Beilinson’s conjecture for $K_2$ of certain (hyper)elliptic curves
Dagstuhl, May 2004

Rob de Jeu
University of Durham
email: rob.de-jeu@durham.ac.uk
website: http://maths.dur.ac.uk/~dma0rdj


Motivation

$k$: a number field.
$\mathcal{O}_k$: the ring of algebraic integers of $k$.

$r_1$: the number of embeddings $k \to \mathbb{R}$
$2r_2$: the number of non-real embeddings $k \to \mathbb{C}$

$[k : \mathbb{Q}] = r_1 + 2r_2$

$\mathcal{O}_k^\ast$ has rank $r = r_1 + r_2 - 1$

Let $\sigma_1, \ldots, \sigma_{r+1}$ be the embeddings of $k$ into $\mathbb{C}$ up to complex conjugation.

If $u_1, \ldots, u_r$ form a $\mathbb{Z}$-basis of $\mathcal{O}_k^\ast$/torsion let

$$R = \frac{2^{r_2}}{[k : \mathbb{Q}]} \det \begin{pmatrix} 1 & \log |\sigma_1(u_1)| & \ldots & \log |\sigma_1(u_r)| \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \log |\sigma_{r+1}(u_1)| & \ldots & \log |\sigma_{r+1}(u_r)| \end{pmatrix}$$

$$\zeta_k(s) = \sum_{(0) \neq I \subseteq \mathcal{O}_k} (\# \mathcal{O}_k/I)^{-s} = \prod_{0 \neq \mathcal{P} \subseteq \mathcal{O}_k} \frac{1}{1 - (\# \mathcal{O}_k/\mathcal{P})^{-s}}$$

$$\text{Res}_{s=1} \zeta_k(s) = \frac{2^{r_1}(2\pi)^{r_2} R |\text{Cl}(\mathcal{O}_k)|}{w \sqrt{\Delta_k}}$$

$\Delta_k =$ the absolute value of the discriminant of $k$.

$w = |\mathcal{O}_{k,\text{tor}}| =$ #roots of unity in $k$. 

2
\[ K_0(\mathcal{O}_k) \cong \mathbb{Z} \oplus \text{Cl}(\mathcal{O}_k) \]
\[ K_1(\mathcal{O}_k) \cong \mathcal{O}_k^* \]
\[ |\text{Cl}(\mathcal{O}_k)| = |K_0(\mathcal{O}_k)_{\text{tor}}| \]
\[ w = |K_1(\mathcal{O}_k)_{\text{tor}}| \]

If \( F \) is a field, then
\[ K_0(F) \cong \mathbb{Z}, \]
\[ K_1(F) \cong F^* = F \setminus \{0\}, \]
\[ K_2(F) \cong F^* \otimes_{\mathbb{Z}} F^*/\langle a \otimes (1 - a), a \in F^* \setminus \{1\} \rangle. \]

The class of \( a \otimes b \) in \( K_2(F) \) is denoted \( \{a, b\} \), so \( K_2(F) \) is generated by symbols \( \{a, b\} \) with \( a, b \) in \( F^* \), and rules
\[ \{a_1a_2, b\} = \{a_1, b\} + \{a_2, b\}, \]
\[ \{a, b_1b_2\} = \{a, b_1\} + \{a, b_2\}, \]
\[ \{a, 1 - a\} = 0 \text{ if } a \neq 0, 1. \]

This implies \( \{a, b\} + \{b, a\} = \{c, -c\} = 0 \) for \( a, b \) and \( c \) in \( F^* \).

**Example**

\[ K_2(\mathbb{Q}) \cong \{\pm 1\} \oplus \bigoplus_{p \text{ prime}} (\mathbb{Z}/p\mathbb{Z})^*. \]
Borel’s theorem

\( k \): number field (hence \( K_{2n-1}(\mathcal{O}_k) \cong K_{2n-1}(k) \) if \( n \geq 2 \))
\( K_n(\mathcal{O}_k) \) is finitely generated for all \( n \geq 0 \).
\( m_n = \) the rank of \( K_n(\mathcal{O}_k) \).

**Theorem (Borel)** \( K_{2n}(\mathcal{O}_k) \) is a finite group if \( n \geq 1 \). For \( n \geq 2, K_{2n-1}(\mathcal{O}_k) \) has rank \( m_{2n-1} = \begin{cases} r_1 + r_2 & \text{if } n \text{ is odd,} \\ r_2 & \text{if } n \text{ is even.} \end{cases} \)
Furthermore, there exists a natural regulator map

\[
K_{2n-1}(\mathcal{O}_k)/\text{torsion} \to \mathbb{R}^{m_{2n-1}}.
\]

The image is a lattice with volume \( V_n \) of a fundamental domain satisfying

\[
V_n \sim_{\mathbb{Q}^*} \frac{\zeta_k(n)}{\pi^n(\frac{|k: \mathbb{Q}|}{-m_{2n-1}})\sqrt{\Delta_k}}
\]

where \( \Delta_k \) is the absolute value of the discriminant of \( k \).

[ \( a \sim_{\mathbb{Q}^*} b \) means \( a = qb \) for some \( q \) in \( \mathbb{Q}^* \).]

**Example**
\( \zeta_{\mathbb{Q}} \) is the Riemann zeta function. For \( n \geq 2, \)
\( K_{2n-1}(\mathbb{Z}) \) is finite for \( n \) even;
\( K_{2n-1}(\mathbb{Z}) \) has rank 1 for \( n \) odd, and \( V_n \sim_{\mathbb{Q}^*} \zeta(n) \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_{2n-1} )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>...</td>
</tr>
<tr>
<td>( \zeta(n) )</td>
<td>( \pi^2/6 )</td>
<td>irrat.</td>
<td>( \pi^4/90 )</td>
<td>???</td>
<td>( \pi^6/945 )</td>
<td>???</td>
<td>...</td>
</tr>
</tbody>
</table>
The case of $K_2$ of a curve over the rationals

Let $C/\mathbb{Q}$ be a (smooth, proper, geometrically irreducible) curve of genus $g$.

**Conjecture (Hasse-Weil)** With $N$ the conductor of $C$, the function

$$L^*(C, s) = \frac{N^{s/2}}{(2\pi)^{gs}} \Gamma(s)^g L(C, s)$$

extends to an entire function of $s$ and satisfies

$$L^*(C, s) = w L^*(C, 2 - s)$$

with $w = +1$ or $-1$.

This conjecture would imply

$$L^{(0)}(C, 0) = \cdots = L^{(g-1)}(C, 0) = 0$$

and

$$\frac{L^{(g)}(C, 0)}{g!} = L^*(C, 0) = w L^*(C, 2) = \frac{wN}{(2\pi)^{2g}} L(C, 2) \neq 0.$$
Tame symbols

Set $F = \mathbb{Q}(C)$ and

$$K_2^T (C) = \text{Ker} \left( K_2(F) \xrightarrow{T} \bigoplus_{x \in C(\overline{Q})} \overline{Q}^* \right)$$

where $T_x$ is the tame symbol for $x$,

$$T_x : \{a, b\} \mapsto (-1)^{\text{ord}_x(a) \text{ord}_x(b)} \frac{a^{\text{ord}_x(b)}}{b^{\text{ord}_x(a)}}(x).$$

Product formula If $\alpha$ is an element of $K_2(F)$ then

$$\prod_{x \in C(\overline{Q})} T_x(\alpha) = 1.$$ 

Similarly one can define

$$K_2^T (C_{\mathbb{Q}}) = \text{Ker} \left( K_2(\overline{\mathbb{Q}}(C)) \xrightarrow{T} \bigoplus_{x \in C(\overline{Q})} \overline{Q}^* \right),$$

$$K_2^T (C_{\mathbb{R}}) = \text{Ker} \left( K_2(\mathbb{R}(C)) \xrightarrow{T} \bigoplus_{x \in C(\mathbb{C})} \mathbb{C}^* \right)$$

and

$$K_2^T (C_{\mathbb{C}}) = \text{Ker} \left( K_2(\mathbb{C}(C)) \xrightarrow{T} \bigoplus_{x \in C(\mathbb{C})} \mathbb{C}^* \right)$$

and the corresponding product formula holds in each case.
Let $\mathcal{C}$ be a regular proper model of $C$ over $\mathbb{Z}$ and $\mathcal{C}_p$ its fibre above the prime $p$. For each irreducible component $\mathcal{D}$ of $\mathcal{C}_p$ let $\mathbb{F}_p(\mathcal{D})$ be its field of rational functions over $\mathbb{F}_p$. Put

$$K^T_2(C) = \text{Ker} \left( K^T_2(C) \xrightarrow{T} \bigoplus_p \bigoplus_{\mathcal{D} \subseteq \mathcal{C}_p} \mathbb{F}_p(\mathcal{D})^* \right)$$

where the map to $\mathbb{F}_p(\mathcal{D})^*$ is given by the tame symbol corresponding to $\mathcal{D}$,

$$T_\mathcal{D} : \{a, b\} \mapsto (-1)^{v_\mathcal{D}(a)v_\mathcal{D}(b)} a^{v_\mathcal{D}(b)} b^{v_\mathcal{D}(a)}(\mathcal{D}),$$

with $v_\mathcal{D}$ the discrete valuation on $F$ given by the order of vanishing along $\mathcal{D}$.

Finally put

$$K_2(C; \mathbb{Z}) = \frac{K^T_2(C)}{\text{torsion}} \subseteq \frac{K^T_2(C)}{\text{torsion}},$$

which is independent of the choice of the model $\mathcal{C}$ of $C$.

[We could have defined $K^T_2(C)$ in a single step as

$$K^T_2(C) = \ker \left( K_2(F) \xrightarrow{T} \bigoplus_{\mathcal{D}} \mathbb{F}(\mathcal{D})^* \right)$$

where $\mathcal{D}$ runs through all irreducible curves on $\mathcal{C}$ and $\mathbb{F}(\mathcal{D})$ stands for the residue field at $\mathcal{D}$.]
Regulator 1-forms

For $a$ and $b$ in $F^*$ put

$$\eta(a, b) = \log |a| \ d \arg b - \log |b| \ d \arg a,$$

a smooth, closed 1-form where $a$ and $b$ have no pole or zero.

- $\eta(a_1 a_2, b) = \eta(a_1, b) + \eta(a_2, b)$
- $\eta(a, b_1 b_2) = \eta(a, b_1) + \eta(a, b_2)$
- $\eta(a, 1 - a) = dD(a)$, where $D(z)$ is the Bloch-Wigner dilogarithm function.

So $\eta$ induces a map $K_2(F) \to \frac{\text{closed 1-forms}}{\text{exact 1-forms}}$.

And if $\alpha$ is in $K_2^T(C)$ then $\eta(\alpha)$ has trivial residues.

Conjecture (Beilinson) Let $C/\mathbb{Q}$ be a curve of genus $g$ as before and let $X = C(\mathbb{C})$.

1. $K_2(C; \mathbb{Z}) \cong \mathbb{Z}^g$.
2. Let

$$R = \frac{1}{(2\pi)^g} | \det \left( \begin{array}{ccc} \int_{\gamma_1} \eta_1 & \cdots & \int_{\gamma_1} \eta_g \\ \vdots & \ddots & \vdots \\ \int_{\gamma_g} \eta_1 & \cdots & \int_{\gamma_g} \eta_g \end{array} \right) |$$

where $\eta_1, \ldots, \eta_g$ are the regulator 1-forms obtained from a basis of $K_2(C; \mathbb{Z})$, $\gamma_1, \ldots, \gamma_g$ give a basis of $H_1(X; \mathbb{Z})^-$ and are chosen such that the $\eta_k$ have no zeroes of poles on them.

Then

$$L^*(C, 0) \sim_{\mathbb{Q}^*} R.$$  

['−' in $H_1(X; \mathbb{Z})^-$ denotes the anti-invariants of $H_1(X; \mathbb{Z})$ under the action of complex conjugation on $X$.]

- $R$ does not depend on any choices.
- There is a similar conjecture over an arbitrary number field.
- $R$ makes sense for any $g$ elements of $K_2^T(C)$. 

8
Elements in $K_2^T(C)$ from torsion points

Let $P_1$, $P_2$ and $P_3$ in $C(\mathbb{Q})$ be distinct and such that all $(P_i) - (P_j)$ are torsion in $\text{Jac}(C)(\mathbb{Q})$. Pick $f_i$ in $\mathbb{Q}(C)^*$ with

$$(f_i) = m_i(P_{i+1}) - m_i(P_{i-1})$$

where $m_i = \text{ord}(((P_{i+1}) - P_{i-1}))$ (all indices modulo 3). Put

$$S_i = \left\{ \frac{f_{i+1}}{f_{i+1}(P_{i+1})}, \frac{f_{i-1}}{f_{i-1}(P_{i-1})} \right\},$$

an element of $K_2^T(C)$. Then $\langle S_1, S_2, S_3 \rangle$ in $K_2^T(C)/\text{torsion}$ is generated by an element $\{P_1, P_2, P_3\}$.

All this works if we replace $\mathbb{Q}$ with $\overline{\mathbb{Q}}$, $\mathbb{R}$ or $\mathbb{C}$ as well.

We cannot get much more from out torsion points, in the following sense.

Assume $S \subseteq C(\overline{\mathbb{Q}})$ and $P_0$ in $S$ are such that $(P) - (P_0)$ is torsion for all $P$ in $S$. Let $V \subseteq K_2(\overline{\mathbb{Q}}(C)) \otimes \mathbb{Q}$ be generated by the $\{f, g\}$ with $|(f)|$ and $|(g)|$ in $S$. Then

$$V \cap K_2^T(C_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q} = \langle \{P_0, Q, R\} \rangle_{\mathbb{Q}}$$

with $Q \neq R$ in $S \setminus \{P_0\}$. 
The case of hyperelliptic curves

Assume $C/\mathbb{Q}$ is hyperelliptic of genus $g$ with a rational ramification point $\infty$. Such a $C$ can be defined by

$$y^2 = t(x)$$

with $t(x)$ in $\mathbb{Q}[x]$ of degree $2g + 1$ without multiple roots.

**Example 1: (2g+1)-torsion**

The curve admits an equation

$$y^2 = -x^{2g+1} + f(x)^2/4 = t(x) \quad \text{or} \quad y^2 + f(x)y + x^{2g+1} = 0$$

with $f(x)$ in $\mathbb{Q}[x]$, $\deg(f(x)) \leq g$, and such that $t(x)$ has no multiple roots.

**Example 2: (2g+2)-torsion**

The curve admits an equation

$$y^2 = -x^{2g+2} + f(x)^2/4 = t(x) \quad \text{or} \quad y^2 + f(x)y + x^{2g+2} = 0$$

with $f(x) = 2x^{g+1} + b_gx^g + \cdots + b_0$ in $\mathbb{Q}[x]$, $b_g \neq 0$, and such that $t(x)$ has no multiple roots.

In both cases we have an equation of the form

$$y^2 + f(x)y + x^d = 0$$

with $d = 2g + 1$ or $2g + 2$ and $t(x) = -x^d + f(x)^2/4$.

Let $O = (0,0)$ and $O' = (0, -f(0))$, the image of $O$ under the hyperelliptic involution.

$(O) - (\infty)$ and $(O') - (\infty)$ both have order $d$ in $\text{Jac}(C)(\mathbb{Q})$:

$(y) = d(O) - d(\infty)$.

Let $T_\beta = (\beta, -f(\beta)/2)$ where $\beta$ is a root in $\overline{\mathbb{Q}}$ of $t(x)$, so

$(T_\beta) - (\infty)$ has order 2 in $\text{Jac}(C)(\overline{\mathbb{Q}})$. 
So if \( t(\beta) = 0 \) then we get in \( K_2^T(\overline{C_\mathbb{Q}})/\text{torsion} \)
\[
\{\infty, O, T_\beta\} = \left\{ \frac{y}{-f(\beta)/2}, \frac{x - \beta}{\beta} \right\}.
\]
But if \( m(x) \) is an irreducible factor of \( t(x) \) in \( \mathbb{Q}[x] \) then
\[
2 \sum_{\beta \text{ with } m(\beta) = 0} \{\infty, O, T_\beta\} = \left\{ \frac{y^2}{x^d}, \frac{m(x)}{m(0)} \right\} \overset{\text{def}}{=} M,
\]
an element of \( K_2^T(C) \).

**Proposition**

Let \( t(x) = m_1(x) \cdots m_k(x) \) be a factorisation of \( t(x) \) into irreducibles in \( \mathbb{Q}[x] \) so we have \( M_1, \ldots, M_k \) in \( K_2^T(C) \).

1. If \( d = 2g + 2 \) and
   \[
   4t(x) = -4x^{2g+2} + f(x)^2
   = (b_g x^g + \cdots + b_0)(4x^{g+1} + b_g x^g + \cdots + b_0)
   = 4 \cdot m_1 \cdots m_l \cdot m_{l+1} \cdots m_k
   \]
   then \( M_1 + \cdots + M_l = M_{l+1} + \cdots + M_k \) in \( K_2^T(C) \).

2. If \( f(0) = b_0 = \pm 1 \) then \( \mathbb{M} = \{-f(0)y, -x\} \) is in \( K_2^T(C) \) and
   \[
   2d \mathbb{M} = \sum_{j=1}^{k} M_j.
   \]

3. If \( W \subseteq K_2^T(\overline{C_\mathbb{Q}}) \otimes \mathbb{Q} \) is generated over \( \mathbb{Q} \) by all \( \{P, Q, R\} \) with \( P, Q \) and \( R \) in \( \{\infty, O, O', T_\beta, \ldots, T_{\beta_{2g+1}}\} \) then it is already generated by the \( \{\infty, O, T_\beta\} \).

4. \( K_2^T(C) \otimes \mathbb{Q} = K_2^T(\overline{C_\mathbb{Q}})^{\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \otimes \mathbb{Q} \subseteq K_2^T(C_\mathbb{Q}) \otimes \mathbb{Q} \) and with \( W \) as in (3),
   \[
   W \cap K_2^T(C) \otimes \mathbb{Q} = \langle M_1, \ldots, M_k \rangle_{\mathbb{Q}}.
   \]
[So (3) and (4) say that the \( M_j \) give essentially everything we can get from the known torsion points.]
So we need $t(x) = -x^d + f(x)^2/4$ as before with many rational factors, e.g.,

\[
-3326400^2x^{12} + (3326400x^6 + 149040x^5 - 150012x^4
+ 188x^3 + 787x^2 - 2x + 1)^2
\]

\[
= (22x - 1)(20x - 1)(18x - 1)(12x - 1)(10x - 1)(x - 1)(7x + 1)(15x + 1)(18x + 1)(23x + 1)(24x + 1).
\]

**Integrality**

**Theorem**
Consider either

\[ f(x) = b_g x^g + \cdots + b_0 \quad \text{and} \quad d = 2g + 1 \]

or

\[ f(x) = 2x^{g+1} + b_g x^g + \cdots + b_0 \quad \text{with} \quad b_g \neq 0 \quad \text{and} \quad d = 2g + 2 \]

with $b_0, \ldots, b_g$ in $\mathbb{Z}$ such that $t(x) = -x^d + f(x)^2/4$ has no multiple root.

Let $m(x)$ be an irreducible factor of $t(x)$ in $\mathbb{Q}[x]$.

1. If $m(x)/m(0)$ is in $\mathbb{Z}[x]$ then the class of $\{y^2/x^d, m(x)/m(0)\}$ is in $K_2(C; \mathbb{Z})$.
2. If $\gcd(b_0, \ldots, b_g) = 1$ and $m(x)/m(0)$ is not in $\mathbb{Z}[x]$ then no non-trivial multiple of the class of $\{y^2/x^d, m(x)/m(0)\}$ is in $K_2(C; \mathbb{Z})$.
3. If $b_0 = \pm 1$ then

\[
M = \{-b_0 y, -x\} \quad \text{is in} \quad K_2(C; \mathbb{Z}) \quad \text{if} \quad d = 2g + 1,
\]

\[ 2M = \{y^2, x\} \quad \text{is in} \quad K_2(C; \mathbb{Z}) \quad \text{if} \quad d = 2g + 2. \]
Limit results

Theorem
Let $g \geq 1$ and fix $v_1 < \cdots < v_{g-1}$ in $\mathbb{R}^*$. If $v_g = a \gg 0$ and

$$f(x) = 2x^{g+1} + \prod_{j=1}^{g} (v_j x + 1)$$

then

$$4t(x) = (4x^{g+1} + \prod_{j=1}^{g} (v_j x + 1)) \prod_{j=1}^{g} (v_j x + 1)$$

has $2g + 1$ distinct real roots. If that is the case then for the curve $C/\mathbb{R}$ of genus $g$ defined by

$$y^2 + f(x)y + x^{2g+2} = 0$$

the elements

$$M_l = \left\{ \frac{y^2}{x^{2g+2}}, v_l x + 1 \right\} \quad (l = 1, \ldots, g)$$

in $K_2^T(C)$ have Beilinson regulator $R = R(a)$ satisfying

$$\lim_{a \to \infty} \frac{R(a)}{(2 \log a)^g} = g + 1.$$ 

Corollary
With the same notation but $v_1 < \cdots < v_{g-1}$ in $\mathbb{N}$ fixed, if $v_g \gg 0$ in $\mathbb{N}$ then the classes of $M_1, \ldots, M_g$ are independent in $K_2(C; \mathbb{Z})$. 

13
Remark

- In this situation with \( g > 1 \), the two curves corresponding
to \( 0 < \nu_1 < \cdots < \nu_g \) and \( 0 < \tilde{\nu}_1 < \cdots < \tilde{\nu}_g \) are isomorphic
over \( \mathbb{Q} \) precisely when either \( \tilde{\nu}_j = \nu_j \) for \( j = 1, \ldots, g \), or \( g \) is
odd and \( \tilde{\nu}_j = \nu_g - \nu_{g-j} \) for \( j = 1, \ldots, g-1 \) and \( \tilde{\nu}_g = \nu_g \).

- The map from the open part of \( \mathbb{Q}^* \) where \( t(x) \) has no multiple roots to the set of isomorphism classes over \( \overline{\mathbb{Q}} \) of hyperelliptic curves over \( \overline{\mathbb{Q}} \) is finite to one, so we get infinitely many isomorphism classes over \( \overline{\mathbb{Q}} \) of curves \( C/\mathbb{Q} \) for which the rank of \( K_2(C; \mathbb{Z}) \) is at least \( g \).

Explicit example:

For the curves \( C_{a,b} \) defined by

\[
y^2 + (2x^3 + (ax + 1)(bx + 1))y + x^6 = 0
\]

with integers \( a \equiv 1 \) and \( b \equiv 2 \) modulo 3, and \( a \gg 0 \) or \( b \gg 0 \):

1. the two elements \( \{y^2/x^6, ax+1\} \) and \( \{y^2/x^6, bx+1\} \) give
classes in \( K_2(C_{a,b}; \mathbb{Z}) \);
2. the Jacobian of \( C_{a,b} \) does not split over \( \overline{\mathbb{Q}} \);
3. \[
\lim_{a \to \infty} \frac{R(C_{a,b})}{(2 \log a)^2} = 3 .
\]
Proposition
Let $k$ be a real quadratic field and let $\mathcal{O}_k$ be its ring of algebraic integers. Fix $v \neq \pm 1$ in $\mathcal{O}_k^*$ as well as $p$ and $q$ in $\mathcal{O}_k$ satisfying $pq = 4$. If $pv^n \neq \pm 2$

$$y^2 + (2x^2 + (pv^n + qv^{-n})x + 1)y + x^4 = 0$$

defines an elliptic curve $C$ over $k$. The classes of

$$\begin{cases} 
    y^2 \\
    x^4
  \end{cases}, \frac{pv^n x + 1}{x^4}$$

and

$$\begin{cases} 
    y^2 \\
    x^4
  \end{cases}, \frac{qv^{-n} x + 1}{x^4}$$

are in $K_2(C; \mathbb{Z})$ and their Beilinson regulator $R = R(n)$ satisfies

$$\lim_{n \to \infty} \frac{R(n)}{n^2} = 4|\log |v||^2 .$$

Here $|v|$ is the absolute value using either embedding of $k$ into $\mathbb{R}$, $|\log |v||$ being independent of the embedding.

Moreover, if the norm of $p$ in $\mathbb{Q}$ does not have absolute value 4 (e.g., when $p = 1$ and $q = 4$), then for $n \gg 0$ the $j$-invariant is neither rational nor an algebraic integer. Therefore, for $n \gg 0$ those curves cannot be obtained from curves over $\mathbb{Q}$ by enlarging the base field, nor do they admit complex multiplication over $\overline{\mathbb{Q}}$. 

15
Some interesting questions

(1) In our tables we often find an integer for $L^*(C, 0)/R(\Lambda)$ so

$$\frac{L^*(C, 0)}{R(K_2(C; \mathbb{Z}))} = (K_2(C; \mathbb{Z}) : \Lambda) \frac{L^*(C, 0)}{R(\Lambda)}$$

might always be an integer. For this we may need more elements in $K_2(C; \mathbb{Z})$. It would be interesting to try and do this, especially for the curve defined by

$$y^2 + (5x^3 - 13x^2 + 7x - 1)y + x^7 = 0$$

where $L^*(C, 0)/R(\Lambda) \approx 1/19$.

(2) If $C/\mathbb{Q}$ is a curve and $F = \mathbb{Q}(C)$, can we decide in an algorithmic way if an element in $K_2(F)$ is trivial or not? More specifically we can consider the following questions.

(a) Can we decide if the relations in the tables as suggested by the computer calculations are true in $K_2(F)$ or not?

(b) The element

$$\{(x - 1)^2, (y + 1)^3(y + 2x - 3)\}$$

is trivial in $K_2(F)$ where $F = \mathbb{Q}(E)$ and $E$ is defined by

$$y^2 = x^3 - 2x^2 + 1.$$ 

How do we write explicitly

$$(x - 1)^2 \otimes [(y + 1)^3(y + 2x - 3)] = \sum_i m_i (f_i \otimes (1 - f_i))$$

in $F^* \otimes F^*$ for some $m_i$ in $\mathbb{Z}$ and $f_i$ in $F \setminus \{0, 1\}$?