# Algebraic $K$-theory and arithmetic

Rob de Jeu

http://www.few.vu.nl/∼jeu

Department of Mathematics

VU University Amsterdam

Amsterdam

The Netherlands

# The Riemann $\zeta$-function

$$\zeta(s) = \sum_{n\geq 1} n^{-s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}} \qquad (\mathrm{Re}(s) > 1)$$

can be extended to a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1$ with residue $1$

| | |
|---|---|
| $\zeta(2) = \pi^2/6$ | $\zeta(3)$ irrational |
| $\zeta(4) = \pi^4/90$ | $\zeta(5)$ ??? |
| $\zeta(6) = \pi^6/945$ | $\zeta(7)$ ??? |
| $\vdots$ | $\vdots$ |

# The $\zeta$-function of a number field

Let $k$ be a number field, i.e., for some $f(X)$ an irreducible polynomial of degree $d$ in $\mathbb{Q}[X]$, and $\alpha$ a root of $f(X)$ in $\mathbb{C}$,

$$k = \mathbb{Q}(\alpha) = \{b_0 + b_1\alpha + \cdots + b_{d-1}\alpha^{d-1},\ \text{all } b_j \text{ in } \mathbb{Q}\}.$$

Let $\mathcal{O}$ be the ring of algebraic integers of $k$: $x \in k$ is an algebraic integer if it is the zero of a polynomial $X^n + a_{n-1}X^{n-1} + \ldots + a_1X + a_0$ with all $a_i$ in $\mathbb{Z}$.

The $\zeta$-function of $k$ is defined by (for $\mathrm{Re}(s) > 1$)

$$\zeta_k(s) = \sum_{\substack{(0)\neq I\subset\mathcal{O} \\ I \text{ an ideal of } \mathcal{O}}} (\#\mathcal{O}/I)^{-s} = \prod_{\substack{0\neq\mathcal{P}\subset\mathcal{O} \\ \mathcal{P} \text{ prime ideal}}} \frac{1}{1-(\#\mathcal{O}/\mathcal{P})^{-s}}.$$

Every non-zero ideal of $\mathcal{O}$ is uniquely (up to ordering) the product of non-zero prime ideals.

$\zeta_k(s)$ can be extended to a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1$

Let $r_1$ the number of embeddings $k \to \mathbb{R}$, $2r_2$ the number of non-real embeddings $k \to \mathbb{C}$, so $d = r_1 + 2r_2$.

$\mathcal{O}^* \cong \mathbb{Z}^r \times \mathbb{Z}/w\mathbb{Z}$ with $r = r_1 + r_2 - 1$ and

$w =$ the number of roots of unity in $k$

Let $\sigma_1, \ldots, \sigma_{r+1}$ be the embeddings of $k$ into $\mathbb{C}$ up to complex conjugaton.

If $u_1, \ldots, u_r$ form a $\mathbb{Z}$-basis of $\mathcal{O}^*/\{\text{roots of unity}\}$, let

$$R = \frac{2^{r_2}}{d} \left| \det \begin{pmatrix} 1 & \log|\sigma_1(u_1)| & \ldots & \log|\sigma_1(u_r)| \\ \vdots & \vdots & & \vdots \\ 1 & \log|\sigma_{r+1}(u_1)| & \ldots & \log|\sigma_{r+1}(u_r)| \end{pmatrix} \right|$$

Then

$$\mathrm{Res}_{s=1}\zeta_k(s) = \frac{2^{r_1}(2\pi)^{r_2} R \; |\mathrm{Cl}(\mathcal{O})|}{w\sqrt{\Delta_k}}$$

- $\mathrm{Cl}(\mathcal{O}) =$ the class group of $\mathcal{O}$ measures (failure of) unique factorization
- $w =$ the number of roots of unity in $k = |\mathcal{O}^*{}_{\text{torsion}}|$
- $\Delta_k$ the absolute value of the discriminant of $k$.

Then

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This is a statement about algebraic $K$-theory:

$K_0(\mathcal{O}) \cong \mathbb{Z} \oplus \mathrm{Cl}(\mathcal{O})$ and $K_1(\mathcal{O}) \cong \mathcal{O}^*$,

so

$|\mathrm{Cl}(\mathcal{O})| = |K_0(\mathcal{O})_{\text{torsion}}| =$ and $w = |K_1(\mathcal{O})_{\text{torsion}}|$.

# Algebraic $K$-theory of a ring: $K_0$

$R$: a commutative ring with identity $1 \neq 0$

$$K_0(R) = \frac{\text{free Abelian group on generators } [M],\ M \text{ a finitely generated projective } R\text{-module}}{\left\langle [P] - [P'] - [P''] \text{ for each exact sequence } 0 \to P' \to P \to P'' \to 0 \right\rangle}.$$

$P$ projective means every surjection $M \to P$ admits a section of $R$-modules, e.g., a free $R$-module. Therefore $P \cong P' \oplus P''$ in the above.

Example

- $F$ a field: $K_0(F) \cong \mathbb{Z}$ via the dimension of a vector space
- $K_0(\mathbb{Z}) \cong \mathbb{Z}$ via rank of a finitely generated Abelian group
- $\mathcal{O} =$ ring of integers in a number field: $K_0(\mathcal{O}) \cong \mathbb{Z} \oplus \mathrm{Cl}(\mathcal{O})$

# Algebraic $K$-theory of a ring: $K_1$

View $GL_n(R) \subset GL_{n+1}(R)$ via $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$.

Let $GL(R) = \bigcup_n GL_n(R)$.

**Definition** $K_1(R) = GL(R)/[GL(R), GL(R)]$

The determinant gives a surjection $K_1(R) \to R^*$ with kernel $SK_1(R)$

**Example**

- $F$ a field: $K_1(F) \cong F^*$
- $K_1(\mathbb{Z}) \cong \mathbb{Z}^* = \{\pm 1\}$
- $\mathcal{O}$ = ring of integers in a number field: $K_1(\mathcal{O}) \cong \mathcal{O}^*$
- If, e.g., $R = \mathbb{Q}[x, y]/(y^2 - x^3 - 3)$ then $R^* = \mathbb{Q}^*$ but $SK_1(R)$ is infinite

# Algebraic $K$-theory of a ring: $K_2$

$K_1(R)$ (matrices or maps) give relations between generators of $K_0(R)$ (=$R$-modules). "So" $K_2(R)$ should involve "relations among the relations for $K_1(R)$".

Definition For $i, j \geq 1$, $i \neq j$,and $r$ in $R$, let $e_{i,j}(r)$ be the elementary matrix with $r$ in position $(i, j)$

Then

$$e_{i,j}(r)e_{i,j}(s) = e_{i,j}(r + s)$$

$$[e_{i,j}(r), e_{j,l}(s)] = e_{i,l}(rs) \text{ if } i \neq l$$

$$[e_{i,j}(r), e_{k,l}(s)] = 1 \text{ if } j \neq k, i \neq l$$

and the $e_{i,j}(r)$ generate the subgroup $[GL(R), GL(R)]$ of $GL(R)$

Definition The Steinberg group St$(R)$ of $R$ is the free group on symbols $x_{i,j}(r)$ with $i, j \geq 1$, $i \neq j$, $r$ in $R$, quotiented out to give the same three relations for the $x_{i,j}(r)$ as for the $e_{i,j}(r)$.

We have a surjective group homomorphism

$$\begin{aligned} \varphi : \mathrm{St}(R) &\rightarrow [GL(R), GL(R)] \\ x_{i,j}(r) &\mapsto e_{i,j}(r) \end{aligned}$$

Definition $K_2(R) = \ker(\varphi)$

Proposition $K_2(R)$ is an Abelian group. In fact, it is the centre of St$(R)$.

# $K_2$ of a field

If $F$ is a field then $K_2(F)$ is an Abelian group written additively, with

generators $\{a, b\}$ for $a, b$ in $F^*$

relations
$$\{a_1a_2, b\} = \{a_1, b\} + \{a_2, b\}$$
$$\{a, b_1b_2\} = \{a, b_1\} + \{a, b_2\}$$
$$\{a, 1-a\} = 0 \text{ if } a \neq 0, 1$$

Then also $\{a, b\} = -\{b, a\}$ and $\{c, -c\} = 0$ for $a$, $b$, $c$ in $F^*$.

If $A$ in $\mathrm{St}(F)$ lifts $\begin{pmatrix} a & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $B$ lifts $\begin{pmatrix} b & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b^{-1} \end{pmatrix}$, then $\{a, b\} = [A, B]$ in $K_2(F)$.

# An example: $K_2(\mathbb{Q})$

**Proposition**

$$K_2(\mathbb{Q}) \xrightarrow{\sim} \{\pm 1\} \times \oplus_{p \text{ prime}} \mathbb{F}_p^*$$

with components

$$T_\infty : K_2(\mathbb{Q}) \to \{\pm 1\} \text{ with } T_\infty(\{a,b\}) = \begin{cases} -1 \text{ if } a,b < 0 \\ \ 1 \text{ otherwise} \end{cases}$$

$T_p : K_2(\mathbb{Q}) \to \mathbb{F}_p^*$ with $T_p(\{a,b\}) = (-1)^{v_p(a)v_p(b)} \frac{a^{v_p(b)}}{b^{v_p(a)}} \text{ modulo } p$
where $v_p(a) \in \mathbb{Z}$ is the number of factors $p$ in $a$
$T_p =$ the tame symbol for $p$

The proof of the proposition is based on repeated rewriting using division with remainder: if $a = qb + r$ with $a, b, q, r$ non-zero integers, then $\{a/r, -qb/r\} = 0$ in $K_2(\mathbb{Q})$.

# Quadratic reciprocity

For $p > 2$ and $\alpha \in K_2(\mathbb{Q})$ we have $T_p(\alpha)^{\frac{p-1}{2}} \in \{\pm 1\} \subseteq \mathbb{F}_p^*$.

Define $\widetilde{T}_2 : K_2(\mathbb{Q}) \to \{\pm 1\}$ as follows.
Write $a = (-1)^i 2^j 5^k \frac{c}{d}$ with $i, k = 0, 1$ and $c$, $d$ integers congruent $1 \bmod 8$, $b = (-1)^I 2^J 5^K \frac{c'}{d'}$ similarly. Then

$$\widetilde{T}_2(\{a, b\}) = (-1)^{iI + jK + kJ}.$$

Identify $\{\pm 1\} \subseteq \mathbb{F}_p^*$ for all primes $p > 2$.

Theorem $\quad T_\infty(\{a, b\}) = \widetilde{T}_2(\{a, b\}) \prod_{\substack{p>2 \\ p \text{ prime}}} T_p(\{a, b\})^{\frac{p-1}{2}}.$

This is equivalent with quadratic reciprocity. E.g., let $p$ and $q$ be distinct odd primes, and put $\left(\frac{p}{q}\right)$ equal to 1 if $p$ is a square modulo $q$, and to $-1$ if not. Equivalently,

$$\left(\frac{p}{q}\right) = p^{\frac{q-1}{2}} \bmod q = T_q(\{p,q\})^{\frac{q-1}{2}}$$

The theorem says that

$$\begin{aligned} 1 &= \widetilde{T}_2(\{p,q\}) T_p(\{p,q\})^{\frac{p-1}{2}} T_q(\{p,q\})^{\frac{q-1}{2}} \\ &= (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}} \left(\frac{q}{p}\right)\left(\frac{p}{q}\right). \end{aligned}$$

# Borel's theorem

Quillen defined Abelian groups $K_n(R)$ ($n \geq 0$) for rings $R$, as well as for algebraic varieties.

Let $k$ be a number field, with $r_1$ real and $2r_2$ non-real embeddings, $d = r_1 + 2r_2$, and ring of algebraic integers $\mathcal{O}$, and $\Delta_k$ the absolute value of the discriminant of $k$

Recall that

- $K_0(\mathcal{O}) \cong \mathbb{Z} \oplus \mathrm{Cl}(\mathcal{O})$
- $K_1(\mathcal{O}) \cong \mathcal{O}^*$ has rank $r_1 + r_2 - 1$

Theorem (Quillen) $K_n(\mathcal{O})$ is finitely generated for all $n \geq 0$.

# Borel's theorem

Theorem (Borel)

(1) $K_{2n}(\mathcal{O})$ is a finite group if $n \geq 1$.

(2) For $n \geq 2$, $K_{2n-1}(\mathcal{O})$ has rank $m_{2n-1} = r_1 + r_2$ if $n$ is odd, and rank $m_{2n-1} = r_2$ if $n$ is even.

(3) There exists a natural regulator map

$$K_{2n-1}(\mathcal{O}) \to \mathbb{R}^{m_{2n-1}} \qquad (n \geq 2).$$

Its image is a lattice with volume of a fundamental domain

$$V_n = q \frac{\zeta_k(n)}{\pi^{n(d-m_{2n-1})}\sqrt{\Delta_k}}$$

with $q$ in $\mathbb{Q}^*$.

# Example: the $K$-theory of $\mathbb{Z}$

$\zeta_{\mathbb{Q}}$ is the Riemann zeta function. For $n \geq 2$:

$K_{2n-1}(\mathbb{Z})$ is finite for $n$ even;

$K_{2n-1}(\mathbb{Z})$ has rank $1$ for $n$ odd, and $V_n = q\zeta(n)$ for some $q \in \mathbb{Q}^*$.

| $n$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | $\ldots$ |
|---|---|---|---|---|---|---|---|
| $m_{2n-1}$ | $0$ | $1$ | $0$ | $1$ | $0$ | $1$ | $\ldots$ |
| $\zeta(n)$ | $\pi^2/6$ | irrational | $\pi^4/90$ | ??? | $\pi^6/945$ | ??? | $\ldots$ |

# Generalizations: curves

For simplicity, let $E$ be defined in $\mathbb{P}^2_{\mathbb{Q}}$ by

$$Y^2Z = X^3 + AXZ^2 + BZ^3$$

with $A$, $B$ in $\mathbb{Z}$, and $4A^3 + 27B^2 \neq 0$.
Let $p \neq 2, 3$ be a prime number not dividing $4A^3 + 27B^2$, then $Y^2Z = X^3 + \overline{A}XZ^2 + \overline{B}Z^3$ defines a "nice" curve $E_p$ in $\mathbb{P}^2_{\mathbb{F}_p}$.
Put

$$a_p = 1 + p - |E_p(\mathbb{F}_p)| \quad \text{and} \quad L_p(E,s) = \frac{1}{1 - a_p p^{-s} + p^{1-2s}}.$$

There is also an $L_p$ for the remaining $p$. Put

$$L(E,s) = \prod_{p \text{ prime}} L_p(E,s) \qquad (\mathrm{Re}(s) > \frac{3}{2}).$$

Let $E_{\mathbb{C}}$ be the curve by the equation in $\mathbb{P}^2_{\mathbb{C}}$, $F$ be the field of meromorphic functions on $E_{\mathbb{C}}$. There is an exact localization sequence

$$K_2(E_{\mathbb{C}}) \to K_2(F) \xrightarrow{T} \oplus_{x \in E_{\mathbb{C}}} \mathbb{C}^*$$

where $T_x$ is the tame symbol for $x$:

$$T_x : \{f, g\} \mapsto (-1)^{\mathrm{ord}_x(f)\,\mathrm{ord}_x(g)} \frac{f^{\mathrm{ord}_x(g)}}{g^{\mathrm{ord}_x(f)}}|_x$$

with $\mathrm{ord}_x(f)$ the order of vanishing of $f$ at $x$.

For two non-zero meromorphic functions $f$ and $g$ on $E_{\mathbb{C}}$, $\log|f|\,\mathrm{d}\arg g - \log|g|\,\mathrm{d}\arg f$ is a closed $1$-form on some $E_{\mathbb{C}} \setminus S$ with $S$ finite. Because

$$\log|z|\,\mathrm{d}\arg(1-z) - \log|1-z|\,\mathrm{d}\arg z = \mathrm{d}P_2(z),$$

where $P_2(z)$ is a $C^\infty$-function on $\mathbb{C} \setminus \{0,1\}$, we get a map

$$\begin{aligned} \mathrm{reg} : K_2(F) &\to \frac{\{\text{closed } 1\text{-forms on some } E_{\mathbb{C}} \setminus S\}}{\{\text{exact } 1\text{-forms on some } E_{\mathbb{C}} \setminus S\}} \\ \{f, g\} &\mapsto \log|f|\,\mathrm{d}\arg g - \log|g|\,\mathrm{d}\arg f. \end{aligned}$$

This fits into a commutative diagram

$$\begin{tikzcd}
 & K_2(E) \arrow[d] & & \\
 & K_2(E_{\mathbb{C}}) \arrow[r] \arrow[d, dotted, "\mathrm{reg}"'] & K_2(F) \arrow[r] \arrow[d, "\mathrm{reg}"'] & \bigoplus_{x \in E_{\mathbb{C}}} \mathbb{C}^* \arrow[d, "\log|\cdot|"'] \\
0 \arrow[r] & H^1_{\mathrm{dR}}(E_{\mathbb{C}};\mathbb{R}) \arrow[r] & H^1_{\mathrm{dR}}(F;\mathbb{R}) \arrow[r, "i\cdot\text{res}"'] & \bigoplus_{x \in E_{\mathbb{C}}} \mathbb{R}
\end{tikzcd}$$

with $H^1_{\mathrm{dR}}(F;\mathbb{R}) = \varinjlim_{S \subset E_{\mathbb{C}}} H^1_{\mathrm{dR}}(E_{\mathbb{C}} \setminus S;\mathbb{R})$ where all $S$ are finite.

# Bloch's result

Theorem (Bloch) Let $E$ be an elliptic curve defined over $\mathbb{Q}$, with complex multiplication. Then there exists an element $\alpha$ in $K_2(E)$ with

$$L'(E,0) = q\frac{1}{2\pi}\int_{E_{\mathbb{C}}} \mathrm{reg}(\alpha)\wedge\omega$$

for some $q$ in $\mathbb{Q}^*$, or, using the functional equation for the $L$-function:

$$\frac{1}{2\pi}L(E,2) = q'\int_{E_{\mathbb{C}}} \mathrm{reg}(\alpha)\wedge\omega.$$

$\omega$ is a non-zero holomorphic form on $E_{\mathbb{C}}$ with $\int_{E(\mathbb{R})}\omega = 1$.

# Beilinson's conjecture for $K_2$

Consider a "nice" curve $C$ defined over $\mathbb{Q}$ with field of rational functions $F = \mathbb{Q}(C)$. We have an exact localization sequence

$$\cdots \to \coprod_{P \in C^{(1)}} K_2(\mathbb{Q}(P)) \to K_2(C) \to K_2(F) \xrightarrow{T} \coprod_{P \in C^{(1)}} \mathbb{Q}(P)^* \to \cdots$$

Fact $K_2$ of a number field is an infinite torsion group.

So $K_2(C)/\mathrm{torsion} = \ker(T)/\mathrm{torsion}$.

Beilinson defines a subgroup

$$K_2(C;\mathbb{Z}) \subseteq \ker(T)/\mathrm{torsion} = K_2(C)/\mathrm{torsion}$$

by considering the reduction of the curve at all primes.

## Conjecture (Beilinson)

(1) The group $K_2(C;\mathbb{Z})$ is a free Abelian group of rank $g$, the genus of $C$.

(2) The pairing

$$H_1(C_{\mathrm{an}};\mathbb{Z})^- \times K_2(C;\mathbb{Z}) \to \mathbb{R}$$
$$(\gamma,\alpha) = \frac{1}{2\pi}\int_\gamma \mathrm{reg}(\alpha)$$

is non-degenerate. $H_1(C_{\mathrm{an}};\mathbb{Z})^- \cong \mathbb{Z}^g$: anti-invariants under complex conjugation on $C_{\mathbb{C}}$

(3) Let $R$ denote the absolute value of the determinant of this pairing with respect to $\mathbb{Z}$-bases of $H_1(C_{\mathrm{an}};\mathbb{Z})^-$ and $K_2(C;\mathbb{Z})$. Then $\frac{1}{(2\pi)^{2g}}L(C,2) = qR$ with $q \in \mathbb{Q}^*$.

# Obtaining elements in $\ker(T)$

For $g > 0$ it is never known that $K_2(C;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is finite dimensional. In practice one finds a subgroup of $K_2(C;\mathbb{Z})$ of the expected rank and uses this to compute the regulator $R$ and check (3).

A construction Suppose $O$, $P$, $Q$ in $C(\mathbb{Q})$ are such that there exist $f_P$ and $f_Q$ in $F^*$ and integers $d, e > 0$ with

$$\mathrm{ord}_P(f_P) = d \qquad \mathrm{ord}_O(f_P) = -d \qquad \mathrm{ord}_x(f_P) = 0 \text{ for } x \neq O, P$$

$$\mathrm{ord}_Q(f_Q) = e \qquad \mathrm{ord}_O(f_Q) = -e \qquad \mathrm{ord}_x(f_P) = 0 \text{ for } x \neq O, Q$$

(i.e., div$(f_P) = d(P) - d(O)$ and div$(f_Q) = e(Q) - e(O)$)

and $f_P(Q) = f_Q(P) = 1$. Then $\alpha = \{f_P, f_Q\}$ is in $\ker(T)$.

# An example for higher genus

For $g > 0$, if $b_1, \ldots, b_g$ in $\mathbb{Q}^*$ are distinct, $\lambda$ is in $\mathbb{Q}^*$,

$$p(x) = 2x^{g+1} + q(x), \quad q(x) = \lambda \prod_{j=1}^{g} (b_j x + 1), \quad r(x) = 4x^{g+1} + q(x),$$

then $y^2 = -4x^{2g+2} + p(x)^2 = q(x)r(x)$ defines a curve over $\mathbb{Q}$ of genus $g$ provided $r(x)$ has no multiple roots, with one point $O$ at infinity. Because

$$\text{div}(y - p(x)) = (2g+2)((0,\lambda)) - (2g+2)(O)$$

$$\text{div}(b_l x + 1) = 2((-1/b_l, 0)) - 2(O)$$

we have elements in $\ker(T)$ for $l = 1, \ldots, g$:

$$\alpha_l \stackrel{\text{def}}{=} 2\left\{\frac{y - p(x)}{-p(-1/b_l)}, b_l x + 1\right\} = 2\left\{\frac{y - p(x)}{2x^{g+1}}, b_l x + 1\right\}$$

**Proposition** If the $b_i$ and $\lambda$ are in $\mathbb{Z}$ then the $\alpha_l$ give classes in $K_2(C;\mathbb{Z})$.

The regulators and $L(C,2)$ can be computed numerically so we can verify (3) in Beilinson's conjecture (relation between $R$ and $L(C,2)$) numerically.

**Example (g=2)** For $b_1 = 1$, $b_2 = 8$, $\lambda = 1$ we get

$$\frac{N}{(2\pi)^4} L(C,2) = 2^{-4} 3^{-1} 7^{-1} R$$

with $N = 2 \cdot 7 \cdot 59$.

**Example (g=4)** For $b_i = i$ ($i = 1, \ldots, 4$) and $\lambda = 1$ we get

$$\frac{N}{(2\pi)^8} L(C,2) = 2^{-6} 5^{-2} R$$

with $N = 2^9 \cdot 3^2 \cdot 17 \cdot 113$.

# Some limit results

Theorem For $g \geq 1$ let $\lambda$ and $b_1, \ldots, b_g$ be in $\mathbb{R}^*$ with all $b_j$ distinct. If $b_1, \ldots, b_g$ are fixed and $|\lambda| \gg 0$, then $r(x)$ has $g+1$ distinct real roots, and on the curve $C$ of genus $g$ defined over $\mathbb{R}$ by $y^2 = -4x^{2g+2} + p(x)^2 = q(x)r(x)$ the regulator $R = R(\lambda)$ of $\alpha_1, \ldots, \alpha_g$ satisfies $\lim_{|\lambda|\to\infty} \frac{R(\lambda)}{(2\log|\lambda|)^g} = 1$.
In particular, if $\lambda$ and all $b_j$ are integers then $K_2(C;\mathbb{Z})$ has rank at least $g$ for $|\lambda| \gg 0$.

Theorem Let $k$ be a real quadratic field and $\mathcal{O}_k$ its ring of algebraic integers. With $v \neq \pm 1$ in $\mathcal{O}_k^*$,

$$y^2 = ((v^n + 4v^{-n})x + 1)(v^n x + 1)(4v^{-n}x + 1)$$

for $n \gg 0$ defines a non-CM elliptic curve over $k$, with $j$-invariant not in $\mathbb{Q}$, having two elements in $K_2(C;\mathbb{Z})$ with $\displaystyle\lim_{n\to\infty} \frac{R(n)}{n^2} = 16\,|\log|v||^2$, $|v|$ computed using either $k \to \mathbb{R}$.