Bloch groups and tessellations of hyperbolic 3-space

Rob de Jeu
r.m.h.de.jeu@vu.nl

Department of Mathematics
Vrije Universiteit Amsterdam
The Netherlands
http://www.few.vu.nl/~jeu

Joint work with David Burns, Herbert Gangl, Alexander Rahm, and Dan Yasaki

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The Riemann ζ-function

$$\zeta(s) = \sum_{n \geq 1} n^{-s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} \qquad (\mathrm{Re}(s) > 1)$$

can be extended to a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1$ with residue 1

$$\begin{array}{ll} \zeta(2) = \pi^2/6 & \zeta(3) \text{ irrational} \\ \zeta(4) = \pi^4/90 & \zeta(5) \text{ ???} \\ \zeta(6) = \pi^6/945 & \zeta(7) \text{ ???} \\ \vdots & \vdots \end{array}$$

The ζ-function of a number field

Let k be a number field, i.e., for some $f(X)$ an irreducible polynomial in $\mathbb{Q}[X]$ of degree d, and α a root of $f(X)$ in $\mathbb{C}$,

$$k = \mathbb{Q}(\alpha) = \{b_0 + b_1\alpha + \cdots + b_{d-1}\alpha^{d-1}, \text{ all } b_j \text{ in } \mathbb{Q}\}$$

the number field generated by α.

Let $\mathcal{O}$ be the ring of algebraic integers of k: $x \in k$ is an algebraic integer if it is the zero of a polynomial $X^n + a_{n-1}X^{n-1} + \ldots + a_1X + a_0$ with all a_i in $\mathbb{Z}$.

The ζ-function of k is defined by (for $\mathrm{Re}(s) > 1$)

$$\zeta_k(s) = \sum_{\substack{(0)\neq I\subset\mathcal{O} \\ I \text{ an ideal of } \mathcal{O}}} (\#\mathcal{O}/I)^{-s} = \prod_{\substack{0\neq\mathcal{P}\subset\mathcal{O} \\ \mathcal{P} \text{ prime ideal}}} \frac{1}{1-(\#\mathcal{O}/\mathcal{P})^{-s}}.$$

Every non-zero ideal of $\mathcal{O}$ is uniquely (up to ordering) the product of non-zero prime ideals.

The ζ-function of a number field

$\zeta_k(s)$ can be extended to a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1$

Let r_1 the number of embeddings $k \to \mathbb{R}$, $2r_2$ the number of non-real embeddings $k \to \mathbb{C}$, so $d = r_1 + 2r_2$.
($r_1 = \#$real roots of $f(X)$, $2r_2 = \#$non-real roots of $f(X)$)

$\mathcal{O}^* \cong \mathbb{Z}^r \times \mathbb{Z}/w\mathbb{Z}$ with $r = r_1 + r_2 - 1$ and

$w =$ the number of roots of unity in k

Let $\sigma_1, \ldots, \sigma_{r+1}$ be the embeddings of k into $\mathbb{C}$ up to complex conjugation.

If $u_1, \ldots, u_r$ form a $\mathbb{Z}$-basis of $\mathcal{O}^*/\{\text{roots of unity}\}$, let

$$R = \frac{2^{r_2}}{d} \left| \det \begin{pmatrix} 1 & \log|\sigma_1(u_1)| & \ldots & \log|\sigma_1(u_r)| \\ \vdots & \vdots & & \vdots \\ 1 & \log|\sigma_{r+1}(u_1)| & \ldots & \log|\sigma_{r+1}(u_r)| \end{pmatrix} \right|$$

The ζ-function of a number field

Then

$$\mathrm{Res}_{s=1}\zeta_k(s) = \frac{2^{r_1}(2\pi)^{r_2} R \; |\mathrm{Cl}(\mathcal{O})|}{w\sqrt{\Delta_k}}$$

• $\mathrm{Cl}(\mathcal{O})$ = the class group of $\mathcal{O}$ (a finite Abelian group which measures (failure of) unique factorization in $\mathcal{O}$)
• w = the number of roots of unity in $k = |\mathcal{O}^*{}_{\mathrm{torsion}}|$
• Δ_k the absolute value of the discriminant of k.

This is a statement about algebraic K-theory:

$K_0(\mathcal{O}) \cong \mathbb{Z} \oplus \mathrm{Cl}(\mathcal{O})$ and $K_1(\mathcal{O}) \cong \mathcal{O}^*$,

so

$|\mathrm{Cl}(\mathcal{O})| = |K_0(\mathcal{O})_{\mathrm{torsion}}|$ and $w = |K_1(\mathcal{O})_{\mathrm{torsion}}|$.

Algebraic K-theory of a ring: K_0

R: a commutative ring with identity $1 \neq 0$

$$K_0(R) = \frac{\begin{array}{l}\text{free Abelian group on generators } [M],\ M \text{ a} \\ \text{finitely generated projective } R\text{-module}\end{array}}{\left\langle \begin{array}{l} [P]-[P']-[P''] \text{ for each exact} \\ \text{sequence } 0 \to P' \to P \to P'' \to 0 \end{array} \right\rangle}.$$

P projective means every surjection $M \to P$ admits a section of R-modules, e.g., a free R-module.
Therefore $P \cong P' \oplus P''$ in the above.

Example

- F a field: $K_0(F) \cong \mathbb{Z}$ via the dimension of a vector space
- $K_0(\mathbb{Z}) \cong \mathbb{Z}$ via rank of a finitely generated Abelian group
- $\mathcal{O}$ = ring of integers in a number field: $K_0(\mathcal{O}) \cong \mathbb{Z} \oplus \mathrm{Cl}(\mathcal{O})$

View $GL_n(R) \subset GL_{n+1}(R)$ via $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$.

Let $GL(R) = \bigcup_n GL_n(R)$.

Definition $K_1(R) = GL(R)/[GL(R), GL(R)]$

Determinant gives a surjection $K_1(R) \to R^*$ with kernel $SK_1(R)$

Example

- F a field: $K_1(F) \cong F^*$
- $K_1(\mathbb{Z}) \cong \mathbb{Z}^* = \{\pm 1\}$
- $\mathcal{O}$ = ring of integers in a number field: $K_1(\mathcal{O}) \cong \mathcal{O}^*$
- If, e.g., $R = \mathbb{Q}[x, y]/(y^2 - x^3 - 3)$ then $R^* = \mathbb{Q}^*$ but $SK_1(R)$ is infinite

Algebraic K-theory of a ring: K_2

$K_1(R)$ (matrices or maps) give relations between generators of $K_0(R)$ (=R-modules). "So" $K_2(R)$ should involve "relations among the relations for $K_1(R)$".

Definition For $i, j \geq 1$, $i \neq j$, and r in R, let $e_{i,j}(r)$ be the elementary matrix with r in position (i,j)

Then

$$e_{i,j}(r)e_{i,j}(s) = e_{i,j}(r+s)$$
$$[e_{i,j}(r), e_{j,l}(s)] = e_{i,l}(rs) \text{ if } i \neq l$$
$$[e_{i,j}(r), e_{k,l}(s)] = 1 \text{ if } j \neq k, i \neq l$$

and the $e_{i,j}(r)$ generate the subgroup $[GL(R), GL(R)]$ of $GL(R)$.

Algebraic K-theory of a ring: K_2

The Steinberg group $\mathrm{St}(R)$ of R is the free group on symbols $x_{i,j}(r)$ with $i, j \geq 1$, $i \neq j$, r in R, quotiented out to give the same three relations for the $x_{i,j}(r)$ as for the $e_{i,j}(r)$.

We have a surjective group homomorphism

$$\begin{aligned} \varphi : \mathrm{St}(R) &\to [GL(R), GL(R)] \\ x_{i,j}(r) &\mapsto e_{i,j}(r) \end{aligned}$$

Definition $K_2(R) = \ker(\varphi)$

Proposition $K_2(R)$ is an Abelian group. It is the centre of $\mathrm{St}(R)$.

K_2 of a field

If F is a field then $K_2(F)$ is an Abelian group written additively, with

$$\begin{array}{ll} \text{generators} & \{a,b\} \text{ for } a,b \text{ in } F^* \\ \text{relations} & \{a_1a_2,b\} = \{a_1,b\} + \{a_2,b\} \\ & \{a,b_1b_2\} = \{a,b_1\} + \{a,b_2\} \\ & \{a,1-a\} = 0 \text{ if } a \neq 0,1 \end{array}$$

Then also $\{a,b\} = -\{b,a\}$ and $\{c,-c\} = 0$ for a, b, c in F^*.

If A in $\mathrm{St}(F)$ lifts $\begin{pmatrix} a & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and B lifts $\begin{pmatrix} b & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b^{-1} \end{pmatrix}$, then $\{a,b\} = [A,B]$ in $K_2(F)$.

Note that $K_2(F) \simeq F^* \otimes F^* / \langle x \otimes (1-x) \rangle$ with $\{a,b\}$ corresponding to the class of $a \otimes b$.

An example: $K_2(\mathbb{Q})$

Proposition

$$K_2(\mathbb{Q}) \xrightarrow{\sim} \{\pm 1\} \times \oplus_{p \text{ prime}} \mathbb{F}_p^*$$

with components

$$T_\infty : K_2(\mathbb{Q}) \to \{\pm 1\} \text{ with } T_\infty(\{a, b\}) = \begin{cases} -1 \text{ if } a, b < 0 \\ 1 \text{ otherwise} \end{cases}$$

$T_p : K_2(\mathbb{Q}) \to \mathbb{F}_p^*$ with $T_p(\{a, b\}) = (-1)^{v_p(a)v_p(b)} \frac{a^{v_p(b)}}{b^{v_p(a)}}$ modulo p
where $v_p(a) \in \mathbb{Z}$ is the number of factors p in a
$T_p =$ the tame symbol for p

The proof of the proposition is based on repeated rewriting using division with remainder: if $a = qb + r$ with a, b, q, r non-zero integers, then $\{a/r, -qb/r\} = 0$ in $K_2(\mathbb{Q})$.

For $p > 2$ and $\alpha \in K_2(\mathbb{Q})$ we have $T_p(\alpha)^{\frac{p-1}{2}} \in \{\pm 1\} \subseteq \mathbb{F}_p^*$.

Define $\widetilde{T}_2 : K_2(\mathbb{Q}) \to \{\pm 1\}$ as follows.
Write $a = (-1)^i 2^j 5^k \frac{c}{d}$ with $i, k = 0, 1$ and c, d integers congruent 1 mod 8, $b = (-1)^I 2^J 5^K \frac{c'}{d'}$ similarly. Then

$$\widetilde{T}_2(\{a, b\}) = (-1)^{iI + jK + kJ}.$$

Identify $\{\pm 1\} \subseteq \mathbb{F}_p^*$ for all primes $p > 2$.

Theorem $\quad T_\infty(\{a, b\}) = \widetilde{T}_2(\{a, b\}) \prod_{\substack{p>2 \\ p \text{ prime}}} T_p(\{a, b\})^{\frac{p-1}{2}}.$

This is equivalent with quadratic reciprocity. E.g., let p and q be distinct odd primes, and put $\left(\frac{p}{q}\right)$ equal to 1 if p is a square modulo q, and to -1 if not. Equivalently,

$$\left(\frac{p}{q}\right) = p^{\frac{q-1}{2}} \bmod q = T_q(\{p,q\})^{\frac{q-1}{2}}$$

The theorem says that

$$\begin{aligned} 1 &= \widetilde{T}_2(\{p,q\})\, T_p(\{p,q\})^{\frac{p-1}{2}}\, T_q(\{p,q\})^{\frac{q-1}{2}} \\ &= (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}} \left(\frac{q}{p}\right)\left(\frac{p}{q}\right). \end{aligned}$$

Borel's theorem

Quillen defined Abelian groups $K_n(R)$ ($n \geq 0$) for rings R, as well as for algebraic varieties.

Let k be a number field, with r_1 real and $2r_2$ non-real embeddings, $d = r_1 + 2r_2$, and ring of algebraic integers $\mathcal{O}$, and let Δ_k be the absolute value of the discriminant of k

Recall that

- $K_0(\mathcal{O}) \cong \mathbb{Z} \oplus \mathrm{Cl}(\mathcal{O})$
- $K_1(\mathcal{O}) \cong \mathcal{O}^*$ has rank $r_1 + r_2 - 1$

Theorem (Quillen) $K_n(\mathcal{O})$ is finitely generated for all $n \geq 0$.

Theorem (Borel)
(1) $K_{2n}(\mathcal{O})$ is a finite group if $n \geq 1$.
(2) For $n \geq 2$, $K_{2n-1}(\mathcal{O})$ has rank $m_{2n-1} = r_1 + r_2$ if n is odd, and rank $m_{2n-1} = r_2$ if n is even.
(3) There exists a natural regulator map

$$K_{2n-1}(\mathcal{O}) \to \mathbb{R}^{m_{2n-1}} \qquad (n \geq 2).$$

Its image is a lattice with (normalized) volume of a fundamental domain

$$R_n(k) = q \frac{\zeta_k(n)}{\pi^{n(d-m_{2n-1})}\sqrt{\Delta_k}}$$

for some q in $\mathbb{Q}^*$.

Example: the K-theory of $\mathbb{Z}$

$\zeta_{\mathbb{Q}}$ is the Riemann zeta function. For $n \geq 2$:

$K_{2n-1}(\mathbb{Z})$ is finite for n even;

$K_{2n-1}(\mathbb{Z})$ has rank 1 for n odd, and $R_n(k) = q\zeta(n)$ for some $q \in \mathbb{Q}^*$.

n	2	3	4	5	6	7	...
m_{2n-1}	0	1	0	1	0	1	...
$\zeta(n)$	$\pi^2/6$	irrational	$\pi^4/90$	???	$\pi^6/945$	???	...

The Lichtenbaum conjecture

Conjecture (Lichtenbaum) Let k be a number field with ring of algebraic integers $\mathcal{O}$. Then for $n \geq 2$ we have

$$\zeta_k^*(1-n) = \pm 2^{?_{k,n}} \frac{|K_{2n-2}(\mathcal{O})|}{|K_{2n-1}(\mathcal{O})_{\text{torsion}}|} R_n(k)$$

where * denotes the first non-vanishing coefficient of $\zeta_k(s)$ in the Taylor expansion at $s = 1-n$, i.e., $\frac{1}{m_{2n-1}!}\zeta_k^{(m_{2n-1})}(1-n)$.

The power of 2 here is still not entirely clear. However, we have

Theorem (based on work of many people)
For $n = 2$ and $k/\mathbb{Q}$ Abelian, we have

$$\zeta_k^*(-1) = (-1)^{r_1+r_2} 2^{r_2} \frac{|K_2(\mathcal{O})|}{|K_3(\mathcal{O})_{\text{torsion}}|} R_2(k)$$

where r_1 is the number of real embeddings of k, and $2r_2$ the number of complex embeddings.

The Lichtenbaum conjecture

The ingredients of this conjecture are difficult to calculate. For any number field k we have:

(1) $K_2(\mathcal{O}) = \ker(K_2(k) \xrightarrow{T} \oplus_{\mathcal{P}\neq 0 \text{ prime}}(\mathcal{O}/\mathcal{P})^*)$

(2) For $n \geq 2$, we have $K_{2n-1}(\mathcal{O}) = K_{2n-1}(k)$

(3) $K_3(k)$ can be described using one subgroup (Milnor K_3) and the resulting quotient, the *indecomposable* K_3 of k, $K_3(k)^{\mathrm{ind}}$

(4) There are formulae for the torsion subgroups of those, so one knows $|K_3(\mathcal{O})_{\text{torsion}}|$

Now for an imaginary quadratic field...

For k imaginary quadratic this means $K_3(k) \simeq \mathbb{Z} \oplus \mathbb{Z}/24\mathbb{Z}$, and

$$\zeta_k'(-1) = -\frac{1}{12}|K_2(\mathcal{O})| \cdot R_2(k)\,.$$

The idea is to find an element in $K_3(k)$/torsion and compute its regulator by comparison with Humbert's classical formula:

$$\mathrm{vol}(\mathrm{PGL}_2(\mathcal{O})\backslash\mathbb{H}) = \frac{1}{8\pi^2}\Delta_k^{\frac{3}{2}}\zeta_k(2)$$

for the action of $\mathrm{PGL}_2(\mathcal{O})$ on hyperbolic 3-space $\mathbb{H}$.

Making $K_3(k)$ more explicit

Let k be any number field (for simplicity).

Theorem (Suslin) Let

$$k^* \otimes_\sigma k^* = \frac{k^* \otimes_{\mathbb{Z}} k^*}{\langle x \otimes y + y \otimes x \rangle}$$

and

$$\mathfrak{p}(k) = \frac{\mathbb{Z}[k^\flat]}{\langle [x] - [y] + [\frac{y}{x}] + [\frac{1-x}{1-y}] - [\frac{1-x^{-1}}{1-y^{-1}}] \text{ with } x \neq y \text{ in } k^\flat \rangle}$$

where $k^\flat = k \setminus \{0, 1\}$. Then the Bloch group

$$B_1(k) = \ker(\mathfrak{p}(k) \to k^* \otimes_\sigma k^*)$$
$$[x] \mapsto x \otimes (1 - x).$$

is isomorphic to $K_3(k)^{\text{ind}}$/a cyclic group of order $2|\mathcal{O}^*_{\text{torsion}}|$.

- $c_k = [x] + [1 - x]$ is in $B_1(k)$ and is independent of x; $6c_k = 0$.
- $[x] + [x^{-1}]$ is annihilated by 2

Making $K_3(k)$ more explicit

Slightly better behaved is the following variation.

Replace $k^* \otimes_\sigma k^*$ with $\tilde{\wedge}^2 k^* = \frac{k^* \otimes k^*}{\langle x \otimes (-x) \rangle}$
and $\mathfrak{p}(k)$ with $\overline{\mathfrak{p}}(k) = \mathfrak{p}(k)/\langle [x] + [1-x], [y] + [y^{-1}] \rangle$.
This gives another Bloch group $B_2(k) = B_1(k)/\langle c_k \rangle$ as the kernel.
This $B_2(k)$ has trivial torsion if $k = \mathbb{Q}$, an imaginary quadratic field, or a cyclotomic field and for those, we have an isomorphism

$$K_3(k)/\text{torsion} \xrightarrow{\sim} B_2(k)$$

Remark The 5-term relation here is

$$0 = \sum_{i=0}^{4} (-1)^i [\mathrm{cr}_2(P_0, \ldots, \widehat{P_i}, \ldots, P_4)]$$

with cr_2 the cross-ratio of 4 points in $\mathbb{P}^1_k$.

The regulator map

Let $k \subset \mathbb{C}$ be an imaginary quadratic field (for simplicity).
So we have $\mathbb{Z} \simeq K_3(k)/\text{torsion} \simeq B_2(k)$.

Theorem There is an injection $B_2(k) \to K_3(k)/\text{torsion}$, such that the composition with the regulator map $K_3(\mathbb{C}) \to \mathbb{R}$ maps $[z]$ to $D(z)$ with $D : \mathbb{C} \setminus \{0, 1\} \to \mathbb{R}$ the Bloch-Wigner dilogarithm,

$$D(z) = \int_{1/2}^{z} \log|w| d\arg(1-w) - \log|1-w| d\arg(w) \,.$$

D satisfies some functional equations:

$$D(z) + D(z^{-1}) = 0 \qquad D(z) + D(1-z) = 0 \qquad D(z) + D(\bar{z}) = 0$$

$$D(x) - D(y) + D(\frac{y}{x}) - D(\frac{1-y}{1-x}) + D(\frac{1-y^{-1}}{1-x^{-1}}) = 0$$

Remark So $K_3(k)/\text{torsion} \xrightarrow{\sim} B_2(k) \to K_3(k)/\text{torsion}$. We know the composition with the regulator map on the right, not on the left...

On to hyperbolic space

Let F be a field, C_n =the free Abelian group on generators $(l_0, \ldots, l_n)$ with $l_i \neq (0,0)$ in F^2 such that if l_i and l_j scale to each other then they are the same.
Leaving out one of the l_j and taking alternating sums of gives the complex in the top row of the commutative diagram

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{\mathrm{d}} & C_4 & \xrightarrow{\mathrm{d}} & C_3 & \xrightarrow{\mathrm{d}} & C_2 & \xrightarrow{\mathrm{d}} & C_1 & \xrightarrow{\mathrm{d}} & C_0 \\
 & & \downarrow & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_2} & & & & \\
 & & 0 & \longrightarrow & \overline{\mathfrak{p}}(F) & \longrightarrow & \tilde{\wedge}^2 F^* & & & &
\end{array}$$

- f_3 and f_2 are G-equivariant.
- f_3 is 0 on a degenerate generator (not all points distinct) and is $[\mathrm{cr}_2(l_0, l_1, l_2, l_3)]$ otherwise. Note $(l_0, l_1, l_2, l_3) \sim_{\mathrm{PGL}_2(F)} \left(\begin{smallmatrix} 1 & 0 & 1 & x \\ 0 & 1 & 1 & 1 \end{smallmatrix}\right)$ for a unique x in $F \setminus \{0, 1\}$, which is the cross-ratio.
- We let f_2 be 0 on degenerate generators, and if l_0, l_1, l_2 are distinct, then $(l_0, l_1, l_2) \sim_{\mathrm{GL}_2(F)} \left(\begin{smallmatrix} 1 & 0 & a \\ 0 & 1 & b \end{smallmatrix}\right)$ for unique a and b in F^*, and we map (l_0, l_1, l_2) to $a \tilde{\wedge} b$.

On to hyperbolic space

$\mathbb{H}^* =$ hyperbolic 3-space $\mathbb{H} \cup \mathbb{P}^1_k$ is acted on by $\Gamma = \mathrm{PGL}_2(\mathcal{O})$.

Yasaki and others: there is a tessellation, preserved by Γ. It consists of 3-cells, 2-cells and 1-cells, vertices in the cusps $\mathbb{P}^1_k$.

• Take a sum over representatives P_i under Γ of the 3-cells:

$$\alpha = \sum_i \frac{24}{|\mathrm{Stab}_\Gamma(P_i)|}[P_i]\,.$$

It has integer coefficients, and the faces (2-cells) formally cancel under the Γ-action.

• Chop all P_i into tetrahedra, obtaining α_T, a formal sum of tetrahedra. The induced triangulation on the faces may no longer match, so take some 'flat' tetrahedra to fix this. We get $\alpha_T + \alpha_F$.

• Map a tetrahedron $[l_0, \ldots, l_3]$ (vertices in $\mathbb{P}^1_k$) to $[\mathrm{cr}_2(l_0, \ldots, l_3)]$. Then $\alpha_T + \alpha_F$ gives an element β in $B_2(k) \simeq \mathbb{Z}$.

On to hyperbolic space

• Mapping this to γ in $K_3(k)^{\mathrm{ind}}_{\mathrm{tf}} \simeq \mathbb{Z}$, we can compute the regulator of γ because $D(\mathrm{cr}_2(l_0, \ldots, l_3))/\pi = \mathrm{vol}[l_0, \ldots, l_3]$, and we have Humbert's formula

$$\mathrm{vol}(\mathrm{PGL}_2(\mathcal{O})\backslash\mathbb{H}) = \frac{1}{8\pi^2}\Delta_k^{\frac{3}{2}}\zeta_k(2)\,.$$

Under the functional equation this relates to $\zeta_k'(-1)$ which equals

$$\zeta_k'(-1) = -\frac{1}{12}|K_2(\mathcal{O})| \cdot R_2(k)$$

by the (in this case known) Lichtenbaum conjecture.

The end of the game

Theorem
(1) γ generates a subgroup of index $|K_2(\mathcal{O})|$ in $K_3(k)/\text{torsion} \simeq \mathbb{Z}$.
(2) slightly easier to calculate is $\gamma - \overline{\gamma} = 2\gamma$ because α_F drops out (the corresponding cross ratios are in $\mathbb{Q} \setminus \{0, 1\}$).

Remark • Belabas and Gangl computed $K_2(\mathcal{O})$ for quite a few k (almost all with k with $\Delta_k < 10^4$).
• We divided $\gamma - \overline{\gamma}$ by $2|K_2(\mathcal{O})|$ for various fields. For all fields for which we computed (with $\Delta_k < 10^5$), we have $\frac{1}{2}(\beta - \overline{\beta})$ in $B_2(k)$ (i.e., just divide all coefficients by 2).

Example For $k = \mathbb{Q}(\sqrt{-303})$ one has $|K_2(\mathcal{O})| = 22$, and dividing $\gamma - \overline{\gamma}$ by 44 was done by finding a suitable element β' in $B_2(k)$, and then computing using generators and relations that $44\beta' = \beta - \overline{\beta}$. (That involved about 1650 5-term relations, symmetrized for the action of S_4 on the cross ratio).

The end of the game

A much simpler example For $k = \mathbb{Q}(\sqrt{-5})$ one has $|K_2(\mathcal{O})| = 1$. There are two 3-cells up to the action of Γ, both triangular prisms. Then starting with $\tilde{\alpha} = 3[P_1] + 2[P_2]$, and simply chopping them into tetrahedra and applying cr_2 gives the element

$$\tilde{\beta} = 7[\frac{\sqrt{-5}+2}{3}] - 3[\frac{-2\sqrt{-5}+5}{3}] + [\frac{3\sqrt{-5}+5}{6}] - 2[\frac{-\sqrt{-5}+7}{6}]$$

which is not in $B_2(k)$, but

$$\beta = 4\tilde{\beta} - 4[3] + 6[5]$$

is. Its image γ in $K_3(k)^{\mathrm{ind}}_{\mathrm{tf}} \simeq \mathbb{Z}$ is a generator because $|K_2(\mathcal{O})| = 1$.