Numerical verification of a conjecture of Perrin-Riou for number fields.

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(Joint work with A. Besser, P. Buckingham and X.-F. Roblot.)

Background

k: a number field of degree d
r_1: the number of real embeddings of k
$2r_2$: the number of complex embeddings of k
$\mathcal{O}_k$: the ring of algebraic integers of k

$K_1(\mathcal{O}_k) \cong \mathcal{O}_k^*$ has rank $r = r_1 + r_2 - 1$
Let $\sigma_1, \ldots, \sigma_{r+1}$ be the embeddings of k into $\mathbb{C}$ up to complex conjugaton.
If $u_1, \ldots, u_r$ form a $\mathbb{Z}$-basis of $\mathcal{O}_k^*/\text{torsion}$, let

$$R = \frac{2^{r_2}}{[k:\mathbb{Q}]} | \det \begin{pmatrix} 1 & \log|\sigma_1(u_1)| & \ldots & \log|\sigma_1(u_r)| \\ \vdots & \vdots & & \vdots \\ 1 & \log|\sigma_{r+1}(u_1)| & \ldots & \log|\sigma_{r+1}(u_r)| \end{pmatrix} |$$

Then

$$\text{Res}_{s=1}\zeta_k(s) = \frac{2^{r_1}(2\pi)^{r_2} R |\text{Cl}(\mathcal{O}_k)|}{w\sqrt{|D_k|}}$$

$D_k =$ the discriminant of k
$w = |\mathcal{O}_{k,\text{torsion}}^*| = $ #roots of unity in k

Theorem (Borel)
For $n \geq 2$:

- $K_{2n-1}(k)$ has rank $r_\pm$ if $(-1)^n = \pm 1$, where $r_+ = r_2$ and $r_- = r_1 + r_2$.
- there is a natural regulator map

$$K_{2n-1}(k) \to \left(\prod_{\sigma : k \to \mathbb{C}} \mathbb{R} \right)^{\mp} \cong \mathbb{R}^{r_\pm}$$

where $\mp$ indicates the part where $x_\sigma = \mp x_{\overline{\sigma}}$, with image a lattice;

- if $R_n(k)$ denotes the volume of a fundamental domain of this image then

$$\frac{\zeta_k(n)\sqrt{|D_k|}}{\pi^{n r_\mp} R_n(k)}$$

is in $\mathbb{Q}^*$.

The conjecture

Interpolation formula

The p-adic L-function $L_{k,p}$ of a totally real number field k satisfies

$$L_{k,p}(n, \omega_k^{1-n}) = E_p(n)\zeta_k(n)$$

for all integers $n < 0$, where $E_p(s) = \prod_{\mathcal{P}|p}(1 - N(\mathcal{P})^{-s})$.

Let $k' \subset \mathbb{C}_p = \hat{\overline{\mathbb{Q}}}_p$ be the Galois closure of k, with completion $\widehat{k'}$. For $\sigma : k \to \widehat{k'}$ consider

$$\mathrm{reg}_{p,\sigma} : K_{2n-1}(k) \cong K_{2n-1}(\mathcal{O}_k) \xrightarrow{\sigma} K_{2n-1}(\mathcal{O}_{k'}) \xrightarrow{\mathrm{reg}_{\mathrm{syn}}} \widehat{k'} \ .$$

For k totally real and $n \geq 2$ let $R_{n,p}(k)$ be the determinant of

$$K_{2n-1}(k)/\text{torsion} \xrightarrow{\prod_\sigma \mathrm{reg}_{p,\sigma}} \widehat{k'}^d$$

with respect to a basis of $K_{2n-1}(k)/\text{torsion}$.

Conjecture (B. Perrin-Riou; A. Besser and R. de Jeu)
For p prime, k a totally real number field, and $n \geq 2$ odd, we have:

- $R_{n,p}(k)$ and $L_{k,p}(n, \omega_k^{1-n})$ are non-zero;
- with $D_k^{1/2,p}$ a square root of D_k in $\mathbb{C}_p$ the quotient

$$\frac{L_{k,p}(n, \omega_k^{1-n}) D_k^{1/2,p}}{E_p(n) R_{n,p}(k)}$$

is in $\mathbb{Q}^*$;

- this is the same rational number as in Borel's theorem.

For the last part we use compatible p-adic and real regulators and roots of D_k:

- fix a $\mathbb{Z}$-basis $\{a_1, \ldots, a_d\}$ of $\mathcal{O}_k$;
- fix a $\mathbb{Z}$-basis $\{A_1, \ldots, A_d\}$ of $K_{2n-1}(k)/\text{torsion}$;
- let $\sigma_i^\infty : k \to \mathbb{C}$ and $\sigma_i^p : k \to \widehat{k'}$ $(i = 1, \ldots, d)$ be the embeddings;
- let

$$D_k^{1/2,\infty} = \det(\sigma_i^\infty(a_j)) \qquad D_k^{1/2,p} = \det(\sigma_i^p(a_j))$$

$$R_n(k) = \det(\sigma_i^\infty(A_j)) \qquad R_{n,p}(k) = \det(\sigma_i^p(A_j))\,.$$

Then

$$\frac{D_k^{1/2,\infty}}{R_n(k)} \quad \text{and} \quad \frac{D_k^{1/2,p}}{R_{n,p}(k)}$$

are invariant under reordering the σ_i^∞ or σ_i^p and transform in the same way if we change the bases of $\mathcal{O}_k$ and $K_{2n-1}(k)/\text{torsion}$.

Remark

- $L_{k,p}(n, \omega_k^{1-n})$ and $R_{n,p}(k)/D_k^{1/2,p}$ are in $\mathbb{Q}_p$.
- In the conjecture we could replace $K_{2n-1}(k)/\text{torsion}$ with $K_{2n-1}(k)_{\mathbb{Q}}$ and use a $\mathbb{Q}$-basis.

Remark

The conjecture is analogous to the result by Colmez that

$$\text{Res}_{s=1}\zeta_{k,p}(s) = \frac{2^d R_{1,p}|\text{Cl}(\mathcal{O}_k)|}{2D_k^{1/2,p}}$$

for the p-adic regulator $R_{1,p}$ of $K_1(\mathcal{O}_k)/\text{torsion} \cong \mathcal{O}_k^*/\text{torsion}$.

Zagier's conjecture: describing $K_{2n-1}(k)$

Let

$$\mathrm{Li}_n(z) = \sum_{j\geq 1} \frac{z^j}{j^n} \qquad (z \text{ in } \mathbb{C} \text{ with } |z| < 1; n \geq 1)$$

- $\mathrm{Li}_1(z) = -\mathrm{Log}(1-z)$
- $\mathrm{Li}'_{n+1}(z) = \mathrm{Li}_n(z)/z$
- $\mathrm{Li}_n(z)$ extends to a multi-valued analytic functions on $\mathbb{C}\setminus\{0,1\}$

On $\mathbb{C} \setminus \{0,1\}$

$$P_n(z) = \pi_{n-1}\left(\sum_{j=0}^{n-1} \frac{b_j}{j!}(2\log|z|)^j \mathrm{Li}_{n-j}(z)\right)$$

is single-valued and satisfies $P_n(z) + (-1)^n P_n(1/z) = 0$.
[b_j = j-th Bernoulli number; π_m = Im for m odd, Re for m even.]

For $n \geq 2$:

- let $B_n(k)$ be a free abelian group on $[x]_n$ $(x \neq 0, 1 \text{ in } k)$
- define

$$\begin{aligned} \tilde{P}_n : B_n(k) &\to \mathbb{R}^d \\ [x]_n &\mapsto (P_n(\sigma(x)))_{\sigma : k \to \mathbb{C}} \end{aligned}$$

- define inductively

$$\mathrm{d}_n : B_n(k) \to \begin{cases} \bigwedge^2_{\mathbb{Z}} k^* & \text{if } n = 2 \\ C_{n-1}(k) \otimes_{\mathbb{Z}} k^* & \text{if } n > 2 \end{cases}$$

$$[x]_n \mapsto \begin{cases} (1-x) \wedge x \text{ if } n = 2 \\ [x]_{n-1} \otimes x \text{ if } n > 2 \end{cases}$$

and

$$C_n(k) = B_n(k)/\mathrm{Ker}(\mathrm{d}_n) \cap \mathrm{Ker}(\tilde{P}_n)$$

Conjecture (Zagier)
If $n \geq 2$ then:
(i) there is an injection

$$\frac{\mathrm{Ker}(\mathrm{d}_n)}{\mathrm{Ker}(\mathrm{d}_n) \cap \mathrm{Ker}(\tilde{P}_n)} \to K_{2n-1}(k)_{\mathbb{Q}}$$

with image a finitely generated group of maximal rank;
(ii) Borel's regulator map is given by $\tilde{P}_n$:

$$\begin{tikzcd}
\frac{\mathrm{Ker}(\mathrm{d}_n)}{\mathrm{Ker}(\mathrm{d}_n) \cap \mathrm{Ker}(\tilde{P}_n)} \arrow[r] \arrow[dr, "\tilde{P}_n"'] & K_{2n-1}(k)_{\mathbb{Q}} \arrow[d, "\mathrm{reg}_{\mathrm{Borel}}"] \\
& \mathbb{R}^d
\end{tikzcd}$$

commutes.

Theorem (R. de Jeu; Beilinson-Deligne)
For $n \geq 2$ there exists an injection as in Zagier's conjecture such that the diagram commutes, with finitely generated image.

Remark
- For $n = 2$ this predates Zagier's conjecture and is due to Bloch and Suslin.
- The image in the theorem has maximal rank for:
  - ⋆ $n = 2$ (Suslin);
  - ⋆ $n = 3$ (Goncharov);
  - ⋆ all $n \geq 2$ if k is cyclotomic.

p-adic polylogarithms: Coleman integration on $\mathbb{P}^1_{\mathbb{C}_p}$

Let:
- $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$
- $|\cdot|_p$: p-adic valution with $|p|_p = p^{-1}$
- $\mathcal{O}$: valuation ring
- $\overline{\mathbb{F}}_p$: residue field

Fix a logarithm $\log : \mathbb{C}_p^* \to \mathbb{C}_p$ such that:
- $\log(ab) = \log(a) + \log(b)$;
- $\log(1+z) =$ usual power series expansion for $|z|_p$ small.

For each x in $\mathbb{P}^1_{\overline{\mathbb{F}}_p}(\overline{\mathbb{F}}_p)$:
- let

$$U_x = \text{residue disc of } x = \{\text{all } y \text{ in } \mathbb{P}^1_{\mathbb{C}_p}(\mathbb{C}_p) \text{ that reduce to } x\}\,,$$

a copy of the maximal ideal of $\mathcal{O}$
- fix a local parameter $t = t_x$ on U_x (e.g., $t_x = z - x$ if $x \neq \infty$, $t_\infty = 1/z$)

For $x \neq 1, \infty$ in $\mathbb{P}^1_{\overline{\mathbb{F}}_p}(\overline{\mathbb{F}}_p)$ let:

- $A(U_x) = \{\sum_{n=0}^{\infty} a_n t^n$ that converge for $|t|_p < 1\}$
- $A_{\log}(U_x) = A(U_x)$
- $\Omega_{\log}(U_x) = A_{\log}(U_x)\mathrm{d}t$

For $x = 1, \infty$ let:

- $A(U_x) = \{\sum_{n=-\infty}^{\infty} a_n t^n$ conv. for $r < |t|_p < 1,$ some $r < 1$ $\}$
- $A_{\log}(U_x) = A(U_x)[\log t]$
- $\Omega_{\log}(U_x) = A_{\log}(U_x)\mathrm{d}t$

Then

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C}_p \arrow[r] & A_{\log}(U_x) \arrow[r, "\mathrm{d}"] & \Omega_{\log}(U_x) \arrow[r] & 0
\end{tikzcd}$$

is exact for each x since $\mathrm{d}\log(t) = \mathrm{d}t/t$.

Theorem (Coleman):
There exists a subspace

$$A_{\mathrm{Col}} \subset \prod_{x \in X(\overline{\mathbb{F}}_p)} A_{\log}(U_x)$$

containing

$$A_{\mathrm{rig}} = \lim_{r \uparrow 1} A_{\mathrm{rig}}(\mathbb{P}^1_{\mathbb{C}_p} \setminus \{z \text{ such that } |z-1|_p \leq r \text{ or } |z|_p \geq 1/r\})$$

and such that, with $\Omega_{\mathrm{Col}} = A_{\mathrm{Col}}\mathrm{d}z$,

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C}_p \arrow[r] & A_{\mathrm{Col}} \arrow[r, "\mathrm{d}"] & \Omega_{\mathrm{Col}} \arrow[r] & 0
\end{tikzcd}$$

is exact.

Definition
For ω in Ω_{Col} and P, Q not in U_1 or U_∞, let

$$\int_P^Q \omega = F_\omega(Q) - F_\omega(P)$$

for any F_ω in A_{Col} with $\mathrm{d}F_\omega = \omega$.

Example
Put $\mathrm{Li}_{n+1}(z) = \int_0^z \mathrm{Li}_n(y)\mathrm{d}\log y$ starting with $\mathrm{Li}_0(z) = \frac{z}{1-z}$.
The $\mathrm{Li}_n(z)$ are characterized in A_{Col} by

- $\mathrm{Li}_n(z) = \sum_{j=1}^\infty \frac{z^j}{j^n}$ for $|z|_p < 1$
- $\mathrm{dLi}_{n+1}(z) = \mathrm{Li}_n(z)\,\mathrm{d}\log(z)$ when $n \geq 0$

Other properties:

- The $\mathrm{Li}_n(z)$ extend to $\mathbb{C}_p \setminus \{1\}$.
- $\mathrm{Li}_n(z^m) = m^{n-1} \sum_{\zeta^m=1} \mathrm{Li}_n(\zeta z)$.

The function

$$P_n^p(z) = \sum_{j=0}^{n-1} c_j \log^j(z)\mathrm{Li}_{n-j}(z)$$

with $c_0 = 1$ satisfies

$$P_n^p(z) + (-1)^n P_n^p(z^{-1}) = 0$$

if $\sum_{j=0}^{n-1} \frac{c_j}{(n-j)!} = 0$.

Theorem (A. Besser and R. de Jeu)
Let k be a number field, and for $\sigma : k \to \widehat{k'}$ let

$$\begin{aligned} C_n^\sigma(\mathcal{O}) &= \langle [x]_n \mid \sigma(x),\, 1-\sigma(x) \text{ are in } \mathcal{O}^* \rangle \\ &\subseteq C_n(k) = \frac{B_n(k)}{\mathrm{Ker}(\mathrm{d}_n) \cap \mathrm{Ker}(\tilde{P}_n)} \,. \end{aligned}$$

Then

$$\begin{aligned} \tilde{P}_n^p : B_n(k) &\to \widehat{k'} \\ [x]_n &\mapsto P_n^p(\sigma(x)) \end{aligned}$$

induces a map

$$\tilde{P}_n^p : C_n^\sigma(\mathcal{O}) \to \widehat{k'}$$

and the solid arrows in

$$\begin{tikzcd} C_n^\sigma(\mathcal{O}) \cap \mathrm{Ker}(\mathrm{d}_n) \arrow[r, hook] \arrow[rrd, "\tilde{P}_n^p"'] & \dfrac{\mathrm{Ker}(\mathrm{d}_n)}{\mathrm{Ker}(\mathrm{d}_n) \cap \mathrm{Ker}(\tilde{P}_n)} \arrow[r] \arrow[rd, dotted, "\tilde{P}_n^p"] & K_{2n-1}(k)_{\mathbb{Q}} \arrow[d, "\mathrm{reg}_{\mathrm{syn}}"] \\ & & \widehat{k'} \end{tikzcd}$$

form a commutative diagram.

Remark

- We conjecture that the dotted arrow exists and that the full diagram commutes.
- This holds for $m[x]_n$ if $x \neq 1$ is an m-th root of unity.

Calculations

We checked the Perrin-Riou/Besser–de Jeu conjecture numerically under the assumption that the dotted arrow in the last diagram exists and that the resulting diagram commutes.

About the calculations:

- To find a subgroup of $\mathrm{Ker}(\mathrm{d}_n)/\mathrm{Ker}(\mathrm{d}_n) \cap \mathrm{Ker}(\tilde{P}_n)$ of rank $\dim_{\mathbb{Q}} K_{2n-1}(k)_{\mathbb{Q}}$ we start with a lot of elements $x \neq 0, 1$ in k and go through the process of Zagier's conjecture.
- We used the computationally simple

$$P_n^p(z) = \mathrm{Li}_n(z) + \log^{n-1}(z) \log(1-z)/n!$$

so that $P_n^p(1/z) = (-1)^{n-1} P_n^p(z)$ and we reduce to $|z|_p \leq 1$.

- If $|z|_p \leq 1$ and $|z-1|_p \geq 1$ then we compute $P_n^p(z)$ using Taylor series around the Teichmüller representatives ζ.
- If $|1-z|_p < 1$ then we do this for

$$\mathrm{Li}_n(z) - \frac{1}{n-1} \log(z) \mathrm{Li}_{n-1}(z)\,,$$

which has a power series expansion on U_1.

- All constants of integration are determined by using

$$\mathrm{Li}_n(z^m) = m^{n-1} \sum_{\zeta^m=1} \mathrm{Li}_n(\zeta z)$$

or by using that

$$\mathrm{Li}_n(z) - p^{-n} \mathrm{Li}_n(z^p) = g_n(1/(1-z))$$

for some $g_n(v)$ in $v\mathbb{Q}[[v]]$, convergent when $|v| < p^{1/(p-1)}$.