Algebraic K-theory, regulators, and cohomology

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The Riemann ζ-function

$$\zeta(s) = \sum_{n \geq 1} n^{-s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} \qquad (\mathrm{Re}(s) > 1)$$

can be extended to a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1$ with residue 1

$$\begin{aligned} \zeta(2) &= \pi^2/6 & \qquad \zeta(3) &\text{ irrational} \\ \zeta(4) &= \pi^4/90 & \qquad \zeta(5) &\ ??? \\ \zeta(6) &= \pi^6/945 & \qquad \zeta(7) &\ ??? \\ &\vdots & &\vdots \end{aligned}$$

The ζ-function of a number field

Let k be a number field, i.e., for some $f(X)$ an irreducible polynomial in $\mathbb{Q}[X]$ of degree d, and α a root of $f(X)$ in $\mathbb{C}$,

$$k = \mathbb{Q}(\alpha) = \{b_0 + b_1\alpha + \cdots + b_{d-1}\alpha^{d-1}, \text{ all } b_j \text{ in } \mathbb{Q}\}$$

the number field generated by α.
Let $\mathcal{O}$ be the ring of algebraic integers of k: $x \in k$ is an algebraic integer if it is the zero of a polynomial

$$X^n + a_{n-1}X^{n-1} + \ldots + a_1X + a_0$$

with all a_i in $\mathbb{Z}$.

The ζ-function of k is defined by (for $\mathrm{Re}(s) > 1$)

$$\zeta_k(s) = \sum_{\substack{(0)\neq I\subset\mathcal{O} \\ I \text{ an ideal of } \mathcal{O}}} (\#\mathcal{O}/I)^{-s} = \prod_{\substack{0\neq\mathcal{P}\subset\mathcal{O} \\ \mathcal{P} \text{ prime ideal}}} \frac{1}{1-(\#\mathcal{O}/\mathcal{P})^{-s}}.$$

Every non-zero ideal of $\mathcal{O}$ is uniquely (up to ordering) the product of non-zero prime ideals.

The ζ-function of a number field

$\zeta_k(s)$ can be extended to a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1$

Let r_1 the number of embeddings $k \to \mathbb{R}$, $2r_2$ the number of non-real embeddings $k \to \mathbb{C}$, so $d = r_1 + 2r_2$.
($r_1 = \#$real roots of $f(X)$, $2r_2 = \#$non-real roots of $f(X)$)

$\mathcal{O}^* \cong \mathbb{Z}^r \times \mathbb{Z}/w\mathbb{Z}$ with $r = r_1 + r_2 - 1$ and

$w =$ the number of roots of unity in k

Let $\sigma_1, \ldots, \sigma_{r+1}$ be the embeddings of k into $\mathbb{C}$ up to complex conjugation.

If $u_1, \ldots, u_r$ form a $\mathbb{Z}$-basis of $\mathcal{O}^*/\{\text{roots of unity}\}$, let

$$R = \frac{2^{r_2}}{d} \left| \det \begin{pmatrix} 1 & \log|\sigma_1(u_1)| & \ldots & \log|\sigma_1(u_r)| \\ \vdots & \vdots & & \vdots \\ 1 & \log|\sigma_{r+1}(u_1)| & \ldots & \log|\sigma_{r+1}(u_r)| \end{pmatrix} \right|$$

The ζ-function of a number field

Then

$$\mathrm{Res}_{s=1}\zeta_k(s) = \frac{2^{r_1}(2\pi)^{r_2}|\mathrm{Cl}(\mathcal{O})|}{w\sqrt{\Delta_k}} \cdot R$$

• $\mathrm{Cl}(\mathcal{O})$ = the class group of $\mathcal{O}$ (a finite Abelian group which measures (failure of) unique factorization in $\mathcal{O}$)
• w = the number of roots of unity in $k = |\mathcal{O}^*{}_{\mathrm{torsion}}|$
• Δ_k the absolute value of the discriminant of k.

This is a statement about algebraic K-theory:

$K_0(\mathcal{O}) \cong \mathbb{Z} \oplus \mathrm{Cl}(\mathcal{O})$ and $K_1(\mathcal{O}) \cong \mathcal{O}^*$,

so

$|\mathrm{Cl}(\mathcal{O})| = |K_0(\mathcal{O})_{\mathrm{torsion}}|$ and $w = |K_1(\mathcal{O})_{\mathrm{torsion}}|$.

Algebraic K-theory of a ring: K_0

R: a commutative ring with identity $1 \neq 0$

$$K_0(R) = \frac{\text{free Abelian group on generators } [M],\ M \text{ a finitely generated projective } R\text{-module}}{\left\langle \begin{array}{l}[P]-[P']-[P''] \text{ for each exact} \\ \text{sequence } 0 \to P' \to P \to P'' \to 0\end{array} \right\rangle}.$$

P projective means every surjection $M \to P$ admits a section of R-modules, e.g., a free R-module.
Therefore $P \cong P' \oplus P''$ in the above.

Example

- F a field: $K_0(F) \cong \mathbb{Z}$ via the dimension of a vector space
- $K_0(\mathbb{Z}) \cong \mathbb{Z}$ via rank of a finitely generated Abelian group
- $\mathcal{O}$ = ring of integers in a number field: $K_0(\mathcal{O}) \cong \mathbb{Z} \oplus \mathrm{Cl}(\mathcal{O})$

Algebraic K-theory of a ring: K_1

View $GL_n(R) \subset GL_{n+1}(R)$ via $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$.

Let $GL(R) = \bigcup_n GL_n(R)$.

Definition $K_1(R) = GL(R)/[GL(R), GL(R)]$

The determinant gives a surjection $K_1(R) \to R^*$, the kernel is denoted $SK_1(R)$

Example

- F a field: $K_1(F) \cong F^*$
- $K_1(\mathbb{Z}) \cong \mathbb{Z}^* = \{\pm 1\}$
- $\mathcal{O}$ = ring of integers in a number field: $K_1(\mathcal{O}) \cong \mathcal{O}^*$
- If, e.g., $R = \mathbb{Q}[x, y]/(y^2 - x^3 - 3)$ then $R^* = \mathbb{Q}^*$ but $SK_1(R)$ is infinite

Algebraic K-theory of a ring: K_2

$K_1(R)$ (matrices or maps) give relations between generators of $K_0(R)$ (=R-modules). "So" $K_2(R)$ should involve "relations among the relations for $K_1(R)$".

Definition For $i, j \geq 1$, $i \neq j$,and r in R, let $e_{i,j}(r)$ be the elementary matrix with r in position (i, j)

Then

$$e_{i,j}(r)e_{i,j}(s) = e_{i,j}(r + s)$$
$$[e_{i,j}(r), e_{j,l}(s)] = e_{i,l}(rs) \text{ if } i \neq l$$
$$[e_{i,j}(r), e_{k,l}(s)] = 1 \text{ if } j \neq k, i \neq l$$

and the $e_{i,j}(r)$ generate the subgroup $[GL(R), GL(R)]$ of $GL(R)$.

Algebraic K-theory of a ring: K_2

The Steinberg group $\mathrm{St}(R)$ of R is the free group on symbols $x_{i,j}(r)$ with $i, j \geq 1$, $i \neq j$, r in R, quotiented out to give the same three relations for the $x_{i,j}(r)$ as for the $e_{i,j}(r)$.

We have a surjective group homomorphism

$$\begin{aligned} \varphi : \mathrm{St}(R) &\to [GL(R), GL(R)] \\ x_{i,j}(r) &\mapsto e_{i,j}(r) \end{aligned}$$

Definition $K_2(R) = \ker(\varphi)$

Proposition $K_2(R)$ Abelian. In fact, it is the centre of $\mathrm{St}(R)$.

K_2 of a field

If F is a field then $K_2(F)$ is an Abelian group written additively, with

$$\begin{array}{ll} \text{generators} & \{a, b\} \text{ for } a, b \text{ in } F^* \\ \text{relations} & \{a_1 a_2, b\} = \{a_1, b\} + \{a_2, b\} \\ & \{a, b_1 b_2\} = \{a, b_1\} + \{a, b_2\} \\ & \{a, 1-a\} = 0 \text{ if } a \neq 0, 1 \end{array}$$

Then also $\{a, b\} = -\{b, a\}$ and $\{c, -c\} = 0$ for a, b, c in F^*.

If A in $\mathrm{St}(F)$ lifts $\begin{pmatrix} a & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and B lifts $\begin{pmatrix} b & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b^{-1} \end{pmatrix}$, then $\{a, b\} = [A, B]$ in $K_2(F)$.

Note that $K_2(F) \simeq F^* \otimes F^* / \langle x \otimes (1-x) \rangle$ with $\{a, b\}$ corresponding to the class of $a \otimes b$.

Proposition

$$K_2(\mathbb{Q}) \xrightarrow{\sim} \{\pm 1\} \times \oplus_{p \text{ prime}} \mathbb{F}_p^*$$

with components

$$T_\infty : K_2(\mathbb{Q}) \to \{\pm 1\} \text{ with } T_\infty(\{a,b\}) = \begin{cases} -1 \text{ if } a, b < 0 \\ 1 \text{ otherwise} \end{cases}$$

$T_p : K_2(\mathbb{Q}) \to \mathbb{F}_p^*$ with $T_p(\{a,b\}) = (-1)^{v_p(a)v_p(b)} \frac{a^{v_p(b)}}{b^{v_p(a)}} \text{ modulo } p$
where $v_p(a) \in \mathbb{Z}$ is the number of factors p in a
$T_p =$ the tame symbol for p

The proof of the proposition is based on repeated rewriting using division with remainder: if $a = qb + r$ with a, b, q, r non-zero integers, then $\{a/r, -qb/r\} = 0$ in $K_2(\mathbb{Q})$.

Quadratic reciprocity

For $p > 2$ and $\alpha \in K_2(\mathbb{Q})$ we have $T_p(\alpha)^{\frac{p-1}{2}} \in \{\pm 1\} \subseteq \mathbb{F}_p^*$.

Define $\widetilde{T}_2 : K_2(\mathbb{Q}) \to \{\pm 1\}$ as follows.
Write $a = (-1)^i 2^j 5^k \frac{c}{d}$ with $i, k = 0, 1$ and c, d integers congruent 1 mod 8, $b = (-1)^I 2^J 5^K \frac{c'}{d'}$ similarly. Then

$$\widetilde{T}_2(\{a, b\}) = (-1)^{iI + jK + kJ}.$$

Identify $\{\pm 1\} \subseteq \mathbb{F}_p^*$ for all primes $p > 2$.

Theorem $\quad T_\infty(\{a, b\}) = \widetilde{T}_2(\{a, b\}) \prod_{\substack{p>2 \\ p \text{ prime}}} T_p(\{a, b\})^{\frac{p-1}{2}}.$

This is equivalent with quadratic reciprocity. E.g., let p and q be distinct odd primes, and put $\left(\frac{p}{q}\right)$ equal to 1 if p is a square modulo q, and to -1 if not. Equivalently,

$$\left(\frac{p}{q}\right) = p^{\frac{q-1}{2}} \bmod q = T_q(\{p,q\})^{\frac{q-1}{2}}$$

The theorem says that

$$\begin{aligned} 1 &= \widetilde{T}_2(\{p,q\})\, T_p(\{p,q\})^{\frac{p-1}{2}}\, T_q(\{p,q\})^{\frac{q-1}{2}} \\ &= (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}} \left(\frac{q}{p}\right)\left(\frac{p}{q}\right). \end{aligned}$$

Borel's theorem

Quillen defined Abelian groups $K_n(R)$ ($n \geq 0$) (1969(?)) for a ring R; later also for an algebraic variety (1973).

Let k be a number field, with r_1 real and $2r_2$ non-real embeddings, $d = r_1 + 2r_2$, and ring of algebraic integers $\mathcal{O}$, and let Δ_k be the absolute value of the discriminant of k

Recall that

- $K_0(\mathcal{O}) \cong \mathbb{Z} \oplus \mathrm{Cl}(\mathcal{O})$
- $K_1(\mathcal{O}) \cong \mathcal{O}^*$ has rank $r_1 + r_2 - 1$

Theorem (Quillen, 1973) $K_n(\mathcal{O})$ is finitely generated for all $n \geq 0$.

Borel's theorem

Theorem (Borel, 1974+1977)
(1) $K_{2n}(\mathcal{O})$ is a finite group if $n \geq 1$.
(2) For $n \geq 2$, $K_{2n-1}(\mathcal{O})$ has rank $m_{2n-1} = r_1 + r_2$ if n is odd, and rank $m_{2n-1} = r_2$ if n is even.
(3) There exists a natural regulator map

$$K_{2n-1}(\mathcal{O}) \to \mathbb{R}^{m_{2n-1}} \qquad (n \geq 2).$$

Its image is a lattice with (normalized) volume of a fundamental domain

$$R_n(k) = q \frac{\zeta_k(n)}{\pi^{n(d-m_{2n-1})}\sqrt{\Delta_k}}$$

for some q in $\mathbb{Q}^*$.

Example: the K-theory of $\mathbb{Z}$

$\zeta_{\mathbb{Q}}$ is the Riemann zeta function. For $n \geq 2$:

$K_{2n-1}(\mathbb{Z})$ is finite for n even;

$K_{2n-1}(\mathbb{Z})$ has rank 1 for n odd, and $R_n(k) = q\zeta(n)$ for some $q \in \mathbb{Q}^*$.

n	2	3	4	5	6	7	...
m_{2n-1}	0	1	0	1	0	1	...
$\zeta(n)$	$\pi^2/6$	irrational	$\pi^4/90$	???	$\pi^6/945$	???	...

The definition of Borel's regulator map

For R as above, Quillen's K-groups can be defined as follows.

There is an H-space $BGL(R)^+$ ("topological group up to homotopy") with a map $BGL(R) \to BGL(R)^+$ inducing an isomorphism $H_*(BLG(R), \mathbb{Z}) \stackrel{\sim}{\to} H_*(BGL(R)^+, \mathbb{Z})$. It satisfies

$$\pi_1(BGL(R)^+) \simeq GL(R)/[GL(R), GL(R)] \simeq K_1(R)$$

as $[GL(R), GL(R)] = E(R)$ in $\pi_1(BGL(R)) = GL(R)$ is its own commutator subgroup.

Then $K_n(R) = \pi_n(BGL(R)^+)$ for $n \geq 1$.

The definition of Borel's regulator map

Borel for $2n-1 \geq 1$ constructs b_n in $H^{2n-1}(GL(\mathbb{C}), \mathbb{R})$ (continuous group cohomology), well-behaved with respect to complex conjugation. For $n = 1$, it is $\log|\cdot|$ on $H_1(GL(\mathbb{C}), \mathbb{Z}) \simeq \mathbb{C}^*$.

If $\sigma : k \to \mathbb{C}$ is an embedding, the σ-component of Borel's regulator map for $n \geq 2$ is the composition

$$K_{2n-1}(k) \stackrel{\sigma_*}{\to} K_{2n-1}(\mathbb{C}) \xrightarrow{\text{Hur}} H_{2n-1}(GL(\mathbb{C}), \mathbb{R}) \xrightarrow{b_n \cap \cdot} \mathbb{R}$$

with Hur the Hurewicz homomorphism $\pi_{2n-1}(BGL(\mathbb{C})^+) \to H_{2n-1}(BGL(\mathbb{C})^+, \mathbb{Z}) = H_{2n-1}(BGL(\mathbb{C}), \mathbb{Z}) = H_{2n-1}(GL(\mathbb{C}), \mathbb{Z})$.

Possible motivations

- The volume of a fundamental domain of a suitable symmetric space is a product of values of ζ_k-values (cf. Humbert's classical formula for an imaginary quadratic field)
- The homology of $BGL(\mathcal{O})^+$ is a wedge algebra on the primitive elements of the homology, i.e., on the image of the K-theory of $\mathcal{O}$, so similar to the previous point

Lichtenbaum conjecture

The functional equation of $\zeta_k(s)$ gives that $\zeta_k(s)$ at $s = 1 - n$ has a zero of order the rank of $K_{2n-1}(\mathcal{O})$ $(n \geq 1)$. Let $\zeta_k^*(1-n)$ be its first non-vanishing coefficient in its Taylor expansion at $s = 1 - n$. Borel's theorem then states that $\zeta_k^*(1-n)R_n(k)^{-1}$ is in $\mathbb{Q}^*$.

Conjecture (Lichtenbaum, 1973!) For $n \geq 2$ we have

$$\zeta_k^*(1-n) = \pm 2^{?_{k,n}} \frac{|K_{2n-2}(\mathcal{O})|}{|K_{2n-1}(\mathcal{O})_{\text{torsion}}|} R_n(k)$$

Known consequence of Quillen-Lichtenbaum conjecture For p an odd prime

$$K_{2n-i}(\mathcal{O}) \otimes \mathbb{Z}_p \xrightarrow{\sim} H^i_{\mathrm{et}}(\mathcal{O}[1/p], \mathbb{Z}_p(n))$$

for $i = 1, 2$ and $n \geq \max(i, 2)$.

Theorem (based on work of many people)
For $n \geq 2$ and $k/\mathbb{Q}$ Abelian, the Lichtenbaum conjecture holds.

Let E be a CM elliptic curve over $\mathbb{Q}$.
Let $E_{\mathbb{C}}$ be the curve obtained by extending the coefficients to $\mathbb{C}$, and F the field of meromorphic functions on $E_{\mathbb{C}}$.

There is an exact localization sequence

$$K_2(E_{\mathbb{C}}) \to K_2(F) \xrightarrow{T} \oplus_{x \in E_{\mathbb{C}}} \mathbb{C}^*$$

where T_x is the tame symbol for x:

$$T_x : \{f, g\} \mapsto (-1)^{\mathrm{ord}_x(f)\,\mathrm{ord}_x(g)} \frac{f^{\mathrm{ord}_x(g)}}{g^{\mathrm{ord}_x(f)}}|_x$$

with $\mathrm{ord}_x(f)$ the order of vanishing of f at x.

For two non-zero meromorphic functions f and g on $E_{\mathbb{C}}$, $\log|f|\,\mathrm{d}\arg g - \log|g|\,\mathrm{d}\arg f$ is a closed 1-form on some $E_{\mathbb{C}} \setminus S$ with S finite. Then

$$\log|z|\,\mathrm{d}\arg(1-z) - \log|1-z|\,\mathrm{d}\arg z = \mathrm{d}D(z),$$

where $D(z)$ is a C^∞-function on $\mathbb{C} \setminus \{0,1\}$, the Bloch-Wigner dilogarithm. This gives a homomorphism

$$\mathrm{reg} : K_2(F) \to \frac{\{\text{closed 1-forms on some } E_{\mathbb{C}} \setminus S\}}{\{\text{exact 1-forms on some } E_{\mathbb{C}} \setminus S\}}$$

$$\{f, g\} \mapsto \log|f|\,\mathrm{d}\arg g - \log|g|\,\mathrm{d}\arg f\,.$$

This fits into a commutative diagram

$$\begin{array}{ccccccc} & & K_2(E) & & & & \\ & & \downarrow & & & & \\ & & K_2(E_{\mathbb{C}}) & \longrightarrow & K_2(F) & \longrightarrow & \bigoplus_{x \in E_{\mathbb{C}}} \mathbb{C}^* \\ & & \vdots\ \mathrm{reg} & & \downarrow\ \mathrm{reg} & & \downarrow\ \log|\cdot| \\ 0 & \longrightarrow & H^1_{\mathrm{dR}}(E_{\mathbb{C}}; \mathbb{R}) & \longrightarrow & H^1_{\mathrm{dR}}(F; \mathbb{R}) & \xrightarrow[i \cdot \mathrm{res}]{} & \bigoplus_{x \in E_{\mathbb{C}}} \mathbb{R} \end{array}$$

with $H^1_{\mathrm{dR}}(F; \mathbb{R}) = \varinjlim_{S \subset E_{\mathbb{C}}} H^1_{\mathrm{dR}}(E_{\mathbb{C}} \setminus S; \mathbb{R})$ where all S are finite.

Bloch's work on CM elliptic curves of $\mathbb{Q}$

Theorem (Bloch; Irvine notes, 1978; published in 2000)
Let E be an elliptic curve defined over $\mathbb{Q}$ with complex multiplication. Then there exists an element α in $K_2(E)$ with

$$L'(E,0) = q\frac{1}{2\pi}\int_{E_{\mathbb{C}}} \mathrm{reg}(\alpha)\wedge\omega$$

for some q in $\mathbb{Q}^*$, or, using the functional equation for the L-function:

$$\frac{1}{2\pi}L(E,2) = q'\int_{E_{\mathbb{C}}} \mathrm{reg}(\alpha)\wedge\omega.$$

ω is a non-zero holomorphic form on $E_{\mathbb{C}}$ with $\int_{E(\mathbb{R})}\omega = 1$.
$L(E,s) =$ "ζ-function for $H^1(E)$"

Beilinson's conjectures on special values of L-functions

Beilinson (1985) "attempted to understand the work by Borel and by Bloch", reinterpreting and vastly generalising their ideas. Crucially, he uses K-theory and a suitable (co)homology theory with their properties, rather than single groups, and relates them using a formalism based on Chern classes (cf. Gillet, 1981). That makes it possible to define regulators on all K-groups at once.

Let $X/\mathbb{Q}$ be smooth and projective. The main ingredients of Beilinson's conjectures are:

- A decomposition of $K_n(X) \otimes \mathbb{Q} = \oplus_{i=0}^{n+\dim(X)} K_n^{(i)}(X)$ with $K_n^{(i)}(X)$ an eigenspace for all Adams operators (Soulé, 1985)
- $\mathrm{reg} : K_n^{(i)}(X) \to H_D^{2i-n}(X_\mathbb{C}, \mathbb{R}(i))^+$ (Deligne cohomology) constructed using Chern classes
- For $n > 1$, this conjecturally induces an isomorphism $\mathrm{reg} : K_n^{(i)}(X)_{\mathrm{int}} \otimes_\mathbb{Q} \mathbb{R} \xrightarrow{\sim} H_D^{2i-n}(X_\mathbb{C}, \mathbb{R}(i))^+$ with int indicating a subgroup of elements "coming from over $\mathbb{Z}$ instead of $\mathbb{Q}$" (cf. $\mathcal{O}_k^* \leftrightarrow k^*$ for a number field k)

Beilinson's conjectures on special values of L-functions

- $H_D^{2i-n}(X_{\mathbb{C}}, \mathbb{R}(i))^+$ is built from $H^{2i-n-1}(X_{\mathbb{C}}, \mathbb{R}(i-1))^+$ and $F^i H_{dR}^{2i-n-1}(X_{\mathbb{R}})$, which identifies $\Lambda H_D^{2i-n}(X_{\mathbb{C}}, \mathbb{R}(i))^+$ with $\Lambda H^{2i-n-1}(X_{\mathbb{C}}, \mathbb{R}(i-1))^+ \otimes (\Lambda F^i H_{dR}^{2i-n-1}(X_{\mathbb{R}}))^{-1}$
- $\Lambda H_D^{2i-n}(X_{\mathbb{C}}, \mathbb{R}(i))^+ \simeq \mathbb{R}$ now has two copies of $\mathbb{Q}$ inside: one from using reg and $\Lambda K_n^{(i)}(X)_{\mathrm{int}}$, and one from using $H^{2i-n-1}(X_{\mathbb{C}}, \mathbb{Q}(i-1))^+$ and $F^i H_{dR}^{2i-n-1}(X_{\mathbb{Q}})$ in the previous point; one is obtained by multiplying the other by an element of $\mathbb{R}^*/\mathbb{Q}^*$, the Beilinson regulator $R_{n,i}$ of $K_n^{(i)}(X)_{\mathrm{int}}$
- Conjecturally, $R_{n,i} \equiv L^*(H^{2i-n-1}(X), i-n)$ in $\mathbb{R}^*/\mathbb{Q}^*$. $L(H^{2i-n-1}(X), s)$ the L-function associated to $H^i(X)$, assumed to satisfy a suitable functional equation, * denotes the first non-vanishing coefficient in the Taylor expansion
- (Also conjectures for $n = 0$, 1, with slightly different ingredients.)

Compatibility of the various regulator maps

- Bloch's regulator map: $X = E$, $n = i = 2$, the cup product in Deligne cohomology gives Bloch's regulator map and Beilinson's coincide (Beilinson's regulator map is "normalized by the logarithm and the whole (co)homology formalism")
- Borel's regulator map: $X = k$, $n = 2m - 1$, $i = m$ with $m \geq 2$: much harder (done by Beilinson (sketchy, difficult to follow), Rapoport, in the end by Burgos Gil): the Borel regulator map is twice the Beilinson regulator map.