Beilinson's conjecture for K_2 of certain (hyper)elliptic curves

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Rob de Jeu

University of Durham

email: `rob.de-jeu@durham.ac.uk`

website: `http://maths.dur.ac.uk/~dma0rdj`

T. Dokchitser, R. de Jeu and D. Zagier. Numerical verification of Beilinson's conjecture for K_2 of hyperelliptic curves (preprint, 2004).

(Available from http://arXiv.org/abs/math/0405040.)

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Motivation

k: a number field.
$\mathcal{O}_k$: the ring of algebraic integers of k.

r_1: the number of embeddings $k \to \mathbb{R}$
$2r_2$: the number of non-real embeddings $k \to \mathbb{C}$
$[k:\mathbb{Q}] = r_1 + 2r_2$
$\mathcal{O}_k^*$ has rank $r = r_1 + r_2 - 1$
Let $\sigma_1, \ldots, \sigma_{r+1}$ be the embeddings of k into $\mathbb{C}$ up to complex conjugation.
If $u_1, \ldots, u_r$ form a $\mathbb{Z}$-basis of $\mathcal{O}_k^*/\text{torsion}$ let

$$R = \frac{2^{r_2}}{[k:\mathbb{Q}]} \left| \det \begin{pmatrix} 1 & \log|\sigma_1(u_1)| & \ldots & \log|\sigma_1(u_r)| \\ \vdots & \vdots & & \vdots \\ 1 & \log|\sigma_{r+1}(u_1)| & \ldots & \log|\sigma_{r+1}(u_r)| \end{pmatrix} \right|$$

$$\zeta_k(s) = \sum_{\substack{(0)\neq I \subset \mathcal{O}_k \\ I \text{ ideal of } \mathcal{O}_k}} (\#\mathcal{O}_k/I)^{-s} = \prod_{\substack{0 \neq \mathcal{P} \subset \mathcal{O}_k \\ \mathcal{P} \text{ prime ideal}}} \frac{1}{1 - (\#\mathcal{O}_k/\mathcal{P})^{-s}}$$

$$\mathrm{Res}_{s=1}\zeta_k(s) = \frac{2^{r_1}(2\pi)^{r_2} R |\mathrm{Cl}(\mathcal{O}_k)|}{w\sqrt{\Delta_k}}$$

Δ_k = the absolute value of the discriminant of k.
$w = |\mathcal{O}_{k,\text{tor}}^*|$ = #roots of unity in k.

$$K_0(\mathcal{O}_k) \cong \mathbb{Z} \oplus \mathrm{Cl}(\mathcal{O}_k)$$
$$K_1(\mathcal{O}_k) \cong \mathcal{O}_k^*$$
$$|\mathrm{Cl}(\mathcal{O}_k)| = |K_0(\mathcal{O}_k)_{\mathrm{tor}}|$$
$$w = |K_1(\mathcal{O}_k)_{\mathrm{tor}}|$$

If F is a field, then

$$\begin{aligned}
&K_0(F) \cong \mathbb{Z}\,,\\
&K_1(F) \cong F^* = F\setminus\{0\}\,,\\
&K_2(F) \cong F^* \otimes_{\mathbb{Z}} F^*/\langle a \otimes (1-a), a \in F^*\setminus\{1\}\rangle.
\end{aligned}$$

The class of $a \otimes b$ in $K_2(F)$ is denoted $\{a, b\}$, so $K_2(F)$ is generated by symbols $\{a, b\}$ with a, b in F^*, and rules

$$\begin{aligned}
&\{a_1 a_2, b\} = \{a_1, b\} + \{a_2, b\}\,,\\
&\{a, b_1 b_2\} = \{a, b_1\} + \{a, b_2\}\,,\\
&\{a, 1-a\} = 0 \text{ if } a \neq 0, 1\,.
\end{aligned}$$

This implies $\{a, b\}+\{b, a\} = \{c, -c\} = 0$ for a, b and c in F^*.

Example

$$K_2(\mathbb{Q}) \cong \{\pm 1\} \oplus \bigoplus_{p \text{ prime}} (\mathbb{Z}/p\mathbb{Z})^*\,.$$

Borel's theorem

k: number field (hence $K_{2n-1}(\mathcal{O}_k) \cong K_{2n-1}(k)$ if $n \geq 2$)
$K_n(\mathcal{O}_k)$ is finitely generated for all $n \geq 0$.
$m_n =$ the rank of $K_n(\mathcal{O}_k)$.

Theorem (Borel) $K_{2n}(\mathcal{O}_k)$ is a finite group if $n \geq 1$. For $n \geq 2$, $K_{2n-1}(\mathcal{O}_k)$ has rank $m_{2n-1} = \begin{cases} r_1 + r_2 & \text{if } n \text{ is odd,} \\ r_2 & \text{if } n \text{ is even.} \end{cases}$
Furthermore, there exists a natural regulator map

$$K_{2n-1}(\mathcal{O}_k)/\text{torsion} \to \mathbb{R}^{m_{2n-1}}.$$

The image is a lattice with volume V_n of a fundamental domain satisfying

$$V_n \sim_{\mathbb{Q}^*} \frac{\zeta_k(n)}{\pi^{n([k:\mathbb{Q}]-m_{2n-1})}\sqrt{\Delta_k}}$$

where Δ_k is the absolute value of the discriminant of k.

[$a \sim_{\mathbb{Q}^*} b$ means $a = qb$ for some q in $\mathbb{Q}^*$.]

Example
$\zeta_{\mathbb{Q}}$ is the Riemann zeta function. For $n \geq 2$:
$K_{2n-1}(\mathbb{Z})$ is finite for n even;
$K_{2n-1}(\mathbb{Z})$ has rank 1 for n odd, and $V_n \sim_{\mathbb{Q}^*} \zeta(n)$.

n	2	3	4	5	6	7	...
m_{2n-1}	0	1	0	1	0	1	...
$\zeta(n)$	$\pi^2/6$	irrat.	$\pi^4/90$	???	$\pi^6/945$	???	...

The case of K_2 of a curve over the rationals

Let $C/\mathbb{Q}$ be a (smooth, proper, geometrically irreducible) curve of genus g.

Conjecture (Hasse-Weil) With N the conductor of C, the function

$$L^*(C,s) = \frac{N^{s/2}}{(2\pi)^{gs}}\,\Gamma(s)^g\,L(C,s)$$

extends to an entire function of s and satisfies

$$L^*(C,s) = wL^*(C,2-s)$$

with $w = +1$ or -1.

This conjecture would imply

$$L^{(0)}(C,0) = \cdots = L^{(g-1)}(C,0) = 0$$

and

$$\frac{L^{(g)}(C,0)}{g!} = L^*(C,0) = wL^*(C,2) = \frac{wN}{(2\pi)^{2g}}L(C,2) \neq 0\ .$$

Tame symbols

Set $F = \mathbb{Q}(C)$ and

$$K_2^T(C) = \operatorname{Ker}\left(K_2(F) \xrightarrow{T} \bigoplus_{x \in C(\overline{\mathbb{Q}})} \overline{\mathbb{Q}}^*\right)$$

where T_x is the tame symbol for x,

$$T_x : \{a, b\} \mapsto (-1)^{\operatorname{ord}_x(a)\operatorname{ord}_x(b)} \frac{a^{\operatorname{ord}_x(b)}}{b^{\operatorname{ord}_x(a)}}(x) \,.$$

Product formula If α is an element of $K_2(F)$ then

$$\prod_{x \in C(\overline{\mathbb{Q}})} T_x(\alpha) = 1 \,.$$

Similarly one can define

$$K_2^T(C_{\overline{\mathbb{Q}}}) = \operatorname{Ker}\left(K_2(\overline{\mathbb{Q}}(C)) \xrightarrow{T} \bigoplus_{x \in C(\overline{\mathbb{Q}})} \mathbb{Q}^*\right) \,,$$

$$K_2^T(C_{\mathbb{R}}) = \operatorname{Ker}\left(K_2(\mathbb{R}(C)) \xrightarrow{T} \bigoplus_{x \in C(\mathbb{C})} \mathbb{C}^*\right)$$

and

$$K_2^T(C_{\mathbb{C}}) = \operatorname{Ker}\left(K_2(\mathbb{C}(C)) \xrightarrow{T} \bigoplus_{x \in C(\mathbb{C})} \mathbb{C}^*\right)$$

and the corresponding product formula holds in each case.

Let $\mathcal{C}$ be a regular proper model of C over $\mathbb{Z}$ and $\mathcal{C}_p$ its fibre above the prime p. For each irreducible component $\mathcal{D}$ of $\mathcal{C}_p$ let $\mathbb{F}_p(\mathcal{D})$ be its field of rational functions over $\mathbb{F}_p$. Put

$$K_2^T(\mathcal{C}) = \mathrm{Ker}\left(K_2^T(C) \xrightarrow{T} \bigoplus_p \bigoplus_{\mathcal{D}\subseteq\mathcal{C}_p} \mathbb{F}_p(\mathcal{D})^* \right)$$

where the map to $\mathbb{F}_p(\mathcal{D})^*$ is given by the tame symbol corresponding to $\mathcal{D}$,

$$T_{\mathcal{D}} : \{a,b\} \mapsto (-1)^{v_{\mathcal{D}}(a)v_{\mathcal{D}}(b)} \frac{a^{v_{\mathcal{D}}(b)}}{b^{v_{\mathcal{D}}(a)}}(\mathcal{D})\,,$$

with $v_{\mathcal{D}}$ the discrete valuation on F given by the order of vanishing along $\mathcal{D}$.
Finally put

$$K_2(C;\mathbb{Z}) = \frac{K_2^T(\mathcal{C})}{\text{torsion}} \subseteq \frac{K_2^T(C)}{\text{torsion}}\,,$$

which is independent of the choice of the model $\mathcal{C}$ of C

[We could have defined $K_2^T(\mathcal{C})$ in a single step as

$$K_2^T(\mathcal{C}) = \ker\left(K_2(F) \xrightarrow{T} \bigoplus_{\mathcal{D}} \mathbb{F}(\mathcal{D})^* \right)$$

where $\mathcal{D}$ runs through all irreducible curves on $\mathcal{C}$ and $\mathbb{F}(\mathcal{D})$ stands for the residue field at $\mathcal{D}$.]

Regulator 1-forms

For a and b in F^* put

$$\eta(a,b) = \log|a|\,\mathrm{d}\arg b - \log|b|\,\mathrm{d}\arg a\,,$$

a smooth, closed 1-form where a and b have no pole or zero.

- $\eta(a_1a_2,b) = \eta(a_1,b) + \eta(a_2,b)$
- $\eta(a,b_1b_2) = \eta(a,b_1) + \eta(a,b_2)$
- $\eta(a,1-a) = dD(a)$, where $D(z)$ is the Bloch-Wigner dilogarithm function.

So η induces a map $K_2(F) \to \dfrac{\text{closed 1-forms}}{\text{exact 1-forms}}$.
And if α is in $K_2^T(C)$ then $\eta(\alpha)$ has trivial residues.

Conjecture (Beilinson) Let $C/\mathbb{Q}$ be a curve of genus g as before and let $X = C(\mathbb{C})$.

(1) $K_2(C;\mathbb{Z}) \cong \mathbb{Z}^g$.

(2) Let

$$R = \frac{1}{(2\pi)^g}\left|\det\begin{pmatrix} \int_{\gamma_1}\eta_1 & \cdots & \int_{\gamma_1}\eta_g \\ \vdots & & \vdots \\ \int_{\gamma_g}\eta_1 & \cdots & \int_{\gamma_g}\eta_g \end{pmatrix}\right|$$

where $\eta_1,\ldots,\eta_g$ are the regulator 1-forms obtained from a basis of $K_2(C;\mathbb{Z})$, $\gamma_1,\ldots,\gamma_g$ give a basis of $H_1(X;\mathbb{Z})^-$ and are chosen such that the η_k have no zeroes of poles on them. Then

$$L^*(C,0) \sim_{\mathbb{Q}^*} R\,.$$

['$-$' in $H_1(X;\mathbb{Z})^-$ denotes the anti-invariants of $H_1(X;\mathbb{Z})$ under the action of complex conjugation on X.]

- R does not depend on any choices.
- There is a similar conjecture over an arbitrary number field.
- R makes sense for any g elements of $K_2^T(C)$.

Elements in $\mathbf{K_2^T(C)}$ from torsion points

Let P_1, P_2 and P_3 in $C(\mathbb{Q})$ be distinct and such that all $(P_i) - (P_j)$ are torsion in $\mathrm{Jac}(C)(\mathbb{Q})$. Pick f_i in $\mathbb{Q}(C)^*$ with

$$(f_i) = m_i(P_{i+1}) - m_i(P_{i-1})$$

where $m_i = \mathrm{ord}((P_{i+1}) - P(_{i-1}))$ (all indices modulo 3). Put

$$S_i = \left\{ \frac{f_{i+1}}{f_{i+1}(P_{i+1})}, \frac{f_{i-1}}{f_{i-1}(P_{i-1})} \right\},$$

an element of $K_2^T(C)$. Then $\langle S_1, S_2, S_3 \rangle$ in $K_2^T(C)/\text{torsion}$ is generated by an element $\{P_1, P_2, P_3\}$.

All this works if we replace $\mathbb{Q}$ with $\overline{\mathbb{Q}}$, $\mathbb{R}$ or $\mathbb{C}$ as well.

We cannot get much more from out torsion points, in the following sense.

Assume $S \subseteq C(\overline{\mathbb{Q}})$ and P_0 in S are such that $(P) - (P_0)$ is torsion for all P in S. Let $V \subseteq K_2(\overline{\mathbb{Q}}(C)) \otimes \mathbb{Q}$ be generated by the $\{f, g\}$ with $|(f)|$ and $|(g)|$ in S. Then

$$V \cap K_2^T(C_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q} = \langle \{P_0, Q, R\} \rangle_{\mathbb{Q}}$$

with $Q \neq R$ in $S \setminus \{P_0\}$.

The case of hyperelliptic curves

Assume $C/\mathbb{Q}$ is hyperelliptic of genus g with a rational ramification point ∞. Such a C can be defined by

$$y^2 = t(x)$$

with $t(x)$ in $\mathbb{Q}[x]$ of degree $2g+1$ without multiple roots.

Example 1: (2g+1)-torsion
The curve admits an equation

$$y^2 = -x^{2g+1} + f(x)^2/4 = t(x) \quad \text{or} \quad y^2 + f(x)y + x^{2g+1} = 0$$

with $f(x)$ in $\mathbb{Q}[x]$, $\deg(f(x)) \leq g$, and such that $t(x)$ has no multiple roots.

Example 2: (2g+2)-torsion
The curve admits an equation

$$y^2 = -x^{2g+2} + f(x)^2/4 = t(x) \quad \text{or} \quad y^2 + f(x)y + x^{2g+2} = 0$$

with $f(x) = 2x^{g+1} + b_g x^g + \cdots + b_0$ in $\mathbb{Q}[x]$, $b_g \neq 0$, and such that $t(x)$ has no multiple roots.

In both cases we have an equation of the form

$$y^2 + f(x)y + x^d = 0$$

with $d = 2g+1$ or $2g+2$ and $t(x) = -x^d + f(x)^2/4$.

Let $O = (0,0)$ and $O' = (0, -f(0))$, the image of O under the hyperelliptic involution.
$(O) - (\infty)$ and $(O') - (\infty)$ both have order d in $\mathrm{Jac}(C)(\mathbb{Q})$: $(y) = d(O) - d(\infty)$.
Let $T_\beta = (\beta, -f(\beta)/2)$ where β is a root in $\overline{\mathbb{Q}}$ of $t(x)$, so $(T_\beta) - (\infty)$ has order 2 in $\mathrm{Jac}(C)(\overline{\mathbb{Q}})$.

So if $t(\beta)=0$ then we get in $K_2^T(C_{\overline{\mathbb{Q}}})/\text{torsion}$

$$\{\infty, O, T_\beta\} = \{\frac{y}{-f(\beta)/2}, \frac{x-\beta}{\beta}\}\,.$$

But if $m(x)$ is an irreducible factor of $t(x)$ in $\mathbb{Q}[x]$ then

$$2\sum_{\substack{\beta \text{ with} \\ m(\beta)=0}} \{\infty, O, T_\beta\} = \left\{\frac{y^2}{x^d}, \frac{m(x)}{m(0)}\right\} \stackrel{\text{def}}{=} M\,,$$

an element of $K_2^T(C)$.

Proposition

Let $t(x) = m_1(x)\cdots m_k(x)$ be a factorisation of $t(x)$ into irreducibles in $\mathbb{Q}[x]$ so we have $M_1,\ldots,M_k$ in $K_2^T(C)$.

(1) If $d = 2g+2$ and

$$\begin{aligned} 4t(x) &= -4x^{2g+2} + f(x)^2 \\ &= (b_g x^g + \cdots + b_0)(4x^{g+1} + b_g x^g + \cdots + b_0) \\ &= 4\cdot m_1\cdots m_l \cdot m_{l+1}\cdots m_k \end{aligned}$$

then $M_1+\cdots+M_l = M_{l+1}+\cdots+M_k$ in $K_2^T(C)$.

(2) If $f(0)=b_0=\pm1$ then $\mathbb{M} = \{-f(0)y, -x\}$ is in $K_2^T(C)$ and $2d\,\mathbb{M} = \sum_{j=1}^k M_j$.

(3) If $W \subseteq K_2^T(C_{\overline{\mathbb{Q}}})\otimes\mathbb{Q}$ is generated over $\mathbb{Q}$ by all $\{P,Q,R\}$ with P, Q and R in $\{\infty, O, O', T_{\beta_1},\ldots,T_{\beta_{2g+1}}\}$ then it is already generated by the $\{\infty, O, T_{\beta_j}\}$.

(4) $K_2^T(C)\otimes\mathbb{Q} = K_2^T(C_{\overline{\mathbb{Q}}})^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}\otimes\mathbb{Q} \subseteq K_2^T(C_{\overline{\mathbb{Q}}})\otimes\mathbb{Q}$ and with W as in (3),

$$W \cap K_2^T(C)\otimes\mathbb{Q} = \langle M_1,\ldots,M_k\rangle_{\mathbb{Q}}\,.$$

[So (3) and (4) say that the M_j give essentially everything we can get from the known torsion points.]

So we need $t(x) = -x^d + f(x)^2/4$ as before with many rational factors, e.g.,

$$
\begin{aligned}
&-3326400^2x^{12} + (3326400x^6 + 149040x^5 - 150012x^4 \\
&\qquad\qquad\qquad\qquad + 188x^3 + 787x^2 - 2x + 1)^2 \\
&= (22x-1)(20x-1)(18x-1)(12x-1)(10x-1) \\
&\qquad (x-1)(7x+1)(15x+1)(18x+1)(23x+1)(24x+1)\,.
\end{aligned}
$$

Integrality

Theorem

Consider either

$$f(x) = b_g x^g + \cdots + b_0 \text{ and } d = 2g+1$$

or

$$f(x) = 2x^{g+1} + b_g x^g + \cdots + b_0 \text{ with } b_g \neq 0 \text{ and } d = 2g+2$$

with $b_0, \ldots, b_g$ in $\mathbb{Z}$ such that $t(x) = -x^d + f(x)^2/4$ has no multiple root.

Let $m(x)$ be an irreducible factor of $t(x)$ in $\mathbb{Q}[x]$.

(1) If $m(x)/m(0)$ is in $\mathbb{Z}[x]$ then the class of $\{y^2/x^d, m(x)/m(0)\}$ is in $K_2(C;\mathbb{Z})$.

(2) If $\gcd(b_0, \ldots, b_g) = 1$ and $m(x)/m(0)$ is not in $\mathbb{Z}[x]$ then no non-trivial multiple of the class of $\{y^2/x^d, m(x)/m(0)\}$ is in $K_2(C;\mathbb{Z})$.

(3) If $b_0 = \pm 1$ then

$$
\begin{aligned}
\mathbb{M} = \{-b_0 y, -x\} &\text{ is in } K_2(C;\mathbb{Z}) \text{ if } d = 2g+1, \\
2\mathbb{M} = \{y^2, x\} &\text{ is in } K_2(C;\mathbb{Z}) \text{ if } d = 2g+2.
\end{aligned}
$$

Limit results

Theorem
Let $g \geq 1$ and fix $v_1 < \cdots < v_{g-1}$ in $\mathbb{R}^*$. If $v_g = a \gg 0$ and

$$f(x) = 2x^{g+1} + \prod_{j=1}^{g}(v_j x + 1)$$

then

$$4t(x) = (4x^{g+1} + \prod_{j=1}^{g}(v_j x + 1)) \prod_{j=1}^{g}(v_j x + 1)$$

has $2g+1$ distinct real roots. If that is the case then for the curve $C/\mathbb{R}$ of genus g defined by

$$y^2 + f(x)y + x^{2g+2} = 0$$

the elements

$$M_l = \left\{ \frac{y^2}{x^{2g+2}}, v_l x + 1 \right\} \qquad (l = 1, \ldots, g)$$

in $K_2^T(C)$ have Beilinson regulator $R = R(a)$ satisfying

$$\lim_{a \to \infty} \frac{R(a)}{(2 \log a)^g} = g + 1 \,.$$

Corollary
With the same notation but $v_1 < \cdots < v_{g-1}$ in $\mathbb{N}$ fixed, if $v_g \gg 0$ in $\mathbb{N}$ then the classes of $M_1, \ldots, M_g$ are independent in $K_2(C; \mathbb{Z})$.

Remark

- In this situation with $g > 1$, the two curves corresponding to $0 < v_1 < \cdots < v_g$ and $0 < \tilde{v}_1 < \cdots < \tilde{v}_g$ are isomorphic over $\mathbb{Q}$ precisely when either $\tilde{v}_j = v_j$ for $j = 1, \ldots, g$, or g is odd and $\tilde{v}_j = v_g - v_{g-j}$ for $j = 1, \ldots, g-1$ and $\tilde{v}_g = v_g$.
- The map from the open part of $\overline{\mathbb{Q}}^{*g}$ where $t(x)$ has no multiple roots to the set of isomorphism classes over $\overline{\mathbb{Q}}$ of hyperelliptic curves over $\overline{\mathbb{Q}}$ is finite to one, so we get infinitely many isomorphism classes over $\overline{\mathbb{Q}}$ of curves $C/\mathbb{Q}$ for which the rank of $K_2(C;\mathbb{Z})$ is at least g.

Explicit example:

For the curves $C_{a,b}$ defined by

$$y^2 + (2x^3 + (ax+1)(bx+1))y + x^6 = 0$$

with integers $a \equiv 1$ and $b \equiv 2$ modulo 3, and $a \gg 0$ or $b \gg 0$:

(1) the two elements $\{y^2/x^6, ax+1\}$ and $\{y^2/x^6, bx+1\}$ give classes in $K_2(C_{a,b};\mathbb{Z})$;

(2) the Jacobian of $C_{a,b}$ does not split over $\overline{\mathbb{Q}}$;

(3)

$$\lim_{a\to\infty} \frac{R(C_{a,b})}{(2\log a)^2} = 3\ .$$

Proposition
Let k be a real quadratic field and let $\mathcal{O}_k$ be its ring of algebraic integers. Fix $v \neq \pm 1$ in $\mathcal{O}_k^*$ as well as p and q in $\mathcal{O}_k$ satisfying $pq = 4$. If $pv^n \neq \pm 2$

$$y^2 + (2x^2 + (pv^n + qv^{-n})x + 1)y + x^4 = 0$$

defines an elliptic curve C over k. The classes of

$$\left\{\frac{y^2}{x^4}, pv^n x + 1\right\}$$

and

$$\left\{\frac{y^2}{x^4}, qv^{-n} x + 1\right\}$$

are in $K_2(C;\mathbb{Z})$ and their Beilinson regulator $R = R(n)$ satisfies

$$\lim_{n\to\infty} \frac{R(n)}{n^2} = 4\left|\log|v|\right|^2 .$$

Here $|v|$ is the absolute value using either embedding of k into $\mathbb{R}$, $|\log|v||$ being independent of the embedding.

Moreover, if the norm of p in $\mathbb{Q}$ does not have absolute value 4 (e.g., when $p = 1$ and $q = 4$), then for $n \gg 0$ the j-invariant is neither rational nor an algebraic integer. Therefore, for $n \gg 0$ those curves cannot be obtained from curves over $\mathbb{Q}$ by enlarging the base field, nor do they admit complex multiplication over $\overline{\mathbb{Q}}$.

Some interesting questions

(1) In our tables we often find an integer for $L^*(C,0)/R(\Lambda)$ so

$$\frac{L^*(C,0)}{R(K_2(C;\mathbb{Z}))} = (K_2(C;\mathbb{Z}) : \Lambda)\,\frac{L^*(C,0)}{R(\Lambda)}$$

might always be an integer. For this we may need more elements in $K_2(C;\mathbb{Z})$. It would be interesting to try and do this, especially for the curve defined by

$$y^2 + (5x^3 - 13x^2 + 7x - 1)y + x^7 = 0$$

where $L^*(C,0)/R(\Lambda) \stackrel{?}{=} 1/19$.

(2) If $C/\mathbb{Q}$ is a curve and $F = \mathbb{Q}(C)$, can we decide in an algorithmic way if an element in $K_2(F)$ is trivial or not? More specifically we can consider the following questions.

(a) Can we decide if the relations in the tables as suggested by the computer calculations are true in $K_2(F)$ or not?

(b) The element

$$\{(x-1)^2, (y+1)^3(y+2x-3)\}$$

is trivial in $K_2(F)$ where $F = \mathbb{Q}(E)$ and E is defined by

$$y^2 = x^3 - 2x^2 + 1\,.$$

How do we write explicitly

$$(x-1)^2 \otimes [(y+1)^3(y+2x-3)] = \sum_i m_i\,(f_i \otimes (1-f_i))$$

in $F^* \otimes F^*$ for some m_i in $\mathbb{Z}$ and f_i in $F \setminus \{0,1\}$?