F is totally real for $|a| \gg 0$

• knowing which cusps of $X_1(N)$ are approached, corresponding to the behaviour of $t = t(u)$ under an embedding $F \to \mathbb{R}$ for $|a| \gg 0$

• comparing the limit behaviour of a root u of $f_a(X)$ with that of τ as τ approaches the corresponding cusp

• understanding complex conjugation on $X_1(N)$, as well as on H_1 of the universal elliptic curve above a real point of $X_1(N)$ (to get a generator of H_1^-)

• reducing to using only J_τ in R_τ, with dominant term given by B_3 in the Fourier expansion

Second construction (numerical results)

$N = 7, 8$: F cubic as in numerical examples of the first construction F is non-Abelian for $a \neq 3$

$N = 7$: we list the rational number $\widetilde{Q}$ for $S_{P,1}, S_{P,2}, S_{P,3}$

$N = 8$: we list the rational number $\widetilde{Q}$ for $2S_{P,1}, 2S_{P,2}, 2S_{P,3}$

$N = 10$: F defined by $f_a(X) = X^4 + aX^3 - aX + 1$, a in $\mathbb{Z} \setminus \{\pm 3\}$

- Galois group of splitting field: D_4 for $a \neq 0$
- two complex places for $a = -2, \ldots, 2$, otherwise totally real
- we list the rational number $\widetilde{Q}$ for $2S_{P,1}, 2S_{P,2}, 2S_{P,3}, 2S_{P,4}$,

Second construction (numerical results)

Some of our data for $N = 7$ red: F not totally real

a	d	c	$L^*(E,0)$	$\widetilde{Q}$
2	-23	$2^3 \cdot 7^2$	3.20759739648506351	7^{-6}
3	7^2	$13 \cdot 29$	14.5301315201187081	7^{-5}
4	257	$2^3 \cdot 41$	235.760168840014734	7^{-4}
5	$17 \cdot 41$	239	1671.96067772426875	$2 \cdot 3 \cdot 5 \cdot 7^{-5}$
6	1489	$2^3 \cdot 13$	4051.92834496448134	7^{-3}
7	2777	83	-6590.94375552556550	$-2 \cdot 5 \cdot 7^{-5} \cdot 11$
8	4729	$2^3 \cdot 41$	114693.828270615380	$2^3 \cdot 3^3 \cdot 7^{-4}$
9	7537	$7^2 \cdot 13$	520366.913326434323	$2 \cdot 3 \cdot 7^{-4} \cdot 137$
10	$7^2 \cdot 233$	$2^3 \cdot 127$	-1485239.71027494934	$-2 \cdot 3^2 \cdot 7^{-4} \cdot 113$
11	$17 \cdot 977$	1471	5790649.98684165696	$2^4 \cdot 3 \cdot 5^2 \cdot 7^{-5} \cdot 41$
12	$97 \cdot 241$	$2^3 \cdot 251$	17255203.9121322960	$2^4 \cdot 3^2 \cdot 7^{-4} \cdot 131$
13	32009	2633	28504752.7830982117	$2^8 \cdot 3 \cdot 7^{-4} \cdot 37$
14	$47 \cdot 911$	$2^3 \cdot 419$	93361926.2369695039	$2^3 \cdot 3 \cdot 7^{-4} \cdot 3571$
15	$73 \cdot 769$	$43 \cdot 97$	192572866.057081271	$2^3 \cdot 3^2 \cdot 7^{-4} \cdot 43 \cdot 53$

Second construction (numerical results)

Some of our data for $N = 8$

a	d	c	$L^*(E,0)$	$\widetilde{Q}$
2	-23	$5 \cdot 137$	5.97110504152047155	2^{-21}
3	7^2	$7 \cdot 113$	31.2948786232840397	2^{-18}
4	257	3^3	25.2202129687784361	$2^{-18} \cdot 3^{-1}$
5	$17 \cdot 41$	$11 \cdot 41$	3130.70411060858445	$2^{-15} \cdot 3$
6	1489	$7 \cdot 13$	3377.15438740388289	2^{-13}
7	2777	$3^3 \cdot 5 \cdot 7$	-110191.314028644712	$-2^{-10} \cdot 3$
8	4729	$17 \cdot 127$	806249.659144856084	$2^{-13} \cdot 11 \cdot 13$
9	7537	$19 \cdot 199$	-3399020.63508445448	$-2^{-12} \cdot 257$
10	$7^2 \cdot 233$	$3^3 \cdot 7 \cdot 31$	9860642.47040826474	$2^{-11} \cdot 3 \cdot 109$
11	$17 \cdot 977$	$23 \cdot 367$	-38313626.2137679483	$-2^{-13} \cdot 4547$
12	$97 \cdot 241$	$5 \cdot 463$	22214626.7118122391	$2^{-14} \cdot 4787$
13	32009	$3^3 \cdot 7$	2759510.81590883242	$2^{-13} \cdot 3 \cdot 7 \cdot 13$
14	$47 \cdot 911$	$7 \cdot 29 \cdot 97$	-549654076.156923184	$-2^{-12} \cdot 3^4 \cdot 311$
15	$73 \cdot 769$	$17 \cdot 31 \cdot 47$	1205314746.12464172	$2^{-9} \cdot 5 \cdot 1289$

Second construction (numerical results)

Data for $N = 10$ red: F two complex places blue: $F = \mathbb{Q}(\zeta_8)$

a	d	c	$L^*(E,0)$	$\widetilde{Q}$
-7	$2^3 \cdot 41^2$	$2^2 \cdot 23^2$	67284.5712909244205	$2^{-11} \cdot 5^{-5}$
-6	$2^6 \cdot 7^2 \cdot 37$	$3^4 \cdot 7^2$	12809909.2599370080	$2^{-9} \cdot 5^{-4} \cdot 13$
-5	$2^3 \cdot 13 \cdot 17^2$	$2^2 \cdot 19^2$	321613.252539691824	$2^{-10} \cdot 5^{-4}$
-4	$2^8 \cdot 17$	17^2	1308.96784301967823	$2^{-10} \cdot 5^{-7}$
-2	$2^6 \cdot 5$	13^2	3.90265959107592883	$2^{-14} \cdot 5^{-9}$
-1	$2^3 \cdot 7^2$	$2^2 \cdot 11^2$	18.1524378610645748	$2^{-14} \cdot 5^{-8}$
0	2^8	3^4	1.29080207928400602	$2^{-14} \cdot 3^{-2} \cdot 5^{-8}$
1	$2^3 \cdot 7^2$	$2^2 \cdot 7^2$	7.41655915683319223	$2^{-15} \cdot 5^{-8}$
2	$2^6 \cdot 5$	5^2	0.604505751430063810	$2^{-14} \cdot 5^{-10}$
4	$2^8 \cdot 17$	7^2	211.227406732423650	$2^{-11} \cdot 5^{-7}$
5	$2^3 \cdot 13 \cdot 17^2$	2^2	825.817965343090665	$2^{-11} \cdot 5^{-7}$
6	$2^6 \cdot 7^2 \cdot 37$	3^4	272030.854985666477	$2^{-9} \cdot 3^2 \cdot 5^{-6}$
7	$2^3 \cdot 41^2$	$2^2 \cdot 5^4$	111421.646021166774	$2^{-10} \cdot 5^{-5}$

Second construction (numerical results)

The formula for $\langle \gamma_0, S_{P,s} \rangle$ is clean, but we also have the $T_{P,s,t}$, and for $N = 8$, 10, elements based on $2P$ (of order $N/2$).

One can analyse the integrality obstruction for those along the lines of the new integrality criterion, and find a subgroup of $K_2^T(E)_{\mathrm{int}}$ of rank 3 ($N = 7, 8$) or 4 ($N = 10$) for which the regulator of a basis gives a rational number that equals $\widetilde{Q}$ multiplied by:

- $N = 7$: 7^6 if F is not totally real, and 7^5 otherwise
- $N = 8$: 2^{10}
- $N = 10$: $2^{10}5^4$

The end!

Are there any questions?