K_4 of curves over number fields

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K_2 of a field

For a field F let $F^\flat = F \setminus \{0, 1\}$. Then

$$K_2(F) = F^* \otimes_{\mathbb{Z}} F^* / \langle x \otimes (1-x) | x \in F^\flat \rangle$$

is an Abelian group written additively with

$$\begin{aligned} \text{generators} \qquad \{a, b\} &= \text{ the class of } a \otimes b \\ \text{relations} \quad \{a_1 a_2, b\} &= \{a_1, b\} + \{a_2, b\} \\ \{a, b_1 b_2\} &= \{a, b_1\} + \{a, b_2\} \\ \{a, 1-a\} &= 0 \text{ if } a \text{ is in } F^\flat \end{aligned}$$

Then also $\{a, b\} = -\{b, a\}$ and $\{c, -c\} = 0$ for a, b, c in F^*.

K-theory of a curve

Let C be an irreducible regular curve over a field k and $F = k(C)$. We have the exact localization sequence

$$\cdots \to \coprod_{P \in C^{(1)}} K_4(k(P)) \to K_4(C) \to K_4(F)$$
$$\xrightarrow{\partial} \coprod_{P \in C^{(1)}} K_3(k(P)) \to K_3(C) \to K_3(F)$$
$$\to \coprod_{P \in C^{(1)}} K_2(k(P)) \to K_2(C) \to K_2(F)$$
$$\xrightarrow{T} \coprod_{P \in C^{(1)}} k(P)^* \to K_1(C) \to F^*$$
$$\xrightarrow{\mathrm{div}} \coprod_{P \in C^{(1)}} \mathbb{Z} \to K_0(C) \to \mathbb{Z} \to 0$$

with the tame symbol $T : K_2(F) \to \coprod_{P \in C^{(1)}} k(P)^*$ given by

$$T_P(\{f, g\}) = (-1)^{\mathrm{ord}_P(f)\mathrm{ord}_P(g)} \frac{f^{\mathrm{ord}_P(g)}}{g^{\mathrm{ord}_P(f)}}|_P$$

K_4 of a curve

We try to approximate $K_2(C)$ as $\ker(T)$ and $K_4(C)$ as $\ker(\partial)$ using

$$\cdots \to \coprod_{P \in C^{(1)}} K_2(k(P)) \to K_2(C) \to K_2(F) \xrightarrow{T} \coprod_{P \in C^{(1)}} k(P)^* \to \cdots$$

$$\cdots \to \coprod_{P \in C^{(1)}} K_4(k(P)) \to K_4(C) \to K_4(F) \xrightarrow{\partial} \coprod_{P \in C^{(1)}} K_3(k(P)) \to \cdots$$

Fact If k is a number field then
(1) all $K_{2n}(k(P))$ $(n \geq 1)$ are infinite torsion groups;
(2) $K_3(k(P))$ is torsion if and only if $k(P)$ is totally real.

Conjecture (Beilinson) if C is complete, regular and geometrically irreducible, and k is a number field, then
(a) $\dim_{\mathbb{Q}} K_4(C)_{\mathbb{Q}} = [k : \mathbb{Q}] \cdot \text{genus}(C) \stackrel{\text{def}}{=} r$ no integrality condition
(b) $R_4(C) = qL^{(r)}(C, -1) \neq 0$, where $q \in \mathbb{Q}^*$, and $R_4(C)$ is a Beilinson regulator of $K_4(C)_{\mathbb{Q}}$ and we assume $L(C, s)$ satisfies a suitable functional equation

Definition of a Beilinson regulator

Let

C_{an}: complex manifold associated to $C(\mathbb{C})$

σ: complex conjugation on C_{an}

$\mathrm{reg} : K_4(C) \to H^1_{\mathrm{dR}}(C_{\mathrm{an}}; \mathbb{R})^+ = \{\xi | \sigma^*(\xi) = \xi\}$
 the Beilinson regulator map

$\{\gamma_1, \dots, \gamma_r\}$: a $\mathbb{Z}$-basis of $H_1(C_{\mathrm{an}}; \mathbb{Z})^+ = \{\gamma | \sigma_*(\gamma) = \gamma\}$,

$\{\alpha_1, \dots, \alpha_r\}$: a $\mathbb{Q}$-basis of $K_4(C)_{\mathbb{Q}}$.

Then

$$R_4(C) = (2\pi i)^{-2r} \det(M) \quad \text{with} \quad M_{j,l} = \int_{\gamma_j} \mathrm{reg}(\alpha_l)$$

A definition using surface integrals

With
$\{\gamma_1', \ldots, \gamma_r'\}$: $\mathbb{Z}$-basis of $H_1(C_{\mathrm{an}}; \mathbb{Z})^- = \{\gamma | \sigma_*(\gamma) = -\gamma\}$,
$\{\omega_1, \ldots, \omega_r\}$: $\mathbb{R}$-basis of $V^+ = \{\omega \text{ in } H^0(C_{\mathrm{an}}; \Omega^1) | \sigma^*(\omega) = \overline{\omega}\}$, let
$T^- = (\int_{\gamma_j'} \omega_l)_{j,l}$
Orient C_{an} such that σ reverses the orientation, and let
$R_{j,l} = \int_{C_{\mathrm{an}}} \mathrm{reg}(\alpha_j) \wedge \overline{\omega_l} \in i\mathbb{R}$.
Then

$$R_4(C) = \frac{\det(R)}{(2\pi i)^{2r} \det(T^-)}$$

Cf. for $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_r$ in K_2: $R_2(C) = \dfrac{\det(R)}{(2\pi i)^r \det(T^+)}$ with
$R_{j,l} = \int_{C_{\mathrm{an}}} \mathrm{reg}(\tilde{\alpha}_j) \wedge \overline{\omega_l} \in i\mathbb{R}$ and $T^+ = (\int_{\gamma_j'} \omega_l)_{j,l}$

For such ω one can extend $K_4(C) \to H^1_{\mathrm{dR}}(C_{\mathrm{an}}; \mathbb{R})^+ \overset{\int \cdot \wedge \overline{\omega}}{\to} i\mathbb{R}$ over
$K_4(C) \to K_4(F) \to K_4(F)/K_3(k) \cup F^*$.

Adams weights

F any field, $F^\flat = F \setminus \{0,1\}$.

- $K_0(F) = \mathbb{Z}$
- $K_1(F) = F^*$
- $K_2(F) \simeq F^* \otimes_{\mathbb{Z}} F^* / \langle (1-x) \otimes x \text{ with } x \in F^\flat \rangle$

- $K_n(F)$ for $n \geq 3$ can have several parts. Write $K_n^{(j)}(F) \subseteq K_n(F)_{\mathbb{Q}}$ for the jth Adams weight eigenspace.
- $K_0(F)_{\mathbb{Q}} = K_0^{(0)}(F) = \mathbb{Q}$
- $K_1(F)_{\mathbb{Q}} = K_1^{(1)}(F) = F^*_{\mathbb{Q}}$
- $K_n(F)_{\mathbb{Q}} = \oplus_{j=2}^n K_n^{(j)}(F)$ for $n \geq 2$
- $K_n^{(n)}(F) \simeq K_n^M(F)_{\mathbb{Q}}$ for $n \geq 2$, with $K_n^M(F) = (F^*)^{\otimes n} / \langle \cdots \otimes (1-x) \otimes x \otimes \ldots \text{ with } x \in F^\flat \rangle$
- Beilinson-Soulé conjecture: $K_n^{(j)}(F) = 0$ for $2j \leq n$ and $n > 0$.

$K_n^{(j)}(F)$ with $j < n$ is not so easy in general.

Theorem(Suslin; actually for $K_3(F)$ modulo image of $K_3^M(F)$) If F is infinite, then there is an exact sequence

$$0 \to K_3^{(2)}(F) \to \frac{\mathbb{Q}[F^\flat]}{\langle \text{5-term relations} \rangle} \xrightarrow{\mathrm{d}} \wedge^2 F_{\mathbb{Q}}^* \to K_2^{(2)}(F) \to 0$$

where the 5-term relations are of the form

$$\sum_{i=1}^{5} (-1)^i [\mathrm{cr}(P_1, \ldots, \widehat{P_i}, \ldots, P_5)]$$

for 5 distinct points P_i in $\mathbb{P}^1_F$, and $\mathrm{d}[x] = (1-x) \wedge x$.

The localization sequence with weights

For $j \in \mathbb{Z}$ we have the exact localization sequences

$$\begin{aligned} \cdots \to \textstyle\coprod_{P \in C^{(1)}} K_4^{(j-1)}(k(P)) \to K_4^{(j)}(C) \to K_4^{(j)}(F) \\ \to \textstyle\coprod_{P \in C^{(1)}} K_3^{(j-1)}(k(P)) \to K_3^{(j)}(C) \to K_3^{(j)}(F) \\ \to \textstyle\coprod_{P \in C^{(1)}} K_2^{(j-1)}(k(P)) \to K_2^{(j)}(C) \to K_2^{(j)}(F) \\ \to \textstyle\coprod_{P \in C^{(1)}} K_1^{(j-1)}(k(P)) \to K_1^{(j)}(C) \to K_1^{(j)}(F) \\ \to \textstyle\coprod_{P \in C^{(1)}} K_0^{(j-1)}(k(P)) \to K_0^{(j)}(C) \to K_0^{(j)}(F) \to 0 \end{aligned}$$

In particular, $0 \to K_2^{(2)}(C) \to K_2^{(2)}(F) \to \coprod_P K_1^{(1)}(k(P))$ and $0 \stackrel{?}{\to} K_4^{(3)}(C) \to K_4^{(3)}(F) \to \coprod_P K_3^{(2)}(k(P))$

Conjecture (Goncharov, Zagier) F any field of characteristic zero (for simplicity). There exist $\mathbb{Q}$-vector spaces $\widetilde{M}_{(n)}(F)$ $(n \geq 2)$ generated by symbols $[f]_n$ with f in $F^\flat$, satisfying certain (unknown) relations, such that

(1) for the cohomological complex in degrees 1, 2

$$\widetilde{\mathcal{M}}_{(2)}(F) : \widetilde{M}_{(2)}(F) \xrightarrow{\mathrm{d}} \bigwedge^2 F^*_{\mathbb{Q}}$$
$$[f]_2 \mapsto (1-f) \wedge f$$

there exists $H^1(\widetilde{\mathcal{M}}_{(2)}(F)) \to K_3(F)$ giving an isomorphism $H^1(\widetilde{\mathcal{M}}_{(2)}(F)) \xrightarrow{\sim} K_3^{(2)}(F) \subseteq K_3(F)_{\mathbb{Q}}$

(1a) If k is a number field then we get, for any $\tau : k \to \mathbb{C}$

$$H^1(\widetilde{\mathcal{M}}_{(2)}(k)) \stackrel{\simeq}{\to} K_3^{(2)}(k) = K_3(k)_{\mathbb{Q}} \stackrel{\tau_*}{\to} K_3(\mathbb{C})_{\mathbb{Q}} \stackrel{\mathrm{reg}}{\to} i\mathbb{R}$$

(reg is Beilinson's regulator map) is given by

$$[x]_2 \mapsto P_2(\tau(x))$$

where, for $z \in \mathbb{C} \setminus \{0, 1\}$

$$P_2(z) = \int_{z_0}^{z} \log|z| \mathrm{d} i \arg(1-z) - \log|1-z| \mathrm{d} i \arg z = iD(z)$$

for any $z_0 \in \mathbb{R} \setminus \{0, 1\}$. ($D$ is the Bloch-Wigner dilogarithm.)

(2) for the cohomological complex in degrees 1, 2, 3

$$\widetilde{\mathcal{M}}_{(3)}(F) : \widetilde{M}_{(3)}(F) \xrightarrow{\mathrm{d}} \widetilde{M}_{(2)}(F) \otimes F^*_{\mathbb{Q}} \xrightarrow{\mathrm{d}} \bigwedge^3 F^*_{\mathbb{Q}}$$

$$[f]_2 \otimes g \mapsto (1-f) \wedge f \wedge g$$

$$[f]_3 \mapsto [f]_2 \otimes f$$

there exists $H^2(\widetilde{\mathcal{M}}_{(3)}(F)) \to K_4(F)_{\mathbb{Q}}$ giving an isomorphism

$$H^2(\widetilde{\mathcal{M}}_{(3)}(F)) \xrightarrow{\sim} K_4^{(3)}(F) \subseteq K_4(F)_{\mathbb{Q}}$$

with an explicit relation (see later) with the boundary map

$$\partial : K_4(F) \to \coprod_{P \in C^{(1)}} K_3(k(P))$$

Some results

Proposition If F has characteristic zero then
(1) there exists an injection (natural up to universal sign)

$$H^1(\widetilde{\mathcal{M}}_{(2)}(F)) \to K_3^{(2)}(F) \subseteq K_3(F)_{\mathbb{Q}}$$

(2) there exists a map (natural up to universal sign)

$$H^2(\widetilde{\mathcal{M}}_{(3)}(F)) \to K_4^{(3)}(F) \subseteq K_4(F)_{\mathbb{Q}}$$

If k is a number field then also
(1a) for $F = k$ (1) is an isomorphism, and for $\tau : k \to \mathbb{C}$,

$$H^1(\widetilde{\mathcal{M}}_{(2)}(k)) \stackrel{\sim}{\to} K_3(k)_{\mathbb{Q}} \stackrel{\tau_*}{\to} K_3(\mathbb{C})_{\mathbb{Q}} \stackrel{\mathrm{reg}}{\to} i\mathbb{R}$$

is given by

$$[x]_2 \mapsto P_2(\tau(x))$$

(2a) if $F = k(C)$ (C as before), ω in V^+, then the composition

$$H^2(\widetilde{\mathcal{M}}_{(3)}(F)) \to K_4(F)_{\mathbb{Q}} \stackrel{\mathrm{reg}}{\to} H^1_{\mathrm{dR}}(F_{\mathrm{an}};\mathbb{R})^+ \xrightarrow{\int \cdot \wedge \overline{\omega}} i\mathbb{R}\,,$$

is induced by

$$[f]_2 \otimes g \mapsto -\tfrac{8}{3} \int_{C_{\mathrm{an}}} P_2 \circ f \,\mathrm{d}\log|g| \wedge \overline{\omega}$$

$$= \tfrac{8}{3} \int_{C_{\mathrm{an}}} \log|g|(\log|f|\mathrm{d}\log|1-f| - \log|f|\mathrm{d}\log|1-f|) \wedge \overline{\omega}$$

Here f denotes the function that on the component of C_{an} corresponding to $\tau : k \to \mathbb{C}$ is obtained by applying τ to the coefficients of f

Proposition The diagram

$$\begin{array}{ccccc}
\widetilde{M}_{(3)}(F) & \xrightarrow{\mathrm{d}} & \widetilde{M}_{(2)}(F) \otimes F^*_{\mathbb{Q}} & \xrightarrow{\mathrm{d}} & \bigwedge^3 F^*_{\mathbb{Q}} \\
\downarrow & & \downarrow \delta & & \downarrow \varepsilon \\
0 & \longrightarrow & \coprod_{P \in C^{(1)}} \widetilde{M}_{(2)}(k(P)) & \xrightarrow{\mathrm{d}} & \coprod_{P \in C^{(1)}} \bigwedge^2 k(P)^*_{\mathbb{Q}}
\end{array}$$

commutes, where

$$[f]_2 \otimes g \overset{\delta_P}{\mapsto} \mathrm{ord}_P(g)[f(P)]_2 \qquad ([0]_2 = [1]_2 = [\infty]_2 = 0)$$

$$f \wedge g \wedge h \overset{\varepsilon_P}{\mapsto} \mathrm{ord}_P(f)\,\overline{g_P} \wedge \overline{h_P} - \mathrm{ord}_P(g)\,\overline{f_P} \wedge \overline{h_P} + \mathrm{ord}_P(h)\,\overline{f_P} \wedge \overline{g_P}$$

with $\overline{f_P} = (f\pi_P^{-\mathrm{ord}_P(f)})_{|P}$ for any fixed uniformizer π_P at P.
ε_P is independent of π_P

Proposition The diagram

$$\begin{array}{ccc} H^2(\widetilde{\mathcal{M}}_{(3)}(F)) & \longrightarrow & K_4(F)_{\mathbb{Q}} \\ \downarrow{\scriptstyle 2\delta} & & \downarrow{\scriptstyle \partial} \\ \coprod_{P\in C^{(1)}} H^1(\widetilde{\mathcal{M}}_{(2)}(k(P))) & \longrightarrow & \dfrac{\coprod_{P\in C^{(1)}} K_3(k(P))_{\mathbb{Q}}}{\partial(K_3(k)\cup F_{\mathbb{Q}}^*)} \end{array}$$

commutes (up to a universal sign)

Let $\widetilde{\mathcal{M}}_{(3)}(C)$ be the total complex associated to the double complex on the previous page, so

$$0 \to H^2(\widetilde{\mathcal{M}}_{(3)}(C)) \to H^2(\widetilde{\mathcal{M}}_{(3)}(F)) \to \coprod_{P\in C^{(1)}} H^1(\widetilde{\mathcal{M}}_{(2)}(k(P)))$$

is exact.

From the commutative diagram (vertical maps are injective)

$$
\begin{array}{ccccc}
H^2(\widetilde{\mathcal{M}}_{(3)}(F)) & \longrightarrow & K_4(F)_{\mathbb{Q}} & \xrightarrow{\mathrm{reg}} & H^1_{\mathrm{dR}}(F_{\mathrm{an}};\mathbb{R})^+ \\
\uparrow & & \uparrow & & \uparrow \\
 & \nearrow & K_4(C)_{\mathbb{Q}} \oplus K_3(k) \cup F^*_{\mathbb{Q}} & & \\
 & & \uparrow & & \\
H^2(\widetilde{\mathcal{M}}_{(3)}(C)) & & K_4(C)_{\mathbb{Q}} & \xrightarrow{\mathrm{reg}} & H^1_{\mathrm{dR}}(C_{\mathrm{an}};\mathbb{R})^+
\end{array}
$$

we define the composition

$$\Phi_C : H^2(\widetilde{\mathcal{M}}_{(3)}(C)) \to K_4(C)_{\mathbb{Q}} \oplus K_3(k) \cup F^*_{\mathbb{Q}} \stackrel{\mathrm{proj}}{\to} K_4(C)_{\mathbb{Q}}$$

Theorem $\mathrm{reg}(\mathrm{im}(\Phi_C)) = \mathrm{reg}(K_4(C)_{\mathbb{Q}})$

Line integrals are calculated faster so want $\mathrm{reg}(\beta)$ as 1-form.
Proposition For f in $F^\flat$ and g in F^* let

$$\psi(f,g) = 2P_2(f)\mathrm{d}i\arg g - \log|f|\log|g|\mathrm{d}\log|1-f| \\ - \frac{1}{3}\log|1-f|(\log|f|\mathrm{d}\log|g| - \log|g|\mathrm{d}\log|f|)$$

With

$$\Psi : H^2(\widetilde{\mathcal{M}}_{(3)}(C)) \to H^1_{\mathrm{dR}}(C_{\mathrm{an}};\mathbb{R})^+ \subset H^1_{\mathrm{dR}}(F_{\mathrm{an}};\mathbb{R})^+ \\ [f]_2 \otimes g \mapsto \psi(f,g)$$

we have $\mathrm{reg}(\Phi_C(\beta)) = \Psi(\beta)$ for β in $H^2(\widetilde{\mathcal{M}}_{(3)}(C))$

Idea of proof $\int_{C_{\mathrm{an}}} \mathrm{reg}(\Phi_C(\beta)) \wedge \overline{\omega} = \int_{C_{\mathrm{an}}} \Psi(\beta) \wedge \overline{\omega}$ for all ω in $H^0(C_{\mathrm{an}}, \Omega^1)$ by Stokes' theorem and the conditions on β.

Some simple examples (over $\mathbb{Q}$)

Simple example 1 For $E:(y-\frac{1}{2})^2=x^3+\frac{1}{4}$ $(N=27)$ we have $y(1-y)=(-x)^3$ so for $\beta=[y]_2\otimes(1-x)-3[x]_2\otimes y$

$$\mathrm{d}(\beta)=(1-y)\wedge y\wedge(1-x)-(1-x)\wedge x^3\wedge y=0\,.$$

$\delta(\beta)$ is supported in $E(\mathbb{Q}(\sqrt{5}))$ so is trivial, and $\beta\in H^2(\widetilde{\mathcal{M}}_{(3)}(C))$. Numerically, $R_4(\Phi_E(\beta))=-\frac{6}{5}L'(E,-1)$.

Simple example 2 For $E:(y-\frac{1}{2})^2=x^3-x^2+\frac{1}{4}$ $(N=11)$ we have $y(1-y)=x^2(1-x)$ and $\beta=[x]_2\otimes y+[y]_2\otimes x$ satisfies $\mathrm{d}(\beta)=0$. $\delta(\beta)$ is supported in $E(\mathbb{Q})$ so is trivial, and $\beta\in H^2(\widetilde{\mathcal{M}}_{(3)}(C))$. Numerically, $R_4(\Phi_E(\beta))=-3L'(E,-1)$.

A tricky construction

If $\{h_1, h_2\} = 0$ in $K_2(F)$ then $h_1 \otimes h_2 = \sum_j c_j(1-f_j) \otimes f_j$ in $F^* \otimes_{\mathbb{Z}} F^*$, with the c_j integers and the f_j in $F^\flat$. Then

$$\beta_1 = [1-h_1]_2 \otimes h_2 + \sum_j c_j [f_j]_2 \otimes (1-h_1)$$

is in $H^2(\widetilde{\mathcal{M}}_{(3)}(F))$ because

$$h_1 \wedge (1-h_1) \wedge h_2 + \sum_j c_j (1-f_j) \wedge f_j \wedge (1-h_1) = 0\,.$$

Also in $H^2(\widetilde{\mathcal{M}}_{(3)}(F))$ is $\beta_2 = \sum_j c_j [f_j]_2 \otimes h_1$, with $\delta(\beta_2)$ in $\coprod_{P \in C^{(1)}} H^1(\widetilde{\mathcal{M}}_{(2)}(k(P)))$ supported where $\mathrm{ord}_P(h_1) \neq 0$.

Drawbacks:

(1) the β_i are not completely explicit: cannot compute the f_j in practice, so cannot compute $\delta(\beta_i)$.

(2) it is difficult to find a non-zero example this way: the obvious trivial elements in $K_2(F)$ $(\{f, 1-f\}, \{f, -f\}, \dots)$ only give $\beta_1 = \beta_2 = 0$.

Advantage(?):

For ω the regulator integral of $\Phi_E(\beta_1)$ equals (so is known!)

$$-4 \int_{C_{\mathrm{an}}} \log|1-h_1| \log|h_2| \mathrm{d} \log|h_1| \wedge \overline{\omega}$$

and of $\Phi_E(\beta_2)$ it equals

$$-4 \int_{C_{\mathrm{an}}} \log|h_1| \log|h_2| \mathrm{d} \log|h_1| \wedge \overline{\omega}$$

Non-trivially trivial elements in $K_2(F)$

Let E/k be an elliptic curve, $F = k(E)$.

Suppose P, Q in $E(k)$ satisfy
(1) $2P = O$, $dQ = O$
(2) $P \neq O$, $2Q \neq O$

Pick f_P and f_Q in F^* with

$$(f_P) = 2(P) - 2(O) \qquad f_P(Q) = 1$$
$$(f_Q) = d(Q) - d(O) \qquad f_Q(P) = 1$$

so $\{f_P, f_Q\}$ is in $\ker(T)$, $T : K_2(F) \to \coprod_{P \in C^{(1)}} k(P)^*$.

Fact Translation on E acts trivially on $\ker(T)/K_2(k)$ with $K_2(k)$ coming from the base field via pullback.

Translating over P we find, modulo $K_2(k)$,

$$\{f_P, f_Q\} \equiv T_P\{f_P, f_Q\} \equiv \{T_P f_P, T_P f_Q\} = \{c f_P^{-1}, T_P f_Q\}$$

with $c = f_P T_P f_P$ in k^*. Hence, for some α in $K_2(k)$,

$$\{f_P, f_Q T_P f_Q\} = \{c, T_P f_Q\} + \alpha$$

If $c^n = 1$ then

$$\{f_P, (f_Q T_P f_Q)^n\} = n\alpha$$

lies in $K_2(k)$ and is trivial:

$$\{f_P, (f_Q T_P f_Q)^n\} = n\{f_P, f_Q T_P f_Q\} = n\{f_P, (1 - f_P)^{-d} f_Q T_P f_Q\}$$

can be pulled back to Q, and $f_P(Q) = 1$.

The elements β_i in $H^2(\widetilde{\mathcal{M}}_{(3)}(F))$ constructed from $\{f_P, (f_Q T_P f_Q)^n\} = 0$ have boundary supported in $G = \langle P, Q\rangle \subseteq E(k)$ and $\sum_{S\in G} T_S\beta_i$ has trivial boundary.

A very explicit example

For $s \neq 0, -\frac{1}{9}, -1$ the point $(0,0)$ has order 6 on

$$E : y^2 + (1-s)xy - (s^2+s)y = x^3 - (s^2+s)x^2$$

With $P = 3(0,0) = (s, s^2)$, $Q = -2(0,0) = (s^2+s, 0)$ we get $f_P = s^{-2}(x-s)$ and $c = f_P T_P f_P = -1/s$.

For $s = 1$ we have $y^2 - 2y = x^3 - 2x^2$, where $c = -1$,

$$f_Q T_P f_Q = (x-y)^3(2x+y-4)^2(x-1)^{-3}(x-2)^{-3}$$

and, in $K_2(\mathbb{Q}(E))$,

$$0 = \{(x-1)^2, f_Q T_P f_Q\} = \{(x-1)^2, (x-y)^3(2x+y-4)^2\}$$

Both β_1 and β_2 are in $H^2(\widetilde{\mathcal{M}}_{(3)}(E))$. Using surface integrals: $R_4(\Phi_E(\beta_1)) = -2R_4(\Phi_E(\beta_2)) = 2L'(E,-1)$ numerically.

A general method

Starting with some generators $[u]_2 \otimes v$ in $\widetilde{M}_{(2)}(F) \otimes F^*_{\mathbb{Q}}$ we can try to find elements in $H^2(\widetilde{\mathcal{M}}_{(3)}(F))$ using a computer.

Advantages:

(1) can aim for elements in K_4 for a given curve
(2) method could work for any curve over any (number) field
(3) the elements are completely explicit

Problems:

(1) not clear which $[u]_2 \otimes v$ lead to an "efficient" calculation
(2) the resulting elements could be big

For an elliptic curve E we start from linear triplet relations $c_1 l_1 l_2 l_3 + c_2 m_1 m_2 m_3 = c_3 n_1 n_2 n_3$ on E with all c_i in k^*, $c_1 l_1 l_2 l_3$ non-constant, all l_i, m_i, n_i equal to 1 or in a finite set of lines $\{A_1, \ldots, A_t\}$. If $u = \frac{c_1 l_1 l_2 l_3}{c_3 n_1 n_2 n_3}$ then $1 - u = \frac{c_2 m_1 m_2 m_3}{c_3 n_1 n_2 n_3}$.

An honest example (over $\mathbb{Q}$)

Example $E : y^2 + y + 1 = x^3 - x \quad (N = 179)$

$E(\mathbb{Q}) = \{O\}$ so $\mathbb{Q}[x, y]/(y^2 + y + 1 - x^3 + x)$ is a UFD and it is easy to calculate in $F^* = \mathbb{Q}(E)^*$. With

$$
\begin{array}{lll}
A_1 = x & A_2 = x + 1 & A_3 = x - 1 \\
A_4 = y & A_5 = y - 1 & A_6 = y + 1 \\
A_7 = y + x & A_8 = y - x & A_9 = y + x + 1 \\
A_{10} = y - x + 1 & A_{11} = y + x - 1 & A_{12} = y - x - 1
\end{array}
$$

yields 32 really distinct exceptional linear triplet relations i.e., that are not polynomial identities. The only prime factors in the c_j are 2, 3. For example

$$4A_1A_2A_3 + (-1)A_5^2 = 3A_6^2$$

The exceptional relations give

$$
\begin{array}{llll}
u_1 = -\dfrac{A_1^2}{A_4 A_6} & u_2 = -\dfrac{A_1 A_7}{A_4 A_{10}} & u_3 = \dfrac{A_1 A_8}{A_4 A_9} & u_4 = -\dfrac{A_1 A_9}{A_6 A_8} \\
u_5 = \dfrac{A_1 A_{10}}{A_6 A_7} & u_6 = \dfrac{A_1^3}{A_4 A_6} & u_7 = \dfrac{A_1^3}{A_8 A_9} & u_8 = \dfrac{A_1^3}{A_6 A_9} \\
u_9 = \dfrac{A_1^3}{A_4 A_8} & u_{10} = \dfrac{A_1^3}{A_9} & u_{11} = -\dfrac{A_1^3}{A_8} & u_{12} = \dfrac{A_1^2 A_3}{A_9 A_{10}} \\
u_{13} = \dfrac{A_1^2 A_3}{A_9} & u_{14} = A_1^2 A_3 & u_{15} = \dfrac{A_1^2 A_4}{A_8^2} & u_{16} = \dfrac{A_1^2 A_4}{A_6^2} \\
u_{17} = \dfrac{A_1^2 A_5}{A_9 A_{10}} & u_{18} = -\dfrac{A_1^2 A_6}{A_9^2} & u_{19} = -\dfrac{A_1^2 A_6}{A_4^2} & u_{20} = \dfrac{A_1 A_2 A_3}{A_5^2} \\
u_{21} = \dfrac{A_1 A_2 A_3}{A_6^2} & u_{22} = \dfrac{A_1 A_2 A_3}{A_6} & u_{23} = \dfrac{4 A_1 A_2 A_3}{3 A_6^2} & u_{24} = A_1 A_2 A_3 \\
u_{25} = A_1 A_4 A_6 & u_{26} = -\dfrac{A_1 A_7 A_8}{A_9} & u_{27} = \dfrac{A_1 A_9 A_{10}}{A_8} & u_{28} = 2\dfrac{A_2^2 A_3}{A_8 A_{10}} \\
u_{29} = -\dfrac{A_2^2 A_3}{A_6 A_7 A_8} & u_{30} = \dfrac{A_2 A_3^2}{A_9 A_{10}} & u_{31} = \dfrac{A_2 A_3 A_8}{A_5 A_6^2} & u_{32} = -\dfrac{A_2 A_5 A_6}{A_7 A_8^2}
\end{array}
$$

We avoid simple relations among the $[u_i]_2$ by using a complement in $\mathbb{Z}[[u_i]_2]$ of the kernel of the composition

$$\mathbb{Z}[[u_i]_2] \to \widetilde{M}_{(2)}(F) \to \bigwedge^2 F^*_{\mathbb{Q}}$$
$$[u_i]_2 \mapsto (1-u_i) \wedge u_i$$

For computing the regulator numerically:

if $\Psi(\beta)$ is defined on $E_{\mathrm{an}} \setminus S$ and γ is a path in $E_{\mathrm{an}} \setminus S$, then $\int_\gamma \mathrm{reg}(\Phi_E(\beta)) = \int_\gamma \Psi(\beta)$

$S = \cup_i (E \cap A_i)$ may be big and "in the way", necessitating large precision when integrating over γ in $H_1(E_{\mathrm{an}}; \mathbb{Z})^+$

Using v_j in $\{A_1, \ldots, A_{12}, 2, 3\}$ we get $32 \times 14 = 448$ generators $[u_i]_2 \otimes v_j$ in $\widetilde{M}_{(2)}(F) \otimes F^*_{\mathbb{Q}}$ and find 129 independent elements in the kernel of

$$\mathrm{d} : \widetilde{M}_{(2)}(F) \otimes F^*_{\mathbb{Q}} \to \bigwedge^3 F^*_{\mathbb{Q}}$$

that also satisfy the boundary condition

$$\ker(\mathrm{d}) \to H^2(\widetilde{\mathcal{M}}_{(3)}(F)) \xrightarrow{\delta} \coprod_{P \in E^{(1)}} H^1(\widetilde{\mathcal{M}}_{(2)}(\mathbb{Q}(P)))$$

including many $[u_i]_2 \otimes u_i$ and $[u_i]_2 \otimes (1 - u_i)$.

Of those only 2 give non-zero regulators, with values $L'(E, -1)$ and $-21L'(E, -1)$. For the first one the element is, modulo $\langle [3]_2 \otimes A_1, \ldots, [3]_2 \otimes A_{12}, [3]_2 \otimes 2, [3]_2 \otimes 3 \rangle$,

$$(2[u_1]_2 + 4[u_2]_2 - 16[u_3]_2 + 43[u_4]_2 + 4[u_5]_2 - 11[u_6]_2 + 16[u_7]_2 + 22[u_8]_2 - 14[u_9]_2 + 34[u_{10}]_2 - 12[u_{11}]_2 + 20[u_{12}]_2 + 14[u_{13}]_2 - 15[u_{14}]_2 - 5[u_{15}]_2 + 26[u_{16}]_2 - 5[u_{17}]_2 - 2[u_{18}]_2 + 25[u_{19}]_2 + [u_{20}]_2 + 56[u_{21}]_2 + 34[u_{22}]_2 - 22[u_{24}]_2 - 12[u_{25}]_2 + 13[u_{26}]_2 - 12[u_{27}]_2 + 2[u_{29}]_2 - 4[u_{30}]_2 + [u_{31}]_2 - 8[u_{32}]_2) \otimes A_1 + (-19[u_2]_2 - 8[u_4]_2 + 22[u_5]_2 + 2[u_6]_2 - [u_7]_2 - 10[u_8]_2 + 9[u_9]_2 - 10[u_{10}]_2 + 5[u_{11}]_2 + 11[u_{12}]_2 + 8[u_{13}]_2 - [u_{14}]_2 + 4[u_{15}]_2 + [u_{16}]_2 + 7[u_{17}]_2 + 2[u_{18}]_2 - 19[u_{19}]_2 + [u_{20}]_2 - 14[u_{21}]_2 - 3[u_{22}]_2 + 11[u_{24}]_2 + [u_{25}]_2 - 3[u_{26}]_2 - 2[u_{27}]_2 - 2[u_{29}]_2 + 6[u_{30}]_2 + [u_{31}]_2 + 6[u_{32}]_2) \otimes A_2 +$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots +$$

$$(10[u_2]_2 + 3[u_3]_2 - 7[u_4]_2 - 8[u_5]_2 - [u_6]_2 - 2[u_9]_2 - 2[u_{10}]_2 + 10[u_{12}]_2 + 3[u_{13}]_2 - 6[u_{14}]_2 + 4[u_{15}]_2 + 5[u_{16}]_2 - 5[u_{17}]_2 - 2[u_{18}]_2 - [u_{19}]_2 + [u_{20}]_2 + [u_{21}]_2 + [u_{22}]_2 + [u_{23}]_2 + [u_{25}]_2 + 3[u_{26}]_2 + 5[u_{27}]_2 - 2[u_{29}]_2 - [u_{30}]_2 - 2[u_{31}]_2 - 3[u_{32}]_2) \otimes A_{10} + (3[u_{20}]_2 + 4[u_{23}]_2) \otimes A_{14}$$

where $A_{13} = 2$ (does not occur in this element) and $A_{14} = 3$

Another example (also over $\mathbb{Q}$)

Example For $E: y^2 + xy + y = x^3 - x^2 + 2x - 1 \quad (N = 802)$ using $A_1, \ldots, A_{21} = x, x+1, x-1, y, y-1, y+1, y+x, y-x, y-x-1, y-x+1, y+x-1, y+x+1, y-x+2, y+x+2, y+x-2, y+2x, y-2x, y+2x+1, y-2x-1, y+2x-1, y-2x+1$ there are 54 really different exceptional relations, giving $u_1 = -2A_1A_8^{-1}A_{18}^{-1}$, $u_2 = A_1A_{10}^{-1}A_{16}^{-1}, \ldots$. For the 134 basis elements β in $\ker(\mathrm{d}) \cap \ker(\delta)$ we get $R_4(\Phi_E(\beta)) = 0$ except for

$$\beta_1 = -67644406137[u_1]_2 \otimes A_1 + 12363841124[u_2]_2 \otimes A_1 + \cdots$$

$$\beta_2 = -1055221213702[u_1]_2 \otimes A_1 - 12521056913[u_2]_2 \otimes A_1 - \cdots$$

$R_4(\Phi_E(\beta_i)) = a_i L'(E, -1)$ for $a_1 = -99577827, a_2 = 21819597$ so

$R_4(\Phi_E(\beta)) = 3L'(E, -1)$ for $\beta = -1711570\beta_1 - 7811071\beta_2 =$

$835818595714399932[u_1]_2 \otimes A_1 + 7664128498987 9143[u_2]_2 \otimes A_1 + \cdots$